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Fourier Transform via Hopf Algebra Formalism

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Abstract

The Fourier transform is the most important and the most studied linear transformation. It appears in many areas in mathematics and one such appearance is as an isomorphism between the algebra $C^\infty(S^1)$ of smooth complex-valued functions on the circle and the algebra $\mathcal{S}(\mathbb{Z})$ of Schwartz sequences with complex entries. Both of these algebras contain dense subalgebras that are isomorphic to rings of regular functions on complex points of two group schemes. The Fourier transform, then, restricts to an isomorphism between these subalgebras. In this article, we will exhibit the Fourier transform as the induced isomorphism between two affine group schemes. Such an induced isomorphism is inevitably an isomorphism between the representing algebras of these affine group schemes. As is well-known in the theory of affine schemes, these representing algebras are Hopf algebras. Finally, we will demonstrate the Fourier transform as an isomorphism between these Hopf algebras.

Keywords: Fourier transform, Hopf algebras

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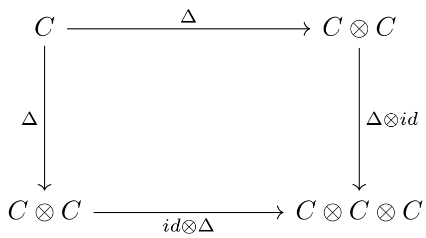
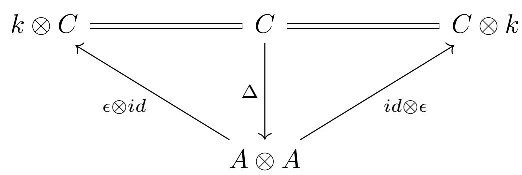
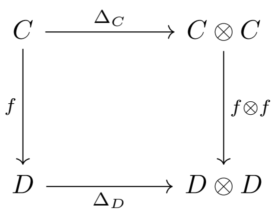
1 Introduction

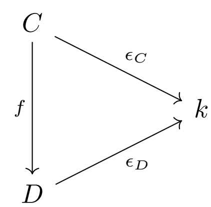
The Fourier transform is an indispensable tool of modern mathematics whose relevance goes deep both into applications and theoretical foundations. Among these, the Fourier transform creates for noncommutative geometry its most important and well-studied example– the *noncommutative torus*. The main result of this article is to demonstrate how the Fourier transform exhibits an isomorphism between two complex Hopf algebras. Let us recall what a Hopf algebra is. Let k be a commutative ring with unity. A *(coassociative, counital) coalgebra* (C, Δ, ϵ) over k is a *comonoid object* in the category of k-modules. More explicitly, this means that C is a k-module equipped with k-module maps $C \xrightarrow{\Delta} C \otimes C$ and $C \xrightarrow{\epsilon} k$ making the following diagrams commute.





The map Δ is called the *coproduct* while ϵ is called the *counit*. Here, the tensor product is understood over k. A k-linear map $(C, \Delta_C, \epsilon_C) \xrightarrow{f} (D, \Delta_D, \epsilon_D)$ is a *coalgebra homomorphism* if the following diagrams





commute. A *bialgebra* $(B, \mu, \eta, \Delta, \epsilon)$ over k is a quintuple in which (B, μ, η) is an associative, unital k-algebra, (B, Δ, ϵ) is a coassociative, counital k-coalgebra such that Δ, ϵ are k-algebra homomorphisms and μ, η are k-coalgebra homomorphisms. Here, $B \otimes B$ and k are equipped with the canonical algebra and coalgebra structures, see for example [9]. Requiring that Δ and η be algebra homomorphisms is equivalent to requiring that μ and η be coalgebra homomorphisms. Now, for a bialgebra $(B, \mu, \eta, \Delta, \epsilon)$ the k-module $End(B)$ gets an associative, unital product $\star$ defined as follows. For any $f, g \in End(B)$, the product $f \star g$ is the composite

$$B \xrightarrow{\Delta} B \otimes B \xrightarrow{f \otimes g} B \otimes B \xrightarrow{\mu} B.$$

It is immediate to check that the unity of $End(B)$ relative to the product $\star$ is $\eta \circ \epsilon$. A *Hopf algebra* H over k is a bialgebra $(H, \mu, \eta, \Delta, \epsilon)$ over k in which $id \in End(H)$ is invertible with respect to the product $\star$, say, with $\star$-inverse S. We call S the *antipode* of H. A homomorphism of Hopf algebras is an algebra and a coalgebra homomorphism commuting with the respective antipodes. In section 3, we will show that the Fourier transform is a Hopf algebra isomorphism.

