

## Leonard $n$ -tuples from normalized elements of the Lie algebra $\mathfrak{sl}_2$

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### Abstract

Leonard pairs and Leonard triples were introduced by Paul Terwilliger and have been useful in studying association schemes. We extend these notions and define Leonard quadruples, and in general, Leonard  $n$ -tuples. We improve on our results from previous work on extending certain Leonard pairs arising from the Lie algebra  $\mathfrak{sl}_2$  to Leonard triples by finding different ways to represent the constraints needed for constructing the extensions. We use these to obtain necessary and sufficient conditions that yield Leonard quadruples, and generally, Leonard  $n$ -tuples of Krawtchouk type from normalized elements of  $\mathfrak{sl}_2$ .

**Keywords:** Leonard pair, Leonard triple, Leonard quadruple, Krawtchouk polynomial, Lie algebra  $\mathfrak{sl}_2$

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## 1 Introduction

This paper is about a linear algebraic object called a Leonard  $n$ -tuple. These objects are generalizations of Leonard pairs, defined and studied by Paul Terwilliger in [4]. Leonard pairs and their generalizations provide a new perspective that has been useful in the classification of commutative association schemes. Association schemes provide a unified framework for the study of codes, designs, finite geometries and other combinatorial objects. The complete classification of commutative association schemes is an ambitious project that will take many more years to complete.

For the rest of the paper,  $\mathbb{K}$  will denote an algebraically closed field of characteristic zero and  $d$  an integer greater than or equal to 3. Let  $M_{d+1}(\mathbb{K})$  denote the  $\mathbb{K}$ -algebra consisting of

all  $d + 1$  by  $d + 1$  matrices with entries in  $\mathbb{K}$  and  $\mathbb{K}^{d+1}$  denote the  $\mathbb{K}$ -vector space consisting of all  $d + 1$  by  $1$  matrices with entries in  $\mathbb{K}$ .

A square matrix is said to be *tridiagonal* if its nonzero entries appear only on the diagonal, the superdiagonal, and the subdiagonal. A tridiagonal matrix is said to be *irreducible* if all the superdiagonal and subdiagonal entries are nonzero.

Before we introduce Leonard  $n$ -tuples, recall the notion of a Leonard pair introduced by Terwilliger in [4].

**Definition 1.** Let  $V$  denote a vector space over  $\mathbb{K}$  with finite positive dimension. By a *Leonard pair* on  $V$ , we mean an ordered pair of linear transformations  $A : V \rightarrow V$  and  $A^* : V \rightarrow V$  that satisfy both the conditions below.

- (i) There exists a basis for  $V$  with respect to which the matrix representing  $A$  is diagonal and the matrix representing  $A^*$  is irreducible tridiagonal.
- (ii) There exists a basis for  $V$  with respect to which the matrix representing  $A^*$  is diagonal and the matrix representing  $A$  is irreducible tridiagonal.

We can restate the definition of a Leonard pair on  $V = \mathbb{K}^{d+1}$  in terms of nonsingular matrices.

**Definition 2.** A *Leonard pair* on  $\mathbb{K}^{d+1}$  is an ordered pair of matrices  $A$  and  $A^*$  in  $M_{d+1}(\mathbb{K})$  that satisfy the conditions below.

- (i) There exists a nonsingular matrix  $S_1$  such that  $S_1^{-1}AS_1$  is diagonal and  $S_1^{-1}A^*S_1$  is irreducible tridiagonal.
- (ii) There exists a nonsingular matrix  $S_2$  such that  $S_2^{-1}A^*S_2$  is diagonal and  $S_2^{-1}AS_2$  is irreducible tridiagonal.

When an ordered pair of matrices  $A, A^*$  forms a Leonard pair on  $\mathbb{K}^{d+1}$  through the nonsingular matrices  $S_1$  and  $S_2$ , we say  $A, A^*$  forms a Leonard pair on  $\mathbb{K}^{d+1}$  via the matrices  $S_1$  and  $S_2$ .

