MATIMYÁS MATEMATIKA
ISSN 0115-6926

Journal of the Mathematical Society of the Philippines
Vol. 47 No. 1 (2024) pp. 21-43

Renormalized solution for a quasilinear elliptic problem posed in a two-component domain

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Abstract

This research aims to prove the existence and uniqueness of the solution for a class of quasilinear elliptic problem posed in a two-component domain with the given data in L^1 and not globally bounded matrix field. Since we have weak data, we cannot use the variational framework for our problem. We then consider the notion of renormalized solution, which was introduced by Di Perna and Lions. For the existence results, we first consider an approximate problem, where the solution approximates the renormalized solution. To show uniqueness, we show that if there are two solutions, then the L^1-norm of their difference is zero.

Keywords: renormalized solution, quasilinear elliptic equation, weak data
2020 MSC: 35A01, 35A02

1 Introduction

In this study, we will prove the existence and uniqueness of the renormalized solution of the following quasilinear elliptic equations:

$$\begin{cases} -\operatorname{div}(B(x,u_1)\nabla u_1) + \lambda u_1 = f & \text{in } \Omega_1, \\ -\operatorname{div}(B(x,u_2)\nabla u_2) + \lambda u_2 = f & \text{in } \Omega_2 \\ u_1 = 0 & \text{on } \partial\Omega, \\ (B(x,u_1)\nabla u_1)\nu_1 = (B(x,u_2)\nabla u_2)\nu_1 & \text{on } \Gamma, \\ (B(x,u_1)\nabla u_1)\nu_1 = -h(x)(u_1 - u_2) & \text{on } \Gamma. \end{cases} \tag{P}$$

The sets Ω_1 and Ω_2 are the two components of the domain $\Omega \subseteq \mathbb{R}^N, N \geq 2$, with Γ being the interface between them. The function f is in $L^1(\Omega)$, h is a nonnegative $L^\infty(\Gamma)$-function, and the matrix field B is elliptic, bounded only on compact subsets of $\mathbb{R}$, and $\lambda > 0$.

When f is in $L^2(\Omega)$, the weak solution of (P) can be obtained by using the Lax-Milgram Theorem and the Schauder's Fixed Point Theorem. The case $\lambda = 0$ and $f \in L^2(\Omega)$ was studied in [1] while the case $\lambda = 0$ and $f \in L^1(\Omega)$ was done in [11, 12].

When we have a weak data (e.g., L^1 data) and a bounded matrix field for the following Dirichlet boundary problem

$$\begin{cases} -\operatorname{div}(A(x,u)\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

a solution in the distribution sense exists but this solution may not be unique as seen in the counterexample by Serrin in [18]. If we remove the boundedness assumption on A, then we cannot expect that $A \in L^1_{loc}$ and thus, a solution in the distribution sense may not even exist. There are frameworks that deal with these kinds of problem: Solutions obtained as limit approximations (SOLA, introduced by Dall'Aglio in [7]), entropy solutions (developed by Benilan et al. in [2]), and renormalized solutions, which is the framework that we choose for this proposed study.

The framework of renormalized solution was introduced by DiPerna and Lions in [10] for first-order equations. It was then further developed by Lions and Murat in [14, 17] for second-order elliptic equations. This framework deals with the existence and uniqueness of solution when the given data is weak (i.e., if the given function is L^1 and/or the matrix field is not bounded). There are various works regarding renormalized solutions; one may refer to the following non-exhaustive list: [2, 3, 4, 5, 8]

The renormalized framework for elliptic problems in a two-component domain was introduced in [11], where the authors proved the existence of a renormalized solution for (P) when $\lambda = 0$. In the mentioned study, the difficulty lies in managing the term with the jump. In fact, looking at Definition 2.3, one will see that there are extra conditions that deal with the integral over the interface Γ. For this paper, we will adapt the arguments used in [11] in passing to the limit to show the existence of a solution.

For our uniqueness result, we look at [12], where the uniqueness of the renormalized solution (in the sense of the definition given in [11]) of (P) with $\lambda = 0$ is shown. In this study, the authors have to resolve an issue regarding the regularity of the solution in $L^1(\Gamma)$. In the current paper, we will use this regularity result in the second major step of the proof of Theorem 3.2, where we prove the uniqueness of the renormalized solution of (P). Moreover, we will use Theorem 3.4 from [9] which states that when we assume a Lipschitz continuity condition on the matrix field B, we will have a function φ with very helpful properties.

To show the existence of a renormalized solution of (P), we will employ the usual technique of considering first an approximate problem (see (12)) and show that the approximate solutions converge to a renormalized solution. As for the uniqueness results, as mentioned above, we first have to show that the trace of a renormalized solution is in $L^1(\Gamma)$ and then follow the steps done in [12]. That is, we suppose that there are two solutions and show that the L^1-norm of the difference of these two solutions is zero. To this aim, we need to choose appropriate test functions.

The paper is organized as follows: Section 2 is for the Preliminaries, where the assumptions and important definitions are discussed. The final section is dedicated to the existence and uniqueness results.

2 Preliminaries

In this section, we define the two-component domain Ω and its components. We also discuss the assumptions for problem (P) and the proper Sobolev space where a solution may exist. Finally, since we are considering the notion of renormalized solution, we present the definition of a renormalized solution of (P).