1.1 Trigonometric Hopf Algebra

Consider the polynomial algebra $\mathbb{R}[c, s]$. Let $H_T = \mathbb{R}[c, s]/(c^2 + s^2 - 1)$ equipped with the quotient unital, associative algebra structure. The coproduct Δ, counit ϵ, and antipode S given as

$$\Delta(c) = c \otimes c - s \otimes s, \quad \epsilon(c) = 1, \quad S(c) = c$$

$$\Delta(s) = s \otimes c + c \otimes s, \quad \epsilon(s) = 0, \quad S(s) = -s$$

turn H into a Hopf algebra. By abuse of notation, the cosets of c and s are denoted by the same symbols. The algebra H_T is called the *trigonometric Hopf algebra*.

1.2 Group Algebras

A large class of Hopf algebras arise from groups, the so-called *group algebras*. Let G be a group and k a commutative unital ring. Denote by kG the linear span of G over k, i.e. kG consists of finite formal linear combinations of elements of G with coefficients from k. Then kG has a natural algebra structure over k with the product given by

$$\left(\sum_{g\in G} \alpha_g g\right)\left(\sum_{g\in G} \beta_g g\right) = \sum_{g\in G}\left(\sum_{h\in G} \alpha_h \beta_{h^{-1}g}\right) g$$

and the unit is given by $1 \cdot e$ where $1 \in k$ is the unity and $e \in G$ is the identity. Furthermore, kG is a Hopf algebra. The coproduct, counit, and antipode is given by

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1}$$

for all $g \in G$.

Consider a general Hopf algebra H. An element $g \in H$ is said to be *group-like* if $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$. The set $Gr(H)$ of all group-like elements in a Hopf algebra H has a natural group structure. When H is free as a k-module or when k is a field, if $Gr(H)$ forms a basis, then H is a group algebra. In general, the set of group-like elements of a Hopf algebra H is a subgroup of the group of units of H.

1.3 Dual Hopf Algebras

The linear dual H^* of a Hopf algebra $(H, \mu, \eta, \Delta, \epsilon, S)$ has a natural Hopf algebra structure. In particular, the structure maps of H^* as a Hopf algebra are $(\Delta^*, \epsilon^*, \mu^*, \eta^*, S^*)$. The group-like elements of H^* are the unital, algebra homomorphisms $H \longrightarrow k$.

Consider $\mathbb{Z}$ as a group. The group-like elements of the dual Hopf algebra $(\mathbb{R}\mathbb{Z})^*$ are the unital, algebra homomorphisms $\mathbb{R}\mathbb{Z} \longrightarrow \mathbb{R}$. Since $\mathbb{R}\mathbb{Z}$ is linearly-generated by $\mathbb{Z}$ and $\mathbb{Z}$ is cyclically generated by $1 \in \mathbb{Z}$, an algebra homomorphism $\mathbb{R}\mathbb{Z} \xrightarrow{\phi} \mathbb{R}$ is completely determined by $\phi(1)$. Unitality implies $\phi(1) = 1$. Thus, $(\mathbb{R}\mathbb{Z})^*$ has only one group-like element.

Now, let us determine the group-like elements of the dual Hopf algebra H_T^*. Again, the group-like elements of H_T^* are the unital, algebra homomorphisms $H_T \xrightarrow{\phi} \mathbb{R}$. One can take ϕ as unital algebra homomorphisms from $\mathbb{R}[c, s]$ to $\mathbb{R}$ vanishing on the ideal $(c^2 + s^2 - 1)$, i.e. $\phi(c)^2 + \phi(s)^2 = 1$. From this description, we get an isomorphism from $U(1)$ to the group

$Gr(H_T^*)$ given explicitly as

$$\begin{aligned} U(1) &\longrightarrow Gr(H_T^*), \\ e^{i\theta} &\longmapsto \phi_\theta, \end{aligned}$$

where ϕ_θ is the map given by $\phi_\theta(c) = \cos\theta$ and $\phi_\theta(s) = \sin\theta$.

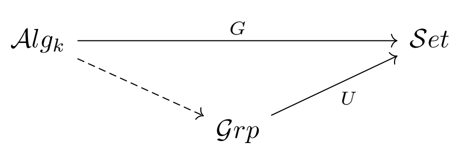
From these arguments, we see that $(\mathbb{R}\mathbb{Z})^*$and H_T^* are not isomorphic as Hopf algebras. By naturality of the dual Hopf algebra structure, this implies that $\mathbb{R}\mathbb{Z}$ and H_T are not isomorphic as Hopf algebras.

