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# Lifespan of Solutions to Weakly Nonlinear Singular Partial Differential Equations

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**Abstract**

Let $(t, x) \in \mathbb{C} \times \mathbb{C}^n$. For a positive integer $m$, we consider the general order singular nonlinear Fuchsian partial differential equation

$$(t\partial_t)^m u = F\left(t, x, \left\{(t\partial_t)^j \partial_x^\alpha u\right\}_{(j,\alpha)\in I_m}\right),$$

where $F$ is holomorphic with respect to all variables, $I_m = \{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^n : j + |\alpha| \leq m, j < m\}$, and $|\alpha|$ is the order of the differential operator $\partial_x^\alpha$. It has been shown that the equation has a unique holomorphic solution that vanishes at $t = 0$ provided that its characteristic exponents are not positive integers at the origin. In this paper, we obtain an estimate of the lifespan of this solution under some smallness conditions on the coefficients of the Taylor expansion of $F$.

**Keywords:** lifespan of solutions, singular nonlinear PDE, majorant functions

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## 1 Introduction

The study of partial differential equations (PDEs) is fundamental to understanding various mathematical concepts and physical phenomena. In addition to classical results such as the existence and uniqueness of solutions, another important area of research concerns the lifespan of a solution, which describes the largest ball on which the solution can be defined.

Numerous authors have studied the relationship of the lifespan of the solution to the size of the initial data of Cauchy Problems. D'Ancona and Spagnolo [2] dealt with the second order nonlinear Cauchy problem

$$\begin{cases}(\partial_t^2 - P(\partial_x))u = f\big(x, \{\partial_x^\alpha u\}_{|\alpha|=1}\big),\\ u(0,x) = \varepsilon\varphi(x),\ \partial_t u(0,x) = \varepsilon\psi(x),\end{cases} \tag{1}$$

where $(t,x) \in \mathbb{R}^{n+1}$, $P(\partial_x) := \sum_{|\alpha|=2} a_\alpha \partial_x^\alpha$ is a second-order linear partial differential operator with coefficients dependent on $x$, the functions $a_\alpha, \varphi, \psi$ and $f$ are real-analytic in all variables, $f(x,0) \equiv 0$, and $|\alpha|$ denotes the sum of the components of $\alpha \in \mathbb{N}^n$. Under some hyperbolicity assumption on the coefficients $a_\alpha$, they were able to show that the lifespan, $T_\varepsilon$, of the solution of (1) satisfies $T_\varepsilon \geq \mu \log\log(1/\varepsilon)$, for some $\mu > 0$.

In a series of papers, Yamane extended this result to more general classes of second order PDEs, each without the hyperbolicity assumption [9, 10, 11, 12]. In particular, his work in 2013 [11] considered the nonlinear Cauchy problem

$$\begin{cases}\big(\partial_t^2 - P(\partial_t, \partial_x)\big)u = f\big(\{\partial_t^j \partial_x^\alpha u\}_{j+|\alpha|\leq 2, j<2}\big),\\ u(0,x) = \phi_1(x),\ \partial_t u(0,x) = \phi_2(x),\end{cases}$$

where $f$ is complex analytic near the origin whose expansion satisfies certain assumptions depending on the presence of $\partial_t$ in the operator $P$. He was able to show that if the initial data, $\phi_1$ and $\phi_2$, satisfy some Cauchy-type estimate dependent on $\varepsilon$, then the lifespan of the solution will be at least $\mu/\varepsilon$, for some $\mu > 0$.

Tolentino et al. [8] further improved Yamane's result by considering the general-order nonlinear Cauchy problem

$$\begin{cases}\partial_t^m u = f\big(\{\partial_t^j \partial_x^\alpha u\}_{(j,\alpha)\in I_m}\big),\\ \partial_t^i u(0,x) = \phi_i(x), \quad i = 0, 1, \ldots, m-1,\end{cases} \tag{2}$$

where $I_m \subseteq \{(j,\alpha) \in \mathbb{N}^{1+n} : j + |\alpha| \leq m, j < m\}$. In their work, they were able to prove a more precise estimate of the lifespan, which was shown to be greater than $\mu/\varepsilon^\sigma$, provided that the initial data satisfy some Cauchy-type estimate. Here, $0 < \sigma \leq 1$ is a quantity determined by the function $f$.