We start with the domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, which is a connected bounded open set with its boundary denoted by $\partial\Omega$. We define its two components Ω_1 and Ω_2 as follows: Ω_2 is an open subset of Ω such that $\overline{\Omega_2} \subset \Omega$ and its boundary denoted by Γ is a Lipschitz continuous boundary, and $\Omega_1 = \Omega \setminus \overline{\Omega_2}$. One can see Γ is the interface between the two components and we can express the domain Ω as the disjoint union

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma.$$

We now prescribe the following assumptions on (P):

(A1) $f \in L^1(\Omega)$

(A2) $h \in L^\infty(\Gamma)$ and there exists $h_0 > 0$ such that $0 < h_0 < h(y)$, for a.e. $y \in \Gamma$

(A3) The matrix field B is a Carathéodory function, that is,

- (a) the map $t \mapsto B(x,t)$ is continuous for a.e. $x \in \Omega$
- (b) the map $x \mapsto B(x,t)$ is measurable for a.e. $t \in \mathbb{R}$,

and B satisfies the following properties:

- (A3.1) $B(x,t)\xi\xi \geq \alpha|\xi|^2$, for a.e. $x \in \Omega$, for any $t \in \mathbb{R}$, and for any $\xi \in \mathbb{R}^N$
- (A3.2) $B(x,t) \in L^\infty(\Omega \times (-k,k))^{N\times N}$, for any $k > 0$
- (A3.3) there exists $M > 0$ such that for any $r, s \in \mathbb{R}$,

$$|B(x,r) - B(x,s)| \leq M|r-s|,$$

that is, B is globally Lipschitz with respect to the second variable.

Since a solution to (P) has a jump, we have to consider the following space. Let V_1 be the space defined by

$$V_1 = \{v \in H^1(\Omega_1) : v = 0 \text{ on } \partial\Omega\} \quad \text{with} \quad \|v\|_{V_1} := \|\nabla v\|_{L^2(\Omega_1)}.$$

Define $V := \{v \equiv (v_1, v_2) : v_1 \in V_1 \text{ and } v_2 \in H^1(\Omega_2)\}$, equipped with the norm

$$\|v\|_V^2 := \|\nabla v_1\|^2_{L^2(\Omega_1)} + \|\nabla v_2\|^2_{L^2(\Omega_2)} + \|v_1 - v_2\|^2_{L^2(\Gamma)}. \tag{1}$$

Proposition 2.1 ([15, 16])**.** *The norm given in* (1) *is equivalent to the norm of* $V_1 \times H^1(\Omega_2)$*, that is, there exist 2 positive constants* c_1, c_2 *such that*

$$c_1\|v\|_V \leq \|v\|_{V_1\times H^1(\Omega_2)} \leq c_2\|v\|_V, \quad \forall v \in V.$$

An important function for the framework of the renormalized solution is the truncation function $T_k : \mathbb{R} \longrightarrow \mathbb{R}$. We define T_k as follows:

$$T_k(t) = \begin{cases} -k, & \text{if } t \leq k \\ t, & \text{if } -k \leq t \leq k \\ k, & \text{if } t \geq k. \end{cases} \tag{2}$$

The following proposition assures us that even if a solution u is not in V, its gradient and trace can be still defined with proper conditions.

Proposition 2.2 ([2, 11])**.** *Let $u = (u_1, u_2) : \Omega \setminus \Gamma \longrightarrow \mathbb{R}$ be a measurable function such that $T_k(u) \in V$ for every $k > 0$. For $i = 1, 2$,*

1. *there exists a unique measurable function $v_i : \Omega_i \longrightarrow \mathbb{R}^N$ such that for all $k > 0$,*

$$\nabla T_k(u_i) = v_i \chi_{\{|u_i|<k\}} \quad a.e. \ in \ \Omega_i,$$

where $\chi_{\{|u_i|<k\}}$ denotes the characteristic function of $\{x \in \Omega_i : |u_i(x)| < k\}$. We define v_i as the gradient of u_i and write $v_i = \nabla u_i$.

2. *if*

$$\sup_{k \geq 1} \frac{1}{k} \|T_k(u)\|_V^2 < \infty,$$

then there exists a unique measurable function $w_i : \Gamma \longrightarrow \mathbb{R}$, for $i = 1, 2$, such that

$$\gamma_i(T_k(u_i)) = T_k(w_i) \quad a.e. \ in \ \Gamma, \tag{3}$$

where $\gamma_i : H^1(\Omega_i) \longrightarrow L^2(\Gamma)$ is the trace operator. We define the function w_i as the trace of u_i on Γ and set

$$\gamma_i(u_i) = w_i.$$

With this proposition, we now present the definition of a renormalized solution of (P).

Definition 2.3. *Let $u = (u_1, u_2) : \Omega \setminus \Gamma \longrightarrow \mathbb{R}$ be a measurable function. Then u is a renormalized solution of* (P) *if*

$$T_k(u) \in V, \quad \forall k > 0; \tag{4a}$$

$$(u_1 - u_2)(T_k(u_1) - T_k(u_2)) \in L^1(\Gamma), \quad \forall k > 0; \tag{4b}$$

$$\lim_{n\to\infty} \frac{1}{n} \int_{\{|u|<n\}} B(x, u)\nabla u \cdot \nabla u \, dx = 0; \tag{5a}$$

$$\lim_{n\to\infty} \frac{1}{n} \int_{\Gamma} (u_1 - u_2)(T_n(u_1) - T_n(u_2)) \, d\sigma = 0; \tag{5b}$$

and for any $S_1, S_2 \in C^1(\mathbb{R})$ (or equivalently for any $S_1, S_2 \in W^{1,\infty}(\mathbb{R})$) with compact *support, u satisfies*

$$\begin{aligned}
&\int_{\Omega_1} S_1(u_1)B(x,u_1)\nabla u_1\cdot\nabla\varphi_1\,dx+\int_{\Omega_1} S_1'(u_1)B(x,u_1)\nabla u_1\cdot\nabla u_1\,\varphi_1\,dx\\
&+\int_{\Omega_2} S_2(u_2)B(x,u_2)\nabla u_2\cdot\nabla\varphi_2\,dx+\int_{\Omega_2} S_2'(u_2)B(x,u_2)\nabla u_2\cdot\nabla u_2\,\varphi_2\,dx\\
&+\lambda\int_{\Omega_1} S_1(u_1)u_1\varphi_1\,dx+\lambda\int_{\Omega_2} S_2(u_2)u_2\varphi_2\,dx\\
&+\int_{\Gamma} h(x)(u_1-u_2)(\varphi_1 S_1(u_1)-\varphi_2 S_2(u_2))\,d\sigma\\
&=\int_{\Omega_1} f\varphi_1 S_1(u_1)\,dx+\int_{\Omega_2} f\varphi_2 S_2(u_2)\,dx,
\end{aligned}\tag{6}$$

for all $\varphi\in V\cap(L^\infty(\Omega_1)\times L^\infty(\Omega_2))$ and S_i', $i=1,2$, denotes the classical derivative of S_i.