2 Group Schemes: A Brief Overview

In this section, let us briefly define what affine group schemes are. For detailed discussions, see [4], [5], and [16]. We will heavily make use of the language of categories, functors, and natural transformations. For a good treatment of these concepts, see [11]. For a given category $\mathscr{C}$ and an object x of $\mathscr{C}$, we will abuse notation by writing $x \in \mathscr{C}$. Now, given $x, y \in \mathscr{C}$ we will denote by $\mathscr{C}(x, y)$ the collection of arrows in $\mathscr{C}$ from x to y. We will ignore set-theoretic difficulties in our use of categories. Let us denote by $\mathcal{S}et$ and $\mathcal{G}rp$, the category of sets and the category of groups, respectively. Denote by $\mathcal{G}rp \xrightarrow{U} \mathcal{S}et$ the forgetful functor.

Let k be a commutative, unital ring. Denote by $\mathcal{A}lg_k$ the category of commutative, unital k-algebras and unital k-algebra homomorphisms. A *scheme X over k* is a representable functor $\mathcal{A}lg_k \xrightarrow{X} \mathcal{S}et$. *Representability* in this situation means that X is naturally isomorphic to $\mathcal{A}lg_k(A, -)$ for some $A \in \mathcal{A}lg_k$, through a natural isomorphism θ. For a detailed discussion on natural isomorphisms, see [11]. We call A the *representing object* of X, or more precisely, the representing algebra of X. Note that in most basic algebraic geometry literature, for example [8] and [15], schemes are defined differently. The definition we presented here is equivalent to the ones found in the [8, 15] using [5, Prop. VI-2]. Also, what we say a scheme *over* k reads a scheme *over* $Spec(k)$ in other literature.

A *group scheme G over k* is a scheme $\mathcal{A}lg \xrightarrow{G} \mathcal{S}et$ that factors through the forgetful functor $\mathcal{G}rp \xrightarrow{U} \mathcal{S}et$, i.e. one with a factorization



It is well-known in the theory of group schemes that the representing algebra A of a group scheme G is a Hopf algebra if and only if G is affine, see for example [16]. We will not give the definition of what an affine scheme is but the reader can take the equivalence just mentioned as a definition. In the remainder of this section, we will describe the group structures on $\mathcal{A}lg_k(A, R)$ for any $R \in \mathcal{A}_k$ realizing the family of isomorphisms $G(R) \xrightarrow{\theta_R} \mathcal{A}lg_k(A, R)$ asserted by the representability assumption. Let us denote by $\mu_A, \eta_A, \Delta_A, \epsilon_A, S_A$

the structure maps on A as a Hopf algebra and denote by μ_R, η_R the multiplication and the unit maps of R, respectively. Using these maps, we can define a product $\star$ on $\mathcal{A}lg_k(A, R)$ as follows. For any $f, g \in \mathcal{A}lg_k(A, R)$, denote by $f \star g$ the composite

$$A \xrightarrow{\Delta_A} A \otimes A \xrightarrow{f \otimes g} R \otimes R \xrightarrow{\mu_R} R.$$

Coassociativity of Δ_A and associativity of μ_R implies that the product $\star$ on $\mathcal{A}lg_k(A, R)$ is associative. The composite $\eta_R \circ \epsilon_A$ is the identity of $\mathcal{A}lg_k(A, R)$ with respect to $\star$. Every $f \in \mathcal{A}lg_k(A, R)$ is invertible with inverse $f \circ S$. Note that in general S is an algebra anti-homomorphism making $f \circ S$ the same. However, since A is commutative S becomes an algebra homomorphism. That $f \circ S$ is the inverse of f follows from the commutativity of the following diagram.

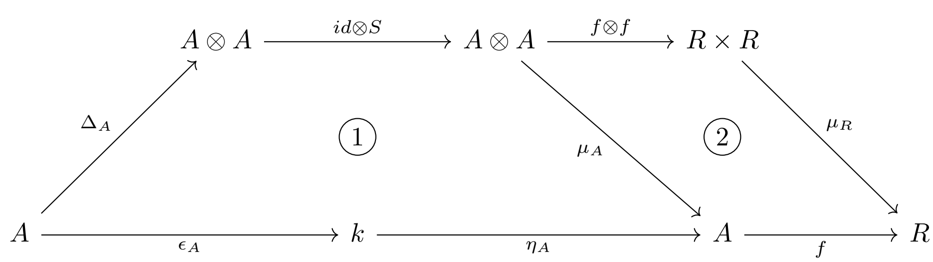


Diagram (1) commutes by definition of the antipode S while diagram (2) commutes since f is an algebra homomorphism. The commutativity of these two implies the commutativity of the outer diagram. In the next section, we will demonstrate two naturally isomorphic group schemes.