**Lemma 3.** Let  $A, A^*$  be a Leonard pair on  $\mathbb{K}^{d+1}$ . Then the following are also Leonard pairs:

- (i)  $A^*, A$
- (ii)  $aA + bI, a^*A^* + b^*I$ , where  $I$  is the identity and  $aa^* \neq 0, b, b^* \in \mathbb{K}$
- (iii)  $Q^{-1}AQ, Q^{-1}A^*Q$ , for a nonsingular matrix  $Q \in M_{d+1}(\mathbb{K})$ .

*Proof.* Let  $S_1, S_2$  be the nonsingular matrices guaranteed by the Definition 2 for the Leonard pair  $A, A^*$ . Then  $A^*, A$  is a Leonard pair on  $\mathbb{K}^{d+1}$ , via the nonsingular matrices  $S_2, S_1$ . The pair  $aA + bI, a^*A^* + b^*I$  is a Leonard pair on  $\mathbb{K}^{d+1}$ , via the same nonsingular matrices  $S_1, S_2$ . The pair  $Q^{-1}AQ, Q^{-1}A^*Q$ , is a Leonard pair on  $\mathbb{K}^{d+1}$ , via the matrices  $P_1 = Q^{-1}S_1, P_2 = Q^{-1}S_2$ .  $\square$

## 2 An example

Here is an example of a Leonard pair given in [4]. Let  $V = \mathbb{K}^4$  and set

$$A = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix} \quad \text{and} \quad A^* = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

The identity matrix  $I$  satisfies condition (ii) of Definition 2, since the matrices  $A$  and  $A^*$  are irreducible tridiagonal and diagonal, respectively. The similarity matrix  $P$  of  $A$  and  $A^*$ , where

$$P = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix},$$

satisfies condition (i) of Definition 2, since the matrices  $P^{-1}A^*P = A$  and  $P^{-1}AP = A^*$  are irreducible tridiagonal and diagonal, respectively. Note that  $P^2 = 8I$ .

In general, the matrices

$$A = \begin{pmatrix} 0 & d & & & \mathbf{0} \\ 1 & 0 & d-1 & & \\ & 2 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ \mathbf{0} & & & d & 0 \end{pmatrix} \quad \text{and} \quad A^* = \text{diag}(d, d-2, \dots, -d)$$

form a Leonard pair on  $\mathbb{K}^{d+1}$  via the identity matrix  $I$  and the matrix  $P = (p_{ij})$ , where the entries of  $(p_{ij})$  are the coefficients of the polynomial  $(1 + \lambda)^{d-i}(1 - \lambda)^i = \sum_{j=0}^d p_{ij}\lambda^j$ . Explicitly

$$p_{ij} = \binom{d}{j} {}_2F_1\left(\begin{matrix} -i & -j \\ -d \end{matrix} \middle| 2\right).$$

We follow the standard notation for hypergeometric series [2]. We remark that the Leonard pairs in the previous example are part of a family of Leonard pairs that correspond to the Krawtchouk polynomials (see [3]).

### 3 Leonard pairs and the Lie algebra $\mathfrak{sl}_2$

Let

$$X = \begin{pmatrix} 0 & d & & & \mathbf{0} \\ & 0 & d-1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ \mathbf{0} & & & & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & & & & \mathbf{0} \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & d & 0 \end{pmatrix}$$

and

$$H = \text{diag}(d, d-2, \dots, -d) \in M_{d+1}(\mathbb{K}).$$

For  $d = 1$ , the matrices  $X$ ,  $Y$  and  $H$  form a basis for the Lie algebra  $\mathfrak{sl}_2 = \{Z \in M_2(\mathbb{K}) \mid \text{trace}(Z) = 0\}$  with multiplication  $[R, S] = RS - SR$  called the bracket or commutator product. When  $d > 1$ , these matrices form a subalgebra isomorphic to the algebra  $\mathfrak{sl}_2$  contained in  $\mathfrak{sl}_{d+1}$ . Note that the matrices  $X + Y$ ,  $H$  give the Leonard pair in the previous example. We fix the notation for these matrices in the succeeding.