Remark 2.4. *Proposition 2.2 assures us that even though we are not sure if $u\in L^1_{loc}\Omega$, the terms with ∇u_i, $i=1,2$, in (6) are well-defined since it is multiplied with $S_i(u_i)$, where S_i has a compact support. In addition, the regularity conditions (4a) and (4b) justifies the validity of the integrals in the renormalized formulation (6) (see [11] for the explanation).*

3 Main Results

In this section, we prove the existence and uniqueness of the renormalized solution of (P). We begin by the following proposition, which is Theorem 3.4 of [9] which will be used in the uniqueness proof.

Proposition 3.1 ([9])**.** *Suppose that (A3.3) holds. Then there exists a function $\varphi\in C^1(\mathbb{R})$ that satisfies the following properties:*

$$\varphi(0)=0 \quad \textit{and} \quad \varphi'\geq 1. \tag{7}$$

In addition, there are constants $\delta>1/2$, $0<k_0<1$, and $L>0$ such that

$$\frac{\varphi'}{(1+|\varphi|)^{2\delta}}\in L^\infty(\mathbb{R}), \tag{8}$$

and for any $r,s\in\mathbb{R}$ satisfying $|\varphi(r)-\varphi(s)|\leq k$, for $0<k<k_0$,

$$\left|\frac{B(x,r)}{\varphi'(r)}-\frac{B(x,s)}{\varphi'(s)}\right|\leq\frac{1}{\varphi'(s)}\frac{Lk}{(1+|\varphi(r)|+|\varphi(s)|)^\delta} \tag{9}$$

and

$$\frac{1}{L}\leq\frac{\varphi'(s)}{\varphi'(r)}\leq L. \tag{10}$$

Theorem 3.2. *Under the assumptions (A1)-(A3), problem* (P) *has a unique renormalized solution.*

Proof. The proof is divided into two main steps. The first step is dedicated for the existence result. Here, we consider an approximate problem (see (12)), where the matrix field B^ε is elliptic and bounded while the given function f^ε is in $L^2(\Omega)$. We will then show that the limit of a subsequence of the sequence of weak solutions $\{u^\varepsilon\}$ to (12) converges pointwise to a renormalized solution u of (P).

The second step is for showing the uniqueness of the renormalized solution. In this part, we suppose that there are two renormalized solutions, say u and v, to (P) and then show that $\|u-v\|_{L^1(\Omega)} = 0$. For this, we need to choose appropriate test functions for the renormalized formulation (6) corresponding to the solutions u and v. These test functions are chosen in such a way that when we subtract the resulting renormalized formulations, we can combine some integrals due to similar terms.

Step 1: Existence result

Let $\varepsilon > 0$. Let $\{f^\varepsilon\}$ be a sequence in $L^2(\Omega)$ such that

$$f^\varepsilon \longrightarrow f \quad \text{strongly in } L^1(\Omega) \text{ as } \varepsilon \longrightarrow 0. \tag{11}$$

Furthermore, define

$$B^\varepsilon(x,t) = B(x, T_{1/\varepsilon}(t)) \in L^\infty(\Omega \times \mathbb{R})^{N\times N},$$

where $T_{1/\varepsilon}$ is given by (2).

Consider the approximate problem:

$$\begin{cases} -\operatorname{div}(B^\varepsilon(x,u_1^\varepsilon)\nabla u_1^\varepsilon) + \lambda u_1^\varepsilon = f^\varepsilon & \text{in } \Omega_1,\\ -\operatorname{div}(B^\varepsilon(x,u_2)\nabla u_2^\varepsilon) + \lambda u_2^\varepsilon = f^\varepsilon & \text{in } \Omega_2\\ u_1 = 0 & \text{on } \partial\Omega,\\ (B^\varepsilon(x,u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1^\varepsilon = (B^\varepsilon(x,u_2^\varepsilon)\nabla u_2^\varepsilon)\nu_1^\varepsilon & \text{on } \Gamma,\\ (B^\varepsilon(x,u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1^\varepsilon = -h(x)(u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma, \end{cases} \tag{12}$$

with variational formulation

$$\begin{cases} \text{Find } u^\varepsilon \in V \text{ such that } \forall \varphi = (\varphi_1,\varphi_2) \in V\\ \displaystyle\int_{\Omega_1} B^\varepsilon(x,u_1^\varepsilon)\nabla u_1^\varepsilon \cdot \nabla\varphi_1\,dx + \int_{\Omega_2} B^\varepsilon(x,u_2^\varepsilon)\nabla u_2^\varepsilon\cdot\nabla\varphi_2\,dx + \lambda\int_{\Omega_1} u_1^\varepsilon\varphi_1\,dx\\ \displaystyle +\lambda\int_{\Omega_2} u_2^\varepsilon\varphi_2\,dx + \int_\Gamma h(x)(u_1^\varepsilon - u_2^\varepsilon)(\varphi_1-\varphi_2)\,d\sigma = \int_\Omega f^\varepsilon\varphi\,dx. \end{cases} \tag{13}$$

One can show by Schauder's fixed point theorem (see e.g., [1]) that this variational formulation has a solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon) \in V$.