3 The Fourier Isomorphism

As we have seen in 1.3, the Hopf algebras H_T and $\mathbb{R}\mathbb{Z}$ are not isomorphic as Hopf algebras. The complexification of $\mathbb{R}\mathbb{Z}$, given by $\mathbb{R}\mathbb{Z} \otimes \mathbb{C}$ is isomorphic to $\mathbb{C}\mathbb{Z}$ as Hopf algebras. As an algebra, $\mathbb{C}\mathbb{Z}$ sits inside the algebra $\mathcal{S}(\mathbb{Z})$ of Schwartz sequences with complex coefficients as a dense subalgebra. On the other hand, the complexification of H_T, given by $H_T \otimes \mathbb{C}$ is isomorphic to $\mathbb{C}[c, s]/(c^2 + s^2 - 1)$ as Hopf algebras. Likewise, $H_T \otimes \mathbb{C}$ sits inside the algebra $C^\infty(S^1)$ of smooth complex-valued functions on the circle as a dense subalgebra. Even though the Hopf algebras $\mathbb{R}\mathbb{Z}$ and H_T are not isomorphic, their complexifications are, as illustrated by the following proposition.

Proposition 1. *The Hopf algebras $H'_T = H_T \otimes \mathbb{C}$ and $\mathbb{R}\mathbb{Z} \otimes \mathbb{C} \cong \mathbb{C}\mathbb{Z}$ are isomorphic via the Fourier transform map.*

Proof. The Fourier transform in this situation takes the form $H'_T \xrightarrow{F} \mathbb{C}\mathbb{Z}$ with $F(c) = \frac{1}{2}(a + a^{-1})$ and $F(s) = \frac{1}{2i}(a - a^{-1})$ where a denotes a generator of $\mathbb{Z}$, extended algebraically over $\mathbb{C}$. The map F is invertible with inverse $\mathbb{C}\mathbb{Z} \to H'_T$, $a \mapsto c + is$. To check that F is an isomorphism of Hopf algebras, it is enough to check that $c + is \in H'_T$ is a group-like element. Indeed, we have

$$\begin{aligned}\Delta(c+is) &= \Delta(c) + i\Delta(s) \\ &= (c \otimes c - s \otimes s) + i(s \otimes c + c \otimes s) \\ &= (c+is) \otimes (c+is)\end{aligned}$$

and $\epsilon(c+is) = \epsilon(c) + i\epsilon(s) = 1$. Here, we abuse notation by writing Δ and ϵ for the unique complex linear extensions of Δ and ϵ, respectively. □

The previous proposition implies, in particular, that the representable functors represented by $H_T \otimes \mathbb{C}$ and $\mathbb{C}\mathbb{Z}$ are isomorphic. Perhaps, this is the fundamental reason why the Fourier transform appears naturally in many places, in particular, in the theory of locally compact abelian groups. However, this is only speculative at this point.

References

[1] Canlubo, C.R.P. *Non-commutative covering spaces and their symmetries*, PhD Thesis, University of Copenhagen, 2017.

[2] Chari, V. and Pressley, A. *A Guide to Quantum Groups*, Cambridge University Press, 1994.

[3] Connes, *A. Noncommutative Geometry*, Academic Press, 2005.

[4] Demazure, M. and Gabriel, P. *Introduction to Algebraic Geometry and Algebraic Groups*, North Holland Publishing Company, 1980.

[5] Eisenbud, D. and Harris, J. *The Geometry of Schemes*, Graduate Texts in Mathematics 197, Springer-Verlag New York, 2000.

[6] Grothendieck, A. and Raynaud, M. *Revêtements étales et groupe fondamental (SGA 1)*, Documents Mathematiques (Paris) [Mathematical Documents (Paris)], 3, Paris, 1971.

[7] Hajac, P. (ed.) *Lectures on Non-commutative Geometry and Quantum Groups.* `https://www.mimuw.edu.pl/~pwit/toknotes/toknotes.pdf`

[8] Hartshorne, R. *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer-Verlag New York, 1977.

[9] Kassel, C. *Quantum Groups*, Graduate Texts in Mathematics, Springer, 1995.

[10] Khalkhali, *M. Basic Noncommutative Geometry*, Second Edition, European Mathematical Society, 2013.

[11] Mac Lane, S. *Categories for the Working Mathematician*, Graduate Texts in Mathematics 5, Springer-Verlag New York, 1978.

[12] Majid, S. *Foundation of Quantum Group Theory*, Cambridge University Press, 1995.

[13] von Neumann, J. *On Rings of Operators. Reduction Theory.* Annals of Mathematics, Second Series, 50 (2): 401485, 1949.

[14] Rudin, W. *Fourier Analysis on Groups*, Interscience Publishers 12, 1962.

[15] Shafarevich, I. *Basic Algebraic Geometry 2: Schemes and Complex Manifolds*, Springer-Verlag Berlin Heidelberg, 1994.

[16] Waterhouse, W. *Introduction to Affine Group Schemes*, Graduate Texts in Mathematics 66, Spriner-Verlag, 1979.

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