**Theorem 4** (See Theorem 20 of [1]). *The matrix  $\alpha H + \beta X + \gamma Y$  is similar to  $\sqrt{\alpha^2 + \beta\gamma}H$  via a matrix  $K \in M_{d+1}(\mathbb{K})$  where  $K^2 = cI$ .*

Note that from Lemma 3, if  $A, A^*$  is a Leonard pair then so is  $aA + bI$ ,  $a^*A^* + b^*I$  provided  $aa^* \neq 0$ . Moreover, the members of the set  $\{(aA + bI, a^*A^* + b^*I) \mid aa^* \neq 0, b, b^* \in \mathbb{K}\}$  of ordered pairs of linear transformations are Leonard pairs provided one element in the set is a Leonard pair. In particular if  $A = \alpha_1 H + \beta_1 X + \gamma_1 Y$  and  $A^* = \alpha_2 H + \beta_2 X + \gamma_2 Y$  for some  $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2 \in \mathbb{K}$ , then the ordered pairs in  $\{(aA + bI, a^*A^* + b^*I) \mid aa^* \neq 0, b, b^* \in \mathbb{K}\}$  are Leonard pairs provided  $B = \frac{1}{\sqrt{\alpha_1^2 + \beta_1\gamma_1}}A$ ,  $B^* = \frac{1}{\sqrt{\alpha_2^2 + \beta_2\gamma_2}}A^*$  is a Leonard pair. Note that both  $B$  and  $B^*$  are similar to  $H$ . With this, we can impose that the eigenvalues of the linear transformation  $\alpha H + \beta X + \gamma Y$  be  $\theta_i = d - 2i$  for  $0 \leq i \leq d$ , so that it is similar to  $H$ , and that  $\alpha^2 + \beta\gamma = 1$ .

We say that the Leonard pair  $A = \alpha_1 H + \beta_1 X + \gamma_1 Y$ ,  $A^* = \alpha_2 H + \beta_2 X + \gamma_2 Y$  is *normalized* if both  $A$  and  $A^*$  are similar to  $H$ . For the rest of the paper we consider normalized Leonard pairs and their extensions.

**Lemma 5.** *Let  $\alpha, \beta, \gamma \in \mathbb{K}$  such that  $\alpha^2 + \beta\gamma = 1$ . The matrices  $H, \alpha H + \beta X + \gamma Y$  form a Leonard pair on  $\mathbb{K}^{d+1}$  if and only if  $\alpha \neq \pm 1$ .*

*Proof.* The first condition is necessary so that  $\alpha H + \beta X + \gamma Y$  is similar to  $H$ . Let  $Q$  be the nonsingular matrix guaranteed by Theorem 4 so that  $Q^{-1}(\alpha H + \beta X + \gamma Y)Q = H$ . Because  $Q^2 = cI$ ,  $Q^{-1}HQ = \alpha H + \beta X + \gamma Y$ . Hence  $S_1 = I, S_2 = Q$  will guarantee that the matrices  $I^{-1}(\alpha H + \beta X + \gamma Y)I$ ,  $Q^{-1}HQ$  are tridiagonal. It is irreducible if  $\beta \neq 0, \gamma \neq 0$ . Equivalently  $\alpha \neq \pm 1$ .  $\square$

**Lemma 6.** *Let  $\alpha, \beta, \gamma \in \mathbb{K}$  such that  $\alpha^2 + \beta\gamma = 1$ . The matrices  $X + Y, \alpha H + \beta X + \gamma Y$  form a Leonard pair on  $\mathbb{K}^{d+1}$  if and only if  $\frac{1}{2}(\beta + \gamma) \neq \pm 1$ .*