In all of these works, they were able to show that as the initial data becomes smaller, the lifespan of the solution becomes longer. We wish to extend these results to singular nonlinear differential equations of general order. In particular, we will consider

$$(t\partial_t)^m u = F\Big(t, x, \big\{(t\partial_t)^j \partial_x^\alpha u\big\}_{(j,\alpha)\in I_m}\Big),$$

where $F(t,x,Z)$ is analytic about the origin such that $F(0,x,0) \equiv 0$. We will also assume that the equation is *Fuchsian*, that is $\partial_{Z_{j,\alpha}} F(0,x,0) \equiv 0$, for any $(j,\alpha)$ such that $\alpha \neq 0$. The local solvability of this equation was studied by Gerard and Tahara [3] in 1993. They proved that if the characteristic exponents, which will be defined later, are not positive integers at the origin, then the existence of a unique local holomorphic solution that satisfies $u(0,x) \equiv 0$

is assured.

Unlike the nonsingular case, the lifespan of solutions to singular PDEs cannot be studied in terms of the Cauchy data. Instead, we consider the size of the coefficients in the Taylor expansion of $F$. In particular, we focus on the coefficients of $t^p$, which play a similar role to that of the Cauchy data in the nonsingular case, as illustrated by the following example.

**Example 1.** *Consider the Cauchy problem*

$$\begin{cases} \partial_t^2 u = \partial_t u + \partial_x^2 u, \\ u(0,x) = \phi(x), \ \partial_t u(0,x) = \psi(x). \end{cases} \tag{3}$$

*For simplicity, suppose $(t,x) \in \mathbb{C}^2$. The Cauchy–Kowalevsky theorem guarantees the existence of a unique holomorphic solution in a neighborhood of the origin of the form $u = \phi + t\psi + tw$, where $w$ is holomorphic and satisfies $w(0,x) = 0$. Multiplying both sides of* (3) *by $t$, we obtain the singular differential equation*

$$\big((t\partial_t)^2 + (t\partial_t)\big)w = t(\phi'' + \psi) + t^2\psi'' + t(w + t\partial_t w + t\partial_x^2 w).$$

This paper aims to investigate the relationship between the growth order of the coefficients in the Taylor expansion of $F$ and the lifespan of the corresponding solution. To this end, we employ a family of majorant functions introduced by Lope and Ona [5]. This family is a modified version of the majorant family originally developed by Pongérard [7].

## 2 Preliminaries

Let $\mathbb{N}$ be the set of non-negative integers and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and multi-variable $x = (x_1, \dots, x_n) \in \mathbb{C}^n$, we define $|\alpha| = \alpha_1 + \dots + \alpha_n$, $|x| = x_1 + \dots + x_n$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, and $\partial_x^\alpha = (\partial_{x_1}^{\alpha_1}, \partial_{x_2}^{\alpha_2}, \dots, \partial_{x_n}^{\alpha_n})$. For a single variable $w \in \mathbb{C}$, $|w|$ will denote the modulus of $w$. Lastly, we denote by $D_r$ the ball of radius $r$ centered at the origin.

The concept of majorants and majorant functions will be used to prove the convergence of formal power series. Here we say that the formal power series $a(x) = \sum a_\alpha x^\alpha$ is majorized by $A(x) = \sum A_\alpha x^\alpha$, denoted as $a(x) \ll A(x)$, if and only if $|a_\alpha| \le A_\alpha$ for each $\alpha$.

We present the family of majorant functions modified by Lope and Tahara [6] from the work of Lax [4]. For each $z \in \mathbb{C}$ and $i \in \mathbb{N}$, define

$$\varphi_i(z) := \frac{1}{4S} \sum_{n=0}^{\infty} \frac{z^n}{(n+1)^{2+i}}, \tag{4}$$

where $S = \frac{\pi^2}{6}$. This family of functions has very interesting properties and will be enumerated in the following proposition. The proofs can be found in [6].

**Proposition 2.** *The following statements are satisfied by $\varphi_i(z)$ for all $i, j \in \mathbb{N}$.*

*(1)* $\varphi_i(z)$ *is convergent for all* $z \in \mathbb{C}$ *with* $|z| < 1$.

*(2)* $\varphi_i(z)\varphi_i(z) \ll 2^i \varphi_i(z)$.

*(3)* $\varphi_j(z) \ll \varphi_i(z)$ *whenever* $j \geq i$.