We claim that for any $\varepsilon > 0$ and for any $k > 0$,

$$\|T_k(u^\varepsilon)\|_V^2 \leq Mk, \quad \text{for some } M > 0. \tag{14}$$

Let $\varepsilon > 0$ and $k > 0$. Choose $\varphi_i = T_k(u_i^\varepsilon)$, $i = 1, 2$, in the variational formulation (13). We then have

$$\begin{aligned}
&\int_{\Omega_1} B^\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon \nabla T_k(u_1^\varepsilon)\,dx + \int_{\Omega_2} B^\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon \nabla T_k(u_2^\varepsilon)\,dx \\
&+ \lambda \int_{\Omega_1} u_1^\varepsilon T_k(u_1^\varepsilon)\,dx + \lambda \int_{\Omega_2} u_2^\varepsilon T_k(u_2^\varepsilon)\,dx \\
&+ \int_\Gamma h(x)(u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon))\,d\sigma_x \\
&= \int_{\Omega_1} f^\varepsilon T_k(u_1^\varepsilon)\,dx + \int_{\Omega_2} f^\varepsilon T_k(u_2^\varepsilon)\,dx.
\end{aligned} \tag{15}$$

Note that for $i = 1, 2$, by (A3.1),

$$\begin{aligned}
\int_{\Omega_i} B^\varepsilon(x, u_i^\varepsilon)\nabla u_i^\varepsilon \nabla T_k(u_i^\varepsilon)\,dx &= \int_{\Omega_i} B^\varepsilon(x, u_i^\varepsilon)\nabla T_k(u_i^\varepsilon) \nabla T_k(u_i^\varepsilon)\,dx \\
&\geq \alpha \|\nabla T_k(u_i^\varepsilon)\|_{L^2(\Omega_i)},
\end{aligned}$$

and since $\lambda > 0$ and T_k is nondecreasing,

$$\lambda \int_{\Omega_i} u_i^\varepsilon T_k(u_i^\varepsilon) \geq 0.$$

Moreover, due to the Lipschitz continuity of T_k with Lipschitz constant 1 and (A2), we have

$$\begin{aligned}
\int_\Gamma h(x)(u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon))\,d\sigma_x &\geq h_0 \int_\Gamma (T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon))^2\,d\sigma_x \\
&= h_0 \|T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)\|_{L^2(\Gamma)}^2.
\end{aligned}$$

It then follows from (15) and the fact that $|T_k| \le k$ that for some $C > 0$,

$$\begin{aligned} C\|T_k(u^\varepsilon)\|_V^2 &\le \alpha\|\nabla T_k(u_1^\varepsilon)\|_{L^2(\Omega_1)}^2 + \alpha\|\nabla T_k(u_2^\varepsilon)\|_{L^2(\Omega_2)}^2 \\ &\qquad + h_0\|T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)\|_{L^2(\Gamma)}^2 \\ &\le \int_{\Omega_1} B^\varepsilon(x, u_1^\varepsilon)\nabla T_k(u_1^\varepsilon)\nabla T_k(u_1^\varepsilon)\, dx \\ &\qquad + \int_{\Omega_2} B^\varepsilon(x, u_2^\varepsilon)\nabla T_k(u_2^\varepsilon)\nabla T_k(u_2^\varepsilon)\, dx \\ &\qquad + \int_\Gamma h(x)(u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon))\, d\sigma_x \\ &\le \int_{\Omega_1} f^\varepsilon T_k(u_1^\varepsilon)\, dx + \int_{\Omega_2} f^\varepsilon T_k(u_2^\varepsilon)\, dx \\ &\le k\|f^\varepsilon\|_{L^1(\Omega)}. \end{aligned} \tag{16}$$

Since $\{f^\varepsilon\}$ is a convergent sequence in $L^1(\Omega)$, it is a bounded sequence. Thus, we have (14).

By the Rellich-Kondrachov Theorem, we know that the embeddings $V \hookrightarrow L^2(\Omega_1) \times L^2(\Omega_2)$ and $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ are compact. Thus, since $\{T_k(u^\varepsilon)\}$ is a bounded sequence in V for all $k > 0$, by a diagonal process, we can find a subsequence of $\{T_k(u^\varepsilon)\}$ (still denoted by ε) such that for all $k > 0$ (countable, e.g. $k \in \mathbb{Q}$), we can find $v_k \in V$ such that as ε tends to zero,

$$\begin{cases} T_k(u^\varepsilon) \rightharpoonup v_k & \text{weakly in } V \\ T_k(u_1^\varepsilon) \rightharpoonup v_{k,1} & \text{weakly in } V_1 \\ T_k(u_2^\varepsilon) \rightharpoonup v_{k,2} & \text{weakly in } H^1(\Omega_2) \\ T_k(u_i^\varepsilon) \to v_{k,i} & \text{strongly in } L^2(\Omega_i) \text{ and a.e. in } \Omega_i,\ i = 1, 2 \\ \gamma_i(T_k(u_i^\varepsilon)) \to \gamma_i(v_{k,i}) & \text{strongly in } L^2(\Gamma) \text{ and a.e. on } \Gamma,\ i = 1, 2. \end{cases} \tag{17}$$

Note that by Poincaré inequality, (1), and (14), we have for all $k > 0$,

$$\begin{aligned} k^2 \operatorname{meas}\{|u_1^\varepsilon| \ge k\} &= \int_{\{|u_1^\varepsilon| \ge k\}} k^2\, dx \le \|T_k(u_1^\varepsilon)\|_{L^2(\Omega_1)}^2 \\ &\le C\|\nabla T_k(u_1^\varepsilon)\|_{L^2(\Omega_1)}^2 \le C\|T_k(u^\varepsilon)\|_V^2 \le CMk, \end{aligned}$$

for some $C > 0$, independent of ε and k.

Similarly, by the definition of the $H^1(\Omega_2)$-norm, Proposition 2.1, and (1), for all $k > 0$,

$$\begin{aligned} k^2 \operatorname{meas}\{|u_2^\varepsilon| \ge k\} &= \int_{\{|u_2\varepsilon| \ge k\}} k^2\, dx \le \|T_k(u_2^\varepsilon)\|_{L^2(\Omega_2)}^2 \\ &\le \|T_k(u_2^\varepsilon)\|_{H^1(\Omega_2)}^2 \le \|T_k(u^\varepsilon)\|_V^2 \le Mk. \end{aligned}$$

That is, for $i = 1, 2$,

$$\text{meas}\{|u_i^\varepsilon| > k\} \le \frac{M}{k} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{18}$$

Moreover, by Poincaré inequality, (1), and (14), for any $k > 0$,

$$\begin{aligned} k^2 \,\text{meas}_\Gamma\{|\gamma_1(u_1^\varepsilon)| \ge k\} &= \int_{\{|\gamma_1(u_1^\varepsilon)|\ge k\}} k^2 \, d\sigma_x = \int_{\{|\gamma_1(u_1^\varepsilon)|\ge k\}} \gamma_1(T_k(u_1^\varepsilon))\, d\sigma_x \\ &\le \|\gamma_1(T_k(u_1^\varepsilon))\|^2_{L^2(\Gamma)} \le C\|\nabla T_k(u_1^\varepsilon)\|^2_{L^2(\Omega_1)} \\ &\le C\|T_k(u^\varepsilon)\|_V^2 \le CMk, \end{aligned}$$

for some positive constant C independent of ε and k.