*Proof.* Let  $P$  be the nonsingular matrix in  $M_{d+1}(\mathbb{K})$  from Lemma 4 such that  $P^{-1}(X + Y)P = H$ . From Lemma 3 (ii),  $X + Y, \alpha H + \beta X + \gamma Y$  is a Leonard pair if and only if  $(P^{-1}(X + Y)P, P^{-1}(\alpha H + \beta X + \gamma Y)P) = (H, P^{-1}(\alpha H + \beta X + \gamma Y)P)$  is also a Leonard pair. We compute  $P^{-1}HP$ ,  $P^{-1}XP$ , and  $P^{-1}YP$ . We have

$$P^{-1}HP = X + Y,$$

$$P^{-1}(X + Y)P = H,$$

$$P^{-1}(X - Y)P = \frac{1}{2}P^{-1}[H, (X + Y)]P = \frac{1}{2}[(X + Y), H] = -X + Y.$$

Hence

$$P^{-1}XP = \frac{1}{2}(H - X + Y),$$

$$P^{-1}YP = \frac{1}{2}(H + X - Y).$$

Therefore

$$P^{-1}(\alpha H + \beta X + \gamma Y)P = \frac{1}{2}(\beta + \gamma)H + \left(\alpha - \frac{1}{2}\beta + \frac{1}{2}\gamma\right)X + \left(\alpha + \frac{1}{2}\beta - \frac{1}{2}\gamma\right)Y.$$

From Lemma 5,  $H, \frac{1}{2}(\beta + \gamma)H + \left(\alpha - \frac{1}{2}\beta + \frac{1}{2}\gamma\right)X + \left(\alpha + \frac{1}{2}\beta - \frac{1}{2}\gamma\right)Y$  is a Leonard pair provided  $\frac{1}{2}(\beta + \gamma) \neq \pm 1$ .  $\square$

**Lemma 7.** *Let  $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2 \in \mathbb{K}$  such that  $\alpha_i^2 + \beta_i\gamma_i = 1$  for  $i = 1, 2$ . The pair  $\alpha_1 H + \beta_1 X + \gamma_1 Y, \alpha_2 H + \beta_2 X + \gamma_2 Y$  is a Leonard pair on  $\mathbb{K}^{d+1}$  if and only if  $\alpha_1\alpha_2 + \frac{1}{2}(\beta_1\gamma_2 + \beta_2\gamma_1) \neq \pm 1$ .*

*Proof.* The proof is similar to the proof of Lemma 6. Let  $Q$  be the nonsingular matrix from Lemma 4 such that  $Q^{-1}(\alpha_1 H + \beta_1 X + \gamma_1 Y)Q = H$ . We compute the following:  $Q^{-1}HQ$ ,  $Q^{-1}(\beta_1 X + \gamma_1 Y)Q$ ,  $Q^{-1}(\beta_1 X - \gamma_1 Y)Q$ ,  $Q^{-1}XQ$ , and  $Q^{-1}YQ$  and obtain

$$Q^{-1}HQ = \alpha_1 H + \beta_1 X + \gamma_1 Y,$$

$$\begin{aligned} Q^{-1}(\beta_1 X + \gamma_1 Y)Q &= H - \alpha_1(\alpha_1 H + \beta_1 X + \gamma_1 Y) \\ &= \beta_1\gamma_1 H - \alpha_1\beta_1 X - \alpha_1\gamma_1 Y, \end{aligned}$$

$$\begin{aligned}
Q^{-1}(\beta_1 X - \gamma_1 Y)Q &= \frac{1}{2}Q^{-1}[H, \beta_1 X + \gamma_1 Y]Q \\
&= \frac{1}{2}[\alpha_1 H + \beta_1 X + \gamma_1 Y, \beta_1 \gamma_1 H - \alpha_1 \beta_1 X - \alpha_1 \gamma_1 Y] \\
&= -\beta_1 X + \gamma_1 Y,
\end{aligned}$$