*(4)* $2^{-2-i}\varphi_i(z) \ll \frac{d}{dz}\varphi_{i+1}(z) \ll \varphi_i(z)$.

*(5) Given any* $0 < \varepsilon < 1$, *there exists a positive constant* $C_{\varepsilon,i}$, *dependent on* $\varepsilon$ *and* $i$, *such that* $(1 - \varepsilon z)^{-1}\varphi_i(z) \ll C_{\varepsilon,i}\varphi_i(z)$.

We present another family of majorant functions, introduced by Pongérard in [7]. This will be used to prove the convergence of the formal power series and in turn the lifespan of the solution. For $i \in \mathbb{N}$ and $s \in \mathbb{N}^*$, define

$$\Phi_i^s(t, z) := \sum_{p=0}^{\infty} t^p \frac{D^{sp}\varphi_i(z)}{(sp)!}, \tag{5}$$

where $\varphi_i$ is the family of functions in (4). The following proposition list some useful properties of $\Phi_i^s(t, z)$. The proofs can be found in [7].

**Proposition 3.** *The following statements are satisfied by* $\varphi_i$ *and* $\Phi_i^s$ *for all* $i$ *and* $s$, *and for a sufficiently small* $R$.

*(1)* $\Phi_i^s\left(t, \frac{z}{R}\right)$ *is convergent for all* $|t|^{1/s} + |z| < R$.

*(2)* $\Phi_i^s(t, z)\Phi_i^s(t, z) \ll 2^i \Phi_i^s(t, z)$.

*(3) For any* $p, q \in \mathbb{N}$, $D^p\varphi_i\left(\frac{z}{R}\right) \ll \frac{p!}{(p+q)!}D^{p+q}\varphi_i\left(\frac{z}{R}\right)$.

*(4)* $\varphi_i\left(t + \frac{z}{R}\right) \ll \Phi_i^s\left(t, \frac{z}{R}\right)$.

*(5) For any* $0 < \varepsilon < 1$, *there exists* $C_{\varepsilon,i} > 0$ *such that*

$$\frac{1}{1 - \varepsilon\left(t + \frac{z}{R}\right)}\Phi_i^s\left(t, \frac{z}{R}\right) \ll C_{\varepsilon,i}\Phi_i^s\left(t, \frac{z}{R}\right).$$

The use of these families of majorant functions is motivated by their favorable interactions with holomorphic functions and with one another. In particular, their ability to reduce products of derivatives and geometric series to constant multiples of $\varphi_i$ and $\Phi_i^s$ proves instrumental in simplifying the nonlinear terms of the partial differential equation under consideration.

# 3 Main Results

Let $m \in \mathbb{N}^*$. Consider the general order singular nonlinear partial differential equation

$$(t\partial_t)^m u = F\left(t, x, \left\{(t\partial_t)^j \partial_x^\alpha u\right\}_{(j,\alpha)\in I_m}\right), \tag{6}$$

where $I_m := \{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^n; j + |\alpha| \le m, j < m\}$. Let $F(t,x,Z)$ be a function defined in a neighborhood containing the polydisk $\Delta = \{(t,x,Z) : |t| \le r_0, |x_i| \le R_0, |Z_{j,\alpha}| \le \rho\}$, for some positive constants $r_0, R_0$ and $\rho$. We will investigate equation (6) under the following assumptions:

(A1) $F(t,x,Z)$ is holomorphic in a neighborhood containing $\Delta$;

(A2) $F(0,x,0) \equiv 0$ in $\Delta_0 := \Delta \cap \{t = 0, Z = 0\}$; and

(A3) $\partial_{Z_{j,\alpha}} F(0,x,0) \equiv 0$ in $\Delta_0$ for $(j,\alpha) \in I_m$ with $|\alpha| > 0$.