Likewise, by the Trace Theorem, Proposition 2.1, and (1), for all $k > 0$,

$$\begin{aligned} k^2 \,\text{meas}_\Gamma\{|\gamma_2(u_2^\varepsilon)| \ge k\} &= \int_{\{|\gamma_2(u_2^\varepsilon)|\ge k\}} k^2 \, d\sigma_x = \int_{\{|\gamma_2(u_2^\varepsilon)|\ge k\}} \gamma_2(T_k(u_2^\varepsilon))\, d\sigma_x \\ &\le \|\gamma_2(T_k(u_2^\varepsilon))\|^2_{L^2(\Gamma)} \le \|T_k(u_2^\varepsilon)\|^2_{H^1(\Omega_2)} \le \|T_k(u^\varepsilon)\|_V^2 \\ &\le Mk. \end{aligned}$$

We then have

$$\text{meas}_\Gamma\{|\gamma_i(u_i^\varepsilon)| \ge k\} \le \frac{M}{k} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty,\ i = 1, 2. \tag{19}$$

Then, using the arguments used in [11] and (14), we can show that for $i = 1, 2$, $\{u_i^\varepsilon\}$ and $\{\gamma_i(u_i^\varepsilon)\}$ are Cauchy in measure. Hence, we can extract a subsequence (still denoted by ε) such that

$$u_i^\varepsilon \longrightarrow u_i \quad \text{a.e. in } \Omega_i,\ i = 1, 2, \tag{20}$$

for some measurable function $u_i : \Omega_i \longrightarrow \overline{\mathbb{R}}$, $i = 1, 2$. By (18) and the continuity of measure, we can conclude that u_i is finite a.e. in Ω_i, for $i = 1, 2$.

We claim that the function $u := (u_1, u_2)$ is a renormalized solution of (P). To this aim, we need to show that u satisfies the conditions in Definition 2.3.

For the regularity conditions (4), we first need to identify the function v_k in (17) with $T_k(u)$. Observe that since T_k is continuous, (20) gives

$$T_k(u_i^\varepsilon) \longrightarrow T_k(u_i) = v_{k,i} \quad \text{a.e. in } \Omega_i,\ i = 1, 2, \tag{21}$$

by the uniqueness of the limit. Hence, $T_k(u) = v_k \in V$, which is (4a).

Furthermore, since $\{\gamma_i(u_i^\varepsilon)\}$ is also Cauchy in measure, we can find a function $\theta_i : \Gamma \longrightarrow \overline{\mathbb{R}}$, for $i = 1, 2$, such that (up to a subsequence)

$$\gamma_i(u_i^\varepsilon) \longrightarrow \theta_i \quad \text{a.e. on } \Gamma,\ i = 1, 2. \tag{22}$$

Note that by (19) and the continuity of measure, the function θ_i is finite a.e. on Γ for $i = 1, 2$. We now show that $\theta_i = \gamma_i(u_i)$, for $i = 1, 2$. Observe that by (17) (specifically, the weak convergence of $T_k(u^\varepsilon)$ to $v_k = T_k(u)$ in V), and (14), we obtain that for any $k > 0$

$$\|T_k(u)\|_V^2 \le \liminf_{\varepsilon\to 0} \|T_k(u^\varepsilon)\|_V^2 \le Mk,$$

that is,

$$\frac{1}{k}\|T_k(u)\|_V^2 \le M, \quad \forall k > 0. \tag{23}$$

Then, we can apply the second assertion in Proposition 2.2 to conclude that $\gamma_i(u_i)$, $i = 1, 2$, is well-defined. Moreover, by (3), (17), (21), and (22), we obtain,

$$T_k(\gamma_i(u_i)) = \gamma_i(T_k(u_i)) = \gamma_i(v_{k,i}) = T_k(\omega_i), \quad \forall k > 0.$$

Then by the continuity of T_k and the uniqueness of the limit, it follows that $\omega_i = \gamma_i(u_i)$, $i = 1, 2$, a.e. on Γ. Thus, for any $k > 0$,

$$u_i^\varepsilon \longrightarrow u_i \quad \textit{and} \quad T_k(u_i^\varepsilon) \longrightarrow T_k(u_i) \quad \text{a.e. on } \Gamma,\ i = 1, 2.$$

Since T_k is nondecreasing, it follows from Fatou's Lemma and (16) that

$$\begin{aligned}\int_\Gamma (u_1 - u_2)(T_k(u_1) - T_k(u_2))\, d\sigma_x &\le \liminf_{\varepsilon\to 0} \int_\Gamma (u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon))\, d\sigma_x \\ &\le Mk < \infty.\end{aligned}$$

This implies that $(u_1 - u_2)(T_k(u_1) - T_k(u_2)) \in L^1(\Gamma)$, which is (4b). Thus, we have shown that u satisfies (4) of Definition 2.3. To show the other two conditions, we need to rewrite (17) with $v_k = T_k(u)$, that is, for all $k > 0$ (countable),

$$\begin{cases} T_k(u^\varepsilon) \rightharpoonup T_k(u) & \text{weakly in } V \\ T_k(u_1^\varepsilon) \rightharpoonup T_k(u_1) & \text{weakly in } V_1 \\ T_k(u_2^\varepsilon) \rightharpoonup T_k(u_2) & \text{weakly in } H^1(\Omega_2) \\ T_k(u_i^\varepsilon) \to T_k(u_i) & \text{strongly in } L^2(\Omega_i) \text{ and a.e. in } \Omega_i,\ i = 1, 2 \\ T_k(u_i^\varepsilon) \to T_k(u_i) & \text{strongly in } L^2(\Gamma) \text{ and a.e. on } \Gamma,\ i = 1, 2. \end{cases} \tag{24}$$