$$Q^{-1}XQ = \frac{1}{2}\gamma_1 H + (-\alpha_1 - 1)X + \frac{-\alpha_1 \gamma_1 + \gamma_1}{\beta_1}Y,$$

$$Q^{-1}YQ = \frac{1}{2}\beta_1 H + \frac{-\alpha_1 \beta_1 + \beta_1}{\gamma_1}X + (-\alpha_1 - 1)Y.$$

Hence

$$\begin{aligned}
Q^{-1}(\alpha_2 H + \beta_2 X + \gamma_2 Y)Q &= \alpha_2(\alpha_1 H + \beta_1 X + \gamma_1 Y) \\
&\quad + \beta_2\left(\frac{1}{2}\gamma_1 H + (-\alpha_1 - 1)X + \frac{-\alpha_1 \gamma_1 + \gamma_1}{\beta_1}Y\right) \\
&\quad + \gamma_2\left(\frac{1}{2}\beta_1 H + \frac{-\alpha_1 \beta_1 + \beta_1}{\gamma_1}X + (-\alpha_1 - 1)Y\right) \\
&= \alpha' H + \beta' X + \gamma' Y.
\end{aligned}$$

We have that  $H, \alpha' H + \beta' X + \gamma' Y$  is a Leonard pair provided  $\alpha' \neq \pm 1$  or  $\alpha_1 \alpha_2 + \frac{1}{2}(\beta_1 \gamma_2 + \beta_2 \gamma_1) \neq \pm 1$ .  $\square$

## 4 Leonard triples

A Leonard triple is a natural extension of a Leonard pair. These objects were also introduced by Terwilliger [4].

**Definition 8.** Let  $V$  denote a vector space over  $\mathbb{K}$  with finite positive dimension. By a *Leonard triple* on  $V$ , we mean an ordered triple of linear transformations  $A : V \rightarrow V$ ,  $A^* : V \rightarrow V$  and  $A^\epsilon : V \rightarrow V$  that satisfy the conditions below.

- (i) There exists a basis for  $V$  for which the matrix representing  $A$  is diagonal and the matrices representing  $A^*$  and  $A^\epsilon$  are irreducible tridiagonal.
- (ii) There exists a basis for  $V$  for which the matrix representing  $A^*$  is diagonal and the matrices representing  $A$  and  $A^\epsilon$  are irreducible tridiagonal.
- (iii) There exists a basis for  $V$  for which the matrix representing  $A^\epsilon$  is diagonal and the matrices representing  $A$  and  $A^*$  are irreducible tridiagonal.

As before, we can restate the definition of a Leonard triple on  $\mathbb{K}^{d+1}$  in terms of nonsingular matrices.

**Definition 9.** Given a positive integer  $d$ , a *Leonard triple* on  $\mathbb{K}^{d+1}$  is an ordered triple of matrices  $A$ ,  $A^*$  and  $A^\epsilon$  in  $M_{d+1}(\mathbb{K})$  that satisfy the conditions below.

- (i) There exists a nonsingular matrix  $S_1$  such that  $S_1^{-1}AS_1$  is diagonal and  $S_1^{-1}A^*S_1$  and  $S_1^{-1}A^\epsilon S_1$  are irreducible tridiagonal.
- (ii) There exists a nonsingular matrix  $S_2$  such that  $S_2^{-1}A^*S_2$  is diagonal and  $S_2^{-1}AS_2$  and  $S_2^{-1}A^\epsilon S_2$  are irreducible tridiagonal.
- (iii) There exists a nonsingular matrix  $S_3$  such that  $S_3^{-1}A^\epsilon S_3$  is diagonal and  $S_3^{-1}AS_3$  and  $S_3^{-1}A^*S_3$  are irreducible tridiagonal.