Under assumptions A1 and A2, we can rewrite the function $F$ as

$$F(t,x,Z) = \sum_{p=1}^{\infty} a_p(x) t^p + \sum_{j<m} b_j(x) Z_{j,0} + \sum_{\substack{p+|q|\ge 2 \\ q \ge 1}} f_{p,q}(x) t^p Z^q,$$

for some holomorphic functions $a_p$, $b_j$ and $f_{p,q}$. Now, define the following index sets:

$$\begin{aligned}
\Lambda_1 &:= \{p \in \mathbb{N}^* : a_p(x) \ne 0\}, \\
\Lambda_2 &:= \{(p,j,\alpha) \in \mathbb{N}^* \times \mathbb{N} \times \mathbb{N}^n : (j,\alpha) \in I_m,\ f_{p,q}(x) \ne 0 \text{ where } |q| = 1\}, \\
\Lambda_3 &:= \{(p,q) \in \mathbb{N} \times \mathbb{N}^N : f_{p,q}(x) \ne 0, |q| \ge 2\}.
\end{aligned}$$

Using these sets, the Taylor expansion of $F$ can further be rewritten into

$$F(t,x,Z) = G_1(t,x) + \sum_{j<m} b_j(x) Z_{j,0} + G_2(t,x,Z) + G_3(t,x,Z),$$

where

$$\begin{aligned}
G_1(t,x) &= \sum_{p\in\Lambda_1} a_p(x) t^p, \\
G_2(t,x,Z) &= \sum_{(p,j,\alpha)\in\Lambda_2} c_{p,j,\alpha}(x) t^p Z_{j,\alpha}, \\
G_3(t,x,Z) &= \sum_{(p,q)\in\Lambda_3} d_{p,q}(x) t^p Z^q.
\end{aligned}$$

We will also assume that (6) is weakly nonlinear, which is given by the following assumption.

(A4) There exist $M > 0$, and non-negative constants $k_{p,j,\alpha}$ and $\ell_{p,q}$ such that

$$c_{p,j,\alpha}(x) \ll \frac{M\varepsilon^{k_{p,j,\alpha}}}{r_0^p \rho} \frac{1}{1-\frac{|x|}{R_0}}, \quad \text{for all } (p,j,\alpha) \in \Lambda_2, \text{ and}$$
$$d_{p,q}(x) \ll \frac{M\varepsilon^{\ell_{p,q}}}{r_0^p \rho^{|q|}} \frac{1}{1-\frac{|x|}{R_0}}, \quad \text{for all } (p,q) \in \Lambda_3.$$

Weakly nonlinear partial differential equations describe systems in which the nonlinear terms are sufficiently small, and thus are regarded as perturbations of some linear equation; this terminology is used, for example, in [1].

Define $\tau$ on $\Lambda_2$ by

$$\tau := \min_{(p,j,\alpha)\in\Lambda_2} \left\{ \frac{k_{p,j,\alpha}}{p} \right\}. \tag{7}$$

To establish an estimate for the lifespan of the solution, we have the following assumption:

(A5) The set $P := \left\{ \frac{\tau p - \ell_{p,q}}{|q|-1} \right\}_{(p,q)\in\Lambda_3}$ is bounded above.

Under assumption (A5), we can define $\omega$ on $\Lambda_3$ by

$$\omega = \max\left\{0, \sup_{(p,q)\in\Lambda_3} \left\{ \frac{\tau p - \ell_{p,q}}{|q|-1} \right\}\right\}. \tag{8}$$

Let $\rho_i(x)$ $(i = 1, 2, \ldots, m)$ be the characteristic exponents of (6), which are the zeroes of the characteristic polynomial of (6) given by

$$C(n, x) = n^m - \sum_{j<m} b_j(x) n^j.$$

Lastly, in line with the solvability results of Gérard and Tahara, we have the following assumption:

(A6) For all $i = 1, 2, \ldots, m$, $\rho_i(0) \notin \mathbb{N}^*$.

We now present our main result.

**Theorem 4.** *Suppose that equation* (6) *satisfies (A1)–(A6). Then, for each $\varepsilon \in (0,1)$ and $R \in (0, R_0/n)$, there exists a positive constant $\mu = \mu(R)$ such that the following holds: If for all $p \in \Lambda_1$,*

$$a_p(x) \ll \frac{M\varepsilon^{\omega+\tau p}}{r_0^p} \frac{1}{1-\frac{|x|}{R_0}}, \tag{9}$$

*then equation* (6) *has a unique solution $u(t,x)$ satisfying $u(0,x) \equiv 0$ that is holomorphic on $D_T \times D_R^n$, with $T = \dfrac{\mu}{\varepsilon^\tau}$ and $T \le r_0$.*

When $\Lambda_2 = \emptyset$, equation (7) can no longer be used to define $\tau$. However, we may choose $\tau = \tau_0$ arbitrarily and then define $\omega$ by equation (8) using the chosen value $\tau = \tau_0$. On the other hand, if $\Lambda_3 = \emptyset$, we define $\omega = 0$.