Moreover, one can show that we also have

$$\nabla T_k(u_i^\varepsilon) \rightharpoonup \nabla T_k(u_i) \quad \text{weakly in } L^2(\Omega_i)^N,\ i = 1, 2. \tag{25}$$

Now, we show that u satisfies the decay conditions (5). First, observe that by the (A3) and (24), for any $n > 0$, we have for $i = 1, 2$,

$$B(x, T_n(u_i^\varepsilon)) \longrightarrow B(x, T_n(u_i)) \quad \text{a.e. in } \Omega_i \text{ and } L^\infty(\Omega_i)\text{-weak-*}. \tag{26}$$

Then, by (25) and the lower semicontinuity of the weak convergence (see Lemma 4.9 of [6]), for $i = 1, 2$,

$$\frac{1}{n}\int_{\{|u_i|<n\}} B(x,u_i)\nabla u_i \nabla u_i \, dx = \frac{1}{n}\int_{\Omega_i} B(x,T_n(u_i))\nabla T_n(u_i)\nabla T_n(u_i)\, dx$$
$$\leq \liminf_{\varepsilon\to 0}\frac{1}{n}\int_{\Omega_i} B(x,T_n(u_i^\varepsilon))\nabla T_n(u_i^\varepsilon)\nabla T_n(u_i^\varepsilon)\, dx.$$

Moreover, since T_n is nondecreasing, by Fatou's Lemma, (20), and (24), we have

$$\frac{1}{n}\int_\Gamma (u_1-u_2)(T_n(u_1)-T_n(u_2))\, d\sigma_x \leq \liminf_{\varepsilon\to 0}\frac{1}{n}\int_\Gamma (u_1^\varepsilon-u_2^\varepsilon)(T_n(u_1^\varepsilon)-T_n(u_2^\varepsilon))\, d\sigma_x.$$

Then, since

$$\frac{1}{n}\int_{\{|u_i|<n\}} B(x,u_i)\nabla u_i\nabla u_i\, dx \quad \text{and} \quad \frac{1}{n}\int_\Gamma (u_1-u_2)(T_n(u_1)-T_n(u_2))\, d\sigma_x$$

are nonnegative, to show (5a) and (5b), it suffices to show that

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{n}\bigg(\int_{\Omega_1} B(x,T_n(u_1^\varepsilon))\nabla T_n(u_1^\varepsilon)\nabla T_n(u_1^\varepsilon)\, dx$$
$$+\int_{\Omega_2} B(x,T_n(u_2^\varepsilon))\nabla T_n(u_2^\varepsilon)\nabla T_n(u_2^\varepsilon)\, dx \tag{27}$$
$$+\int_\Gamma (u_1^\varepsilon-u_2^\varepsilon)(T_n(u_1^\varepsilon)-T_n(u_2^\varepsilon))\, d\sigma_x\bigg) = 0.$$

Writing (13) with $\varphi_i = \frac{1}{n}T_n(u_i^\varepsilon)$, $i = 1, 2$, we have

$$\frac{1}{n}\int_{\Omega_1} B(x,T_n(u_1^\varepsilon))\nabla u_1^\varepsilon\nabla T_n(u_1^\varepsilon)\, dx$$
$$+\frac{1}{n}\int_{\Omega_2} B(x,T_n(u_2^\varepsilon))\nabla u_2^\varepsilon\nabla T_n(u_2^\varepsilon)\, dx + \frac{\lambda}{n}\int_{\Omega_1} u_1^\varepsilon T_n(u_1^\varepsilon)\, dx$$
$$+\frac{\lambda}{n}\int_{\Omega_2} u_2^\varepsilon T_n(u_2^\varepsilon)\, dx + \frac{1}{n}\int_\Gamma h(x)(u_1^\varepsilon-u_2^\varepsilon)(T_n(u_1^\varepsilon)-T_n(u_2^\varepsilon))\, d\sigma_x \tag{28}$$
$$= \frac{1}{n}\int_{\Omega_1} f^\varepsilon T_n(u_1^\varepsilon)\, dx + \int_{\Omega_2} f^\varepsilon T_n(u_2^\varepsilon)\, dx.$$

Observe that since T_n is nondecreasing and $\lambda > 0$, we have

$$\frac{\lambda}{n}\int_{\Omega_i} u_i^\varepsilon T_n(u_i^\varepsilon)\, dx \geq 0, \quad i = 1, 2. \tag{29}$$

Now, we take the limit of the term

$$\frac{1}{n}\int_{\Omega_i} f^\varepsilon T_n(u_i^\varepsilon)\, dx, \quad i = 1, 2,$$

first as ε tends to zero then as n goes to infinity. Note that by the fact that $|T_n| \leq n$ and (24), we have

$$T_n(u_i^\varepsilon) \longrightarrow T_n(u_i) \quad L^\infty(\Omega_i)\text{-weak-*},\ i = 1, 2.$$

It follows from (11) that

$$\lim_{\varepsilon \to 0} \frac{1}{n} \int_{\Omega_i} f^\varepsilon T_n(u_i^\varepsilon)\, dx = \frac{1}{n} \int_{\Omega_i} f T_n(u_i)\, dx, \quad i = 1, 2.$$

Moreover, observe that

$$\left| f \frac{T_n(u_i)}{n} \right| \leq |f| \in L^1(\Omega_i), \quad i = 1, 2,$$

and since u_i, $i = 1, 2$, is finite a.e.,

$$\frac{T_n(u_i)}{n} \longrightarrow 0 \quad \text{a.e. in } \Omega_i,\ i = 1, 2.$$

Thus, by the Lebesgue Dominated Convergence Theorem (LDCT), we have, for $i = 1, 2$,

$$\lim_{n \to \infty} \lim_{\varepsilon \to 0} \frac{1}{n} \int_{\Omega_i} f^\varepsilon T_n(u_i^\varepsilon)\, dx = \lim_{n \to \infty} \frac{1}{n} \int_{\Omega_i} f T_n(u_i)\, dx = 0, \quad i = 1, 2.$$

Combining this with (28) and (29), we have (27).