When an ordered triple of matrices  $A, A^*, A^\epsilon$  forms a Leonard triple  $\mathbb{K}^{d+1}$  through the nonsingular matrices  $S_1, S_2$  and  $S_3$ , we say  $A, A^*, A^\epsilon$  form a Leonard triple  $\mathbb{K}^{d+1}$  via the matrices  $S_1, S_2$  and  $S_3$ . We note that if  $A, A^*, A^\epsilon$  form a Leonard triple on  $\mathbb{K}^{d+1}$ , then  $(A, A^*)$ ,  $(A, A^\epsilon)$  and  $(A^*, A^\epsilon)$  are Leonard pairs on  $\mathbb{K}^{d+1}$ .

## 5 Extending a given Leonard pair to a Leonard triple

We review some results of [1]. Recall that if  $A, A^*$  is a Leonard pair and  $A, A^*, A^\epsilon$  is a Leonard triple, we say that  $A^\epsilon$  extends the Leonard pair  $A, A^*$  to a Leonard triple.

**Theorem 10** (See Theorem 16 of [1]). *If  $A^\epsilon$  extends the Leonard pair  $A = X + Y, A^* = H$  to a Leonard triple,  $A^\epsilon$  is in the  $\mathbb{K}$ -linear span of  $I, X, Y, H$ .*

**Theorem 11** (See Theorem 21 of [1]). *The matrices  $X + Y, H, \alpha H + \beta X + \gamma Y$  form a Leonard triple on  $\mathbb{K}^{d+1}$  if the following are satisfied:*

- (i)  $\alpha^2 + \beta\gamma = 1$ ,
- (ii)  $\beta, \gamma \neq 0$ , and
- (iii)  $2\alpha \neq \pm(\beta - \gamma)$ .

We note here that since  $\beta$  and  $\gamma$  are not zero and  $\alpha^2 + \beta\gamma = 1$ , we have  $\alpha \neq \pm 1$ . From the proof of Lemma 6, observe that (iii) is equivalent to  $\frac{1}{2}(\beta + \gamma) \neq \pm 1$ .

## 6 Leonard $n$ -tuples

We generalize the concept of Leonard pairs and Leonard triples.

**Definition 12.** Let  $V$  denote a vector space over  $\mathbb{K}$  with finite positive dimension and  $n \geq 2$ . By a *Leonard  $n$ -tuple* on  $V$ , we mean an ordered  $n$ -tuple of linear transformations  $\{A_i : V \rightarrow V\}_{i=1}^n$ , such that for each  $i \in \{1, \dots, n\}$  there exists a basis for  $V$  for which the matrix representing  $A_i$  is diagonal and the matrices representing  $A_j$  for all  $j \neq i$  are irreducible tridiagonal.

We can restate the definition of a Leonard  $n$ -tuple on  $\mathbb{K}^{d+1}$  in terms of nonsingular matrices.

**Definition 13.** Given a positive integer  $d$  and  $n \geq 2$ , a Leonard  $n$ -tuple on  $\mathbb{K}^{d+1}$  is an ordered  $n$ -tuple of matrices  $\{A_i\}_{i=1}^n$  in  $M_{d+1}(\mathbb{K})$ , such that for each  $i$  there exists a nonsingular matrix  $S_i$  such that  $S_i^{-1}A_iS_i$  is diagonal and  $S_i^{-1}A_jS_i$  is irreducible tridiagonal for each  $j \neq i$ .

When  $n = 2, 3$  this is just the definition of a Leonard pair and Leonard triple. When  $n = 4$  we call the 4-tuple a Leonard quadruple. We note that any permutation of a Leonard  $n$ -tuple  $A_1, A_2, \dots, A_n$  on  $V$  is also a Leonard  $n$ -tuple on  $V$ .