An interesting case arises when $G_2$ and $G_3$ are independent of the variable $t$ and $\Lambda_1$ is finite. D'Ancona and Spagnolo referred to this as the autonomous case. In this setting, we dispense with the existence of $r_0$ in assumption (A4) and (9). Moreover, $\Lambda_2 = \emptyset$ and $\tau p - \ell_{p,q} \leq 0$, so we can set $\tau = 1$ and $\omega = 0$. Consequently, $T$ is no longer bounded above by any constant.

Before proceeding to the proof, we illustrate the result with the following example.

**Example 5.** *Let $0 < \varepsilon < 1$, $0 < R < 1$ and $(t, x) \in \mathbb{C}^2$. Consider*

$$\big((t\partial_t)^2 + 1\big)u = \frac{2t}{1 - \sqrt{\varepsilon}t - x} + 3\sqrt{\varepsilon}t\partial_x u + 2\varepsilon t u \partial_x u. \tag{10}$$

*The characteristic exponents of (10) are $\pm i \notin \mathbb{N}^*$, and hence (10) admits a unique holomorphic solution $u$ satisfying $u(0, x) \equiv 0$. From (7) and (8), we obtain $\tau = \frac{1}{2}$ and $\omega = 0$. Indeed, the unique solution is given by $u(t, x) = t(1-\sqrt{\varepsilon}\,t-x)^{-1}$, whose lifespan is $T_\varepsilon = (1-R)/\varepsilon^{1/2}$.*

*Proof.* Suppose that equation (6) admits a formal power series solution of the form $u(t, x) = \sum_{k=1}^{\infty} u_k(x)t^k$. By substituting this ansatz into the equation and equating the coefficients of like powers of $t$, we derive the following recursive relation that uniquely determines the coefficients $u_k(x)$:

$$u_1(x) = \frac{a(x)}{C(1, x)}$$

and for each $n \geq 2$,

$$u_n(x) = \frac{1}{C(n, x)} F_n\left(u_1, \dots, u_{n-1}, u_1', \dots, u_{n-1}', \{f_{p,q}(x)\}_{p+|q|\leq n}\right),$$

for some polynomial $F_n$. Clearly, each coefficient $u_n$ is well-defined near the origin provided that $C(n, x)$ is nonvanishing. By assumption (A6), $R_0$ can be chosen sufficiently small so that $C(n, x) \neq 0$ for any $n \in \mathbb{N}^*$ and $x \in \mathbb{C}^n$ satisfying $|x_i| \leq R_0$. In fact, there exists a positive constant $A$ such that

$$\left|\frac{n^m + 1}{C(n, x)}\right| \leq A \tag{11}$$

for all $n \in \mathbb{N}^*$ and for all $x \in \mathbb{C}^n$ with $|x_i| \leq R_0$.

It remains to show the convergence of the formal power series. Using (9) and assumption (A4), we can show that any formal power series $W(t, x)$ satisfying

$$\begin{aligned}\big((t\partial_t)^m + 1\big)W \gg{}& \frac{AM\varepsilon^\omega}{1 - \frac{|x|}{R_0}} \sum_{p\in\Lambda_1} \left(\frac{\varepsilon^\tau t}{r_0}\right)^p + \frac{AM}{1 - \frac{|x|}{R_0}} \sum_{(p,j,\alpha)\in\Lambda_2} \frac{t^p \varepsilon^{k_{p,j,\alpha}}}{r_0^p \rho} (t\partial_t)^j \partial_x^\alpha W \\ &+ \frac{AM}{1 - \frac{|x|}{R_0}} \sum_{(p,q)\in\Lambda_3} \frac{t^p \varepsilon^{\ell_{p,q}}}{r_0^p \rho^{|q|}} \prod_{(j,\alpha)\in I_m} \left[(t\partial_t)^j \partial_x^\alpha W\right]^{q_{j,\alpha}}\end{aligned} \tag{12}$$

and $W(0, x) \equiv 0$, is a majorant of $u(t, x)$. To prove Theorem 4, we only need to find a holomorphic solution to (12), which is given by the following proposition.