Now we show that u satisfies the renormalized formulation (6). Let $S_1, S_2 \in C^1(\mathbb{R})$ with compact support and let $\varphi = (\varphi_1, \varphi_2) \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$. For $n > 0$, define the function $h_n : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$h_n(r) = \begin{cases} 0, & \text{if } r \leq -2n, \\ \dfrac{r}{n} + 2, & \text{if } -2n \leq r \leq -n \\ 1, & \text{if } -n \leq r \leq n \\ -\dfrac{r}{n} + 2, & \text{if } n \leq r \leq 2n \\ 0, & \text{if } r \geq 2n. \end{cases} \tag{30}$$

One can observe that for any $n > 0$, h_n satisfies

$$\operatorname{supp} h_n \subset [-2n, 2n], \quad |h_n| \leq 1 \quad \text{and} \quad |h_n'| \leq \frac{1}{n}. \tag{31}$$

It follows that $h_n(u_i^\varepsilon) = h_n(T_{2n}(u_i^\varepsilon)) \in H^1(\Omega_i) \cap L^\infty(\Omega_i)$, for $i = 1, 2$. Thus, if we define the function

$$\psi_i = \varphi_i S_i(u_i) h_n(u_i^\varepsilon), \quad i = 1, 2,$$

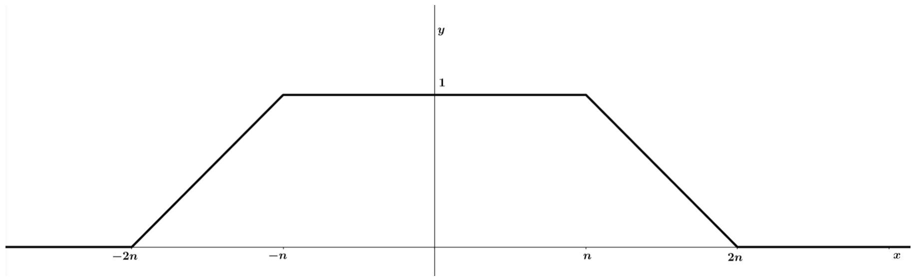

Figure 1: The graph of h_n

then

$$\psi = (\psi_1, \psi_2) \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2)).$$

We can then use this function φ as a test function for (13) to obtain

$$A_1^{\varepsilon,n} + A_2^{\varepsilon,n} + B_1^{\varepsilon,n} + B_2^{\varepsilon,n} + C_1^{\varepsilon,n} + C_2^{\varepsilon,n} + D_1^{\varepsilon,n} + D_2^{\varepsilon,n} + E^{\varepsilon,n} = F_1^{\varepsilon,n} + F_2^{\varepsilon,n}, \qquad (32)$$

where, for $i = 1, 2$,

$$A_i^{\varepsilon,n} = \int_{\Omega_i} B^\varepsilon(x, u_i^\varepsilon)\nabla u_i^\varepsilon \nabla\varphi_i S_i(u_i) h_n(u_i^\varepsilon)\, dx$$

$$B_i^{\varepsilon,n} = \int_{\Omega_i} B^\varepsilon(x, u_i^\varepsilon)\nabla u_i^\varepsilon \nabla u_i S'(u_i)\varphi_i h_n(u_i^\varepsilon)\, dx$$

$$C_i^{\varepsilon,n} = \int_{\Omega_i} B^\varepsilon(x, u_i^\varepsilon)\nabla u_i^\varepsilon \nabla u_i^\varepsilon h_n'(u_i^\varepsilon)\varphi_i S_i(u_i)\, dx$$

$$D_i^{\varepsilon,n} = \lambda \int_{\Omega_i} u_i^\varepsilon \varphi_i S_i(u_i) h_n(u_i^\varepsilon)\, dx$$

$$E^{\varepsilon,n} = \int_{\Gamma} h(x)(u_1^\varepsilon - u_2^\varepsilon)(\varphi_1 S_1(u_1) h_n(u_1) - \varphi_2 S_2(u_2) h_n(u_2))\, d\sigma_x$$

$$F_i^{\varepsilon,n} = \int_{\Omega_i} f^\varepsilon \varphi_i S_i(u_i) h_n(u_i^\varepsilon).$$

Using similar arguments as in [11], we can obtain the following when we take the limit as ε

tends to zero first then as n goes to infinity:

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} A_i^\varepsilon = \int_{\Omega_i} B(x,u_i)\nabla u_i \nabla\varphi_i S_i(u_i), \quad i=1,2$$
$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} B_i^\varepsilon = \int_{\Omega_i} B(x,u_i)\nabla u_i \nabla u_i\varphi_i S_i'(u_i), \quad i=1,2$$
$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0} C_i^{\varepsilon,n} = 0, \quad i=1,2 \tag{33}$$
$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} E^{\varepsilon,n} = \int_{\Gamma} h(x)(u_1-u_2)(\varphi_1 S_1(u_1)-\varphi_2 S_2(u_2))\,d\sigma_x$$
$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} F_i^{\varepsilon,n} = \int_{\Omega_i} f\varphi_i S_i(u_i)\,dx, \quad i=1,2.$$

It then remains study the limit of the term $D_i^{\varepsilon,n}$, $i=1,2$. We first look at the limit as ε goes to zero. Observe that

$$u_i^\varepsilon h_n(u_i^\varepsilon) = T_{2n}(u_i^\varepsilon)h_n(u_i^\varepsilon), \quad i=1,2,$$

with

$$|T_{2n}(u_i^\varepsilon)h_n(u_i^\varepsilon)| \le 2n, \quad \forall n>0,\ i=1,2.$$

Since T_{2n} and h_n are continuous, and (20), we have for $i=1,2$,

$$T_{2n}(u_i^\varepsilon)h_n(u_i^\varepsilon) \longrightarrow T_{2n}(u_i)h_n(u_i) = u_i h_n(u_i) \quad \text{a.e. in } \Omega_i \text{ and } L^\infty(\Omega_i)\text{-weak-*}. \tag{34}$$