## 7 Leonard quadruples from Leonard pairs constructed from the standard basis of the Lie algebra $\mathfrak{sl}_2$

In this section we consider the construction of Leonard quadruples. Given a Leonard pair  $A_1, A_2$  we want to find  $A_3, A_4 \in M_{d+1}(\mathbb{K})$  such that the matrices  $A_1, A_2, A_3, A_4$  form a Leonard quadruple. We will use the next two lemmas to find  $A_3$  and  $A_4$ .

**Lemma 14.** *Let  $A_1, A_2, A_3, A_4 \in \text{End}(V)$ . The following are equivalent:*

- (i)  $A_1, A_2, A_3, A_4$  is a Leonard quadruple.
- (ii) Each 3-subset of  $A_1, A_2, A_3, A_4$  is a Leonard triple.
- (iii) Each 2-subset of  $A_1, A_2, A_3, A_4$  is a Leonard pair.

*Proof.* The following are clear by definition (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii). We now prove (iii)  $\Rightarrow$  (i). Note that if  $\mathcal{B} = \{e_0, \dots, e_d\}$  is a basis for which the matrix representing  $A_1$  is diagonal and  $A_2$  is irreducible tridiagonal, then  $\mathcal{B}' = \{\alpha_0 e_0, \dots, \alpha_d e_d\}$  and  $\mathcal{B}'' = \{\beta_0 e_d, \dots, \beta_d e_0\}$  where  $\alpha_i, \beta_i \neq 0$  for  $0 \leq i \leq d$  are the only bases for which the matrix representing  $A_1$  is diagonal and the matrix representing  $A_2$  is irreducible tridiagonal. Since  $A_1, A_3$  and  $A_1, A_4$  are also Leonard pairs, via the either  $\mathcal{B}'$  or  $\mathcal{B}''$  the matrix representing  $A_1$  is diagonal and the matrix representing  $A_3$  and  $A_4$  are irreducible tridiagonal. Hence via the basis  $\mathcal{B}'$  or  $\mathcal{B}''$ , the matrix representing  $A_1$  is diagonal and the matrices representing  $A_2, A_3, A_4$  are irreducible tridiagonal. We use the same argument to show that there are bases  $\mathcal{B}_i$  for  $i = 2, 3, 4$  for which the matrix representing  $A_i$  is diagonal and the matrices representing the other three are irreducible tridiagonal.  $\square$

**Lemma 15.** *The quadruple  $A_1, A_2, A_3, A_4$  is a Leonard quadruple on  $\mathbb{K}^{d+1}$  if and only if the following are satisfied:*

- (i)  $A_1, A_2, A_3$  is a Leonard triple.
- (ii)  $A_1, A_2, A_4$  is also a Leonard triple.

(iii)  $A_3, A_4$  is a Leonard pair.

*Proof.* If  $A_1, A_2, A_3, A_4$  is a Leonard quadruple, then each 3-subset is a Leonard triple, in particular  $A_1, A_2, A_3$  and  $A_1, A_2, A_4$ . Also each 2-subset is a Leonard pair, in particular  $A_3, A_4$  is a Leonard pair. For the reverse direction we show that each 2-subset is a Leonard pair and use Lemma 14 to show that it is a Leonard quadruple. Since  $A_1, A_2, A_3$  is a Leonard triple, each 2-subset is a Leonard pair so  $A_1, A_2, A_1, A_3$  and  $A_2, A_3$  are Leonard pairs. Because  $A_1, A_2, A_4$  is also a Leonard triple  $A_1, A_4$  and  $A_2, A_4$  are also Leonard pairs. It is given that  $A_3, A_4$  is a Leonard pair. The result follows.  $\square$

We return to our Leonard pair  $A_1 = X + Y$  and  $A_2 = H$  and obtain the following result.