**Proposition 6.** *Let $r \in (0, r_0)$ and $R \in (0, R_0)$. There exists $L > 0$ and $c \in (0,1)$ such that the function*

$$W(t,x) = L\varepsilon^{\omega+\tau} t \Phi_m^m\Big(\frac{\varepsilon^\tau t}{cr}, \frac{|x|}{R}\Big) \tag{13}$$

*satisfies the majorant relation* (12).

Let $\varepsilon > 0$, $r \in (0, r_0)$ and $R \in (0, R_0/n)$. Choose $R' \in (nR, R_0)$. Observe that Proposition 6 with Proposition 3 (1) imply that the formal solution $u(t,x)$ will converge when

$$\Big|\frac{\varepsilon^\tau t}{cr}\Big|^{1/m} + |x_1 + \cdots + x_n| < R'.$$

If $x \in D_R^n$, then $|x_1 + \cdots + x_n| \le nR$. Thus the formal solution converges on $D_T \times D_R^n$ when $|t| < T = \mu\varepsilon^{-\tau}$, where $\mu(R) = cr(R' - nR)^m$. This completes the proof of Theorem 4.

To prove Proposition 6, we will substitute $W(t,x)$ as defined in (13) to (12) and compare the constant coefficients of both sides of the relation to find the constants $R, L$ and $c$. For brevity, we let $(T,X) = (\frac{\varepsilon^\tau t}{cr}, \frac{|x|}{R})$. We start with the left-hand side. Observe that

$$\begin{aligned}
(t\partial_t)^m W &\gg Lcr\varepsilon^\omega \sum_{p=1}^{\infty} (p+1)^m \Big(\frac{\varepsilon^\tau t}{cr}\Big)^{p+1} \frac{D^{mp}\varphi_m\big(\frac{|x|}{R}\big)}{(mp)!} \\
&\gg S_m Lcr\varepsilon^\omega \sum_{p=1}^{\infty} (p+1)^m \Big(\frac{\varepsilon^\tau t}{cr}\Big)^{p+1} \frac{D^{mp-m}\varphi_0\big(\frac{|x|}{R}\big)}{(mp)\cdots(mp-m+1)((mp-m)!)} \\
&\gg \frac{S_m Lcr\varepsilon^\omega}{m^m} \sum_{p=1}^{\infty} \Big(\frac{\varepsilon^\tau t}{cr}\Big)^{p+1} \frac{D^{mp-m}\varphi_0\big(\frac{|x|}{R}\big)}{(mp-m)!} \\
&= \frac{S_m L(\varepsilon^\tau t)^2 \varepsilon^\omega}{m^m cr} \Phi_0^m(T,X),
\end{aligned}$$

where $S_m = \big(\frac{1}{2}\big)^{\sum_{n=0}^{m-1} 2+(m-n)}$ is the constant obtained when we continuously used Proposition 2 (4). Hence,

$$\big((t\partial_t)^m + 1\big) W \gg \frac{S_m L(\varepsilon^\tau t)^2 \varepsilon^\omega}{m^m cr} \Phi_0^m(T,X) + L\varepsilon^{\omega+\tau} t \Phi_m^m(T,X). \tag{14}$$

On the other end, we majorize each sum on the right-hand side of the majorant relation (12) separately. To estimate the first sum, by Proposition 3 (4) and (5) we have

$$\begin{aligned}
\frac{AM\varepsilon^\omega}{1 - \frac{|x|}{R_0}} \sum_{p \in \Lambda_1} \Big(\frac{\varepsilon^\tau t}{r_0}\Big)^p &\ll \frac{AM\varepsilon^{\omega+\tau} t}{r_0} \frac{1}{1 - \frac{|x|}{R_0}} \frac{1}{1 - \frac{\varepsilon^\tau t}{r_0}} \\
&\ll \frac{AM\varepsilon^{\omega+\tau} t}{r_0} \frac{4S}{1 - \xi_1(T+X)} \varphi_m(T+X), \\
&\ll \frac{4AC_{\xi_1,m} MS}{r_0} \varepsilon^{\omega+\tau} t \Phi_m^m(T,X),
\end{aligned}$$

where $\xi_1 = \max\{\frac{cr}{r_0}, \frac{R}{R_0}\}$. To estimate the second and third sums, we first note that by