So by (34),

$$\lim_{\varepsilon\to 0} D_i^{\varepsilon,n} = \lambda\int_{\Omega_i} u_i h_n(u_i)\varphi_i S_i(u_i)\,dx.$$

Now, to pass to the limit as n goes to infinity, we use the LDCT. Note that

$$|u_i h_n(u_i)\varphi_i S_i| = |T_{2n}(u_i)h_n(u_i)\varphi_i S_i| \le 2n\|\varphi_i\|_{L^\infty(\Omega_i)}\|S_i\|_{L^\infty(\mathbb{R})}, \quad i=1,2,$$

and

$$h_n(u_i) \longrightarrow 1 \quad \text{a.e. in } \Omega_i,\ i=1,2.$$

We then have for $i=1,2$,

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} D_i^{\varepsilon,n} = \lim_{n\to\infty}\lambda\int_{\Omega_i} u_i h_n(u_i)\varphi_i S_i(u_i)\,dx = \lambda\int_{\Omega_i} u_i\varphi_i S_i(u_i)\,dx.$$

Thus, combining this with (32) and (33), we obtain the renormalized formulation (6). This ends the proof of existence.

Step 2: Uniqueness result

Let u and v be renormalized solutions of (P). We first claim that u and v are in $L^1(\Omega)$ by using the estimates obtained by Boccardo and Gallouët in [5].

For any $k \in \mathbb{N}$, define $p_k : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$p_k(r) = \begin{cases} -1, & \text{if } r \leq -k-1 \\ s+k, & \text{if } -k-1 \leq r \leq -k \\ 0, & \text{if } -k \leq r \leq k \\ s-k, & \text{if } k \leq r \leq k+1 \\ 1, & \text{if } r \geq k+1. \end{cases}$$

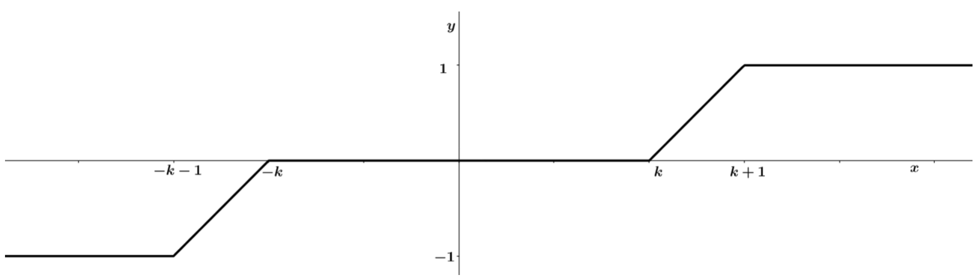

Figure 2: Graph of p_k

We can then obtain

$$\int_{\{k<|u_i|<k+1\}} |\nabla u_i|^2 \, dx \leq \|f\|_{L^1(\Omega)} \tag{35}$$

by using similar arguments used in [12] (see equation (3.11) with $\varphi = p_k$ in the proof of Lemma 3.1 of [12]).

Once we have (35), we can then use the method used in the proof of Lemma 1 of [5] to conclude that for $i = 1, 2$,

$$u_i \in W^{1,p}(\Omega_i), \quad \forall 1 < p < \frac{N}{N-1}. \tag{36}$$

We can also argue the same way for the renormalized solution v, that is,

$$v_i \in W^{1,p}(\Omega_i), \quad \forall 1 < p < \frac{N}{N-1}. \tag{37}$$

With this regularity of renormalized solutions, to show that $u = v$ a.e. in Ω, we can just show that $\|u_i - v_i\|_{L^1(\Omega_i)} = 0$ for $i = 1, 2$.

To achieve this, let n and k be positive real numbers. Let h_n be the function defined in (30) and T_k be the truncation function as in (2). As the usual in the proof of uniqueness of solutions, we want to use $u-v$ as the test function φ for the renormalized formulation. But this is not possible since we don't have enough regularity on $u-v$. The next possible course of action is to instead consider $T_k(u-v)$. However, regularity (4a) from Definition 2.3 for the solutions u and v does not imply that $T_k(u-v)$ also belongs to V. So, to assure us that we have the proper test function φ for the renormalized formulation, we instead consider the function of the form

$$T_k(T_n(u)-T_n(v)).$$

As for the functions S_i, $i=1,2$, in (6), we use $S_1=S_2=h_n$ for the corresponding formulations for u and v since we want to take advantage of the properties of h_n listed in (31) in conjunction with the condition (5) of Definition 2.3.

Now, since we want to be able to combine some integrals once we operate on the resulting renormalized formulations, we then let

$$\varphi = h_n(v)\frac{T_k(T_{2n}(u)-T_{2n}(v))}{k}$$

for the renormalized formulation for the solution u and

$$\varphi = h_n(u)\frac{T_k(T_{2n}(u)-T_{2n}(v))}{k}$$

for the renormalized formulation for the solution v. The factor $\dfrac{1}{k}$ is to ensure that this function is bounded by a constant and so we can use the Lebesgue Dominated Convergence Theorem in passing to the limit of some integrals below.

We now proceed with the proof of uniqueness. Let us denote by (♠) the renormalized formulation (6) for u written with

$$\varphi = h_n(v)\frac{T_k(T_{2n}(u)-T_{2n}(v))}{k} \quad \text{and} \quad S_1=S_2=h_n$$

and denote by (♣) the renormalized formulation for v written with

$$\varphi = h_n(u)\frac{T_k(T_{2n}(u)-T_{2n}(v))}{k} \quad \text{and} \quad S_1=S_2=h_n.$$

Observe that since $\operatorname{supp} h_n \subset [-2n,2n]$,

$$h_n(u_i)h_n(v_i)T_k(T_{2n}(u_i)-T_{2n}(v_i)) = h_n(u_i)h_n(v_i)T_k(u_i-v_i), \quad i=1,2.$$

Then, subtracting (♣) from (♠), we obtain

$$\begin{aligned} G_1^{k,n} + G_2^{k,n} + H_1