**Theorem 16.** *The matrices  $A_1 = X + Y, A_2 = H, A_3 = \alpha_3 H + \beta_3 X + \gamma_3 Y, A_4 = \alpha_4 H + \beta_4 X + \gamma_4 Y$  form a Leonard quadruple on  $\mathbb{K}^{d+1}$  if and only if the following conditions are satisfied:*

- (i)  $\alpha_i^2 + \beta_i \gamma_i = 1$  for  $i = 3, 4$ .
- (ii)  $\alpha_i \neq \pm 1$  for  $i = 3, 4$ .
- (iii)  $\frac{1}{2}(\beta_i + \gamma_i) \neq \pm 1$  for  $i = 3, 4$ .
- (iv)  $\alpha_3 \alpha_4 + \frac{1}{2}(\beta_3 \gamma_4 + \beta_4 \gamma_3) \neq \pm 1$ .

*Proof.* The proof is straightforward from Lemma 15, Theorem 11 and Lemma 7.  $\square$

## 8 Leonard $n$ -tuples from normalized elements of $\mathfrak{sl}_2$

In this last section, we determine when a given set of normalized elements of  $\mathfrak{sl}_2$  forms a Leonard  $n$ -tuple. The next theorem gives conditions on the  $n$ -tuples.

**Theorem 17.** *Let  $n \geq 2$ . For an  $n$ -tuple of linear transformations  $A_1, \dots, A_n$ , the following are equivalent:*

- (i)  $A_1, \dots, A_n$  is a Leonard  $n$ -tuple.
- (ii) Each 3-subset of  $A_1, \dots, A_n$  is a Leonard triple.
- (iii) Each 2-subset of  $A_1, \dots, A_n$  is a Leonard pair.

*Proof.* The proof is similar to the proof of Lemma 14 and is omitted.  $\square$

**Theorem 18.** *Let  $n \geq 2$ . For each  $i \in \{1, \dots, n\}$ , let  $\alpha_i, \beta_i, \gamma_i \in \mathbb{K}$  such that  $\alpha_i^2 + \beta_i \gamma_i = 1$  and let  $A_i = \alpha_i H + \beta_i X + \gamma_i Y$ . The  $n$ -tuple  $A_1, \dots, A_n$  is a Leonard  $n$ -tuple on  $\mathbb{K}^{d+1}$  if and only if for distinct  $i, j \in \{1, \dots, n\}$ ,  $\alpha_i \alpha_j + \frac{1}{2}(\beta_i \gamma_j + \beta_j \gamma_i) \neq \pm 1$ .*

*Proof.* Because of Theorem 17, we just need to show each pair is a Leonard pair. The result follows from Lemma 7.  $\square$

We end this paper by providing an example.

**Example 19.** Let  $n \geq 2$ . Let  $A_k = \frac{1}{k}X + kY$  for  $k = 1, \dots, n-1$  and  $A_n = H$ . Then  $A_1, \dots, A_n$  is a Leonard  $n$ -tuple on  $\mathbb{K}^{d+1}$ .

For each  $k \in \{1, \dots, n-1\}$ , note that  $\alpha_k^2 + \beta_k \gamma_k = 1$ . For distinct  $i, j \in \{1, \dots, n-1\}$ ,  $A_i, A_j$  is a Leonard pair since  $\alpha_i \alpha_j + \frac{1}{2}(\beta_i \gamma_j + \beta_j \gamma_i) = \frac{i^2 + j^2}{2ij} \neq \pm 1$ . For  $i \in \{1, \dots, n-1\}$ , we show that  $A_i, A_n$  is a Leonard pair. Observe that  $\alpha_i \alpha_n + \frac{1}{2}(\beta_i \gamma_n + \beta_n \gamma_i) = 1 \cdot 0 + \frac{1}{2}(\frac{1}{i} \cdot 0 + 0 \cdot i) = 0$  which is not equal to  $\pm 1$ . By Theorem 18,  $A_1, \dots, A_n$  is a Leonard  $n$ -tuple on  $\mathbb{K}^{d+1}$ .

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