Proposition 2 (3) and (4), and Proposition 3 (3), the following majorant relation is true for any $(j,\alpha)\in I_m$:

$$\begin{aligned}(t\partial_t)^j\partial_x^\alpha W &= \frac{Lcr\varepsilon^\omega}{R^{|\alpha|}}\sum_{p=0}^{\infty}(p+1)^j\Big(\frac{\varepsilon^\tau t}{cr}\Big)^{p+1}\frac{D^{mp+|\alpha|}\varphi_m\big(\frac{|x|}{R}\big)}{(mp)!}\\ &\ll \frac{Lcr\varepsilon^\omega}{R^{|\alpha|}}\sum_{p=0}^{\infty}(p+1)^j\Big(\frac{\varepsilon^\tau t}{cr}\Big)^{p+1}\frac{D^{mp}\varphi_{m-|\alpha|}\big(\frac{|x|}{R}\big)}{(mp)!}\\ &\ll \frac{Lcr\varepsilon^\omega}{R^{|\alpha|}}\sum_{p=0}^{\infty}(p+1)^j\Big(\frac{\varepsilon^\tau t}{cr}\Big)^{p+1}\frac{D^{mp+j}\varphi_{m-|\alpha|}\big(\frac{|x|}{R}\big)}{(mp+j)!}\\ &\ll \frac{Lcr\varepsilon^\omega}{R^{|\alpha|}}\sum_{p=0}^{\infty}(p+1)^j\Big(\frac{\varepsilon^\tau t}{cr}\Big)^{p+1}\frac{D^{mp}\varphi_{m-(j+|\alpha|)}\big(\frac{|x|}{R}\big)}{(mp+1)^j(mp)!}\\ &\ll \frac{L\varepsilon^{\omega+\tau}t}{R^m}\Phi_0^m(T,X).\end{aligned}$$

For the second sum, using the above estimate, and by multiplying both the numerator and denominator by $\varepsilon^\tau$, we obtain

$$\begin{aligned}&\frac{AM}{1-\frac{|x|}{R_0}}\sum_{(p,j,\alpha)\in\Lambda_2}\frac{t^p\varepsilon^{k_{p,j,\alpha}}}{r_0^p\rho}(t\partial_t)^j\partial_x^\alpha W\\ &\qquad\ll \frac{AM}{1-\frac{|x|}{R_0}}\sum_{(p,j,\alpha)\in\Lambda_2}\Big(\frac{\varepsilon^\tau t}{r_0}\Big)^p\varepsilon^{-\tau p+k_{p,j,\alpha}+\omega}\frac{L\varepsilon^\tau t}{\rho R^m}\Phi_0^m(T,X).\end{aligned}$$

Since $\tau\le\frac{k_{p,j,\alpha}}{p}$ for all $(p,j,\alpha)\in\Lambda_2$, it follows that $-\tau p+k_{p,j,\alpha}+\omega\ge\omega$. Hence, by another application of Proposition 3 (5), we get

$$\begin{aligned}&\frac{AM}{1-\frac{|x|}{R_0}}\sum_{(p,j,\alpha)\in\Lambda_2}\frac{t^p\varepsilon^{k_{p,j,\alpha}}}{r_0^p\rho}(t\partial_t)^j\partial_x^\alpha W\\ &\qquad\ll \frac{AM}{1-\frac{|x|}{R_0}}\frac{L\varepsilon^{\omega+\tau}t}{\rho R^m}\Phi_0^m(T,X)\frac{N\varepsilon^\tau t}{r_0(1-\frac{\varepsilon^\tau t}{r_0})}\\ &\qquad\ll \frac{ALMN\varepsilon^{\omega+2\tau}t^2}{r_0\rho R^m}\frac{1}{1-\xi_2(T+X)}\Phi_0^m(T,X)\\ &\qquad\ll \frac{AC_{\xi_2,0}LMN}{r_0\rho R^m}\varepsilon^{\omega+2\tau}t^2\Phi_0^m(T,X),\end{aligned}$$

where $\xi_2=\max\{\frac{cr}{r_0},\frac{R}{R_0}\}$. For the last sum, by Proposition 3 (2), we obtain

$$\begin{aligned}&\frac{AM}{1-\frac{|x|}{R_0}}\sum_{(p,q)\in\Lambda_3}\frac{t^p\varepsilon^{\ell_{p,q}}}{r_0^p\rho^{|q|}}\prod_{(j,\alpha)\in I_m}\big[(t\partial_t)^j\partial_x^\alpha W\big]^{q_{j,\alpha}}\\ &\qquad\ll \frac{AM}{1-\frac{|x|}{R_0}}\sum_{(p,q)\in\Lambda_3}\Big(\frac{\varepsilon^\tau t}{r_0}\Big)^p\frac{\varepsilon^{-\tau p+\ell_{p,q}}}{\rho^{|q|}}\prod_{(j,\alpha)\in I_m}\Big[\frac{L\varepsilon^{\omega+\tau}}{R^m}\Phi_0^m(T,X)\Big]^{q_{j,\alpha}}\end{aligned}$$

$$\ll \frac{AM}{1-\frac{|x|}{R_0}} \sum_{(p,q)\in\Lambda_3} \Big(\frac{\varepsilon^\tau t}{r_0}\Big)^p \varepsilon^{-\tau p+\ell_{p,q}+\omega|q|} \Big(\frac{NL\varepsilon^{\tau t}}{\rho R^m}\Big)^{|q|} \Phi_0^m(T,X).$$

Since $\omega \geq \frac{\tau p-\ell_{p,q}}{|q|-1}$, it follows that $\omega \leq -\tau p + \ell_{p,q} + \omega|q|$. Thus,

$$\begin{aligned}
&\frac{AM}{1-\frac{|x|}{R_0}} \sum_{(p,q)\in\Lambda_3} \frac{t^p\varepsilon^{\ell_{p,q}}}{r_0^p\rho^{|q|}} \prod_{(j,\alpha)\in I_m} \big[(t\partial_t)^j\partial_x^\alpha W\big]^{q_{j,\alpha}} \\
&\qquad \ll \frac{AM\varepsilon^\omega}{1-\frac{|x|}{R_0}} \sum_{(p,q)\in\Lambda_3} \Big(\frac{\varepsilon^\tau t}{r_0}\Big)^p \Big(\frac{NL\varepsilon^\tau t}{\rho R^m}\Big)^{|q|} \Phi_0^m(T,X) \\
&\qquad \ll AM\varepsilon^{\omega+2\tau}t^2\Big(\frac{1}{r_0}+\frac{NL}{\rho R^m}\Big)^2 \frac{1}{1-T(\frac{cr}{r_0}+\frac{NLcr}{\rho R^m})-X(\frac{R}{R_0})} \Phi_0^m(T,X) \\
&\qquad \ll AM\varepsilon^{\omega+2\tau}t^2\Big(\frac{1}{r_0}+\frac{NL}{\rho R^m}\Big)^2 \frac{1}{1-\xi_3(T+X)} \Phi_0^m(T,X) \\
&\qquad \ll AC_{\xi_3,0}M\varepsilon^{\omega+2\tau}t^2\Big(\frac{1}{r_0}+\frac{NL}{\rho R^m}\Big)^2 \Phi_0^m(T,X).
\end{aligned}$$

Here, in order to apply Proposition 3 (5), it is necessary that $\xi_3 = \max\{\frac{cr}{r_0}+\frac{NLcr}{\rho R^m}, \frac{R}{R_0}\} < 1$.

Therefore, the function $W(t,x)$ defined in (13) satisfies (12) if we choose $L$ large enough so that

$$L \geq \frac{4AC_{\xi_1,m}MS}{r_0},$$

and then choosing $c$ small enough so that

$$\frac{S_mL}{m^m cr} \geq \frac{AC_{\xi_2,0}LMN}{r_0\rho R^m} + AC_{\xi_3,0}M\Big(\frac{1}{r_0}+\frac{NL}{\rho R^m}\Big)^2 \quad \text{and} \quad \frac{1}{r_0}+\frac{NL}{\rho R^m} < \frac{1}{cr}.$$

□

# 4 Conclusions

We have shown that, as the coefficients in the Taylor expansion of $F$ become smaller, the lifespan of the corresponding solution becomes longer. Moreover, the lifespan depends primarily on the powers of $t$ and the sizes of the coefficients of the nonlinear terms, rather than on the differential operator $(t\partial_t)^j\partial_x^\alpha$. It would be interesting to determine conditions under which the lifespan also depends on the operator $(t\partial_t)^j\partial_x^\alpha$. Another natural direction for future research is to investigate whether the present results can be extended to other classes of formal power series.

# Conflict of interest

The authors declare that they have no conflict of interest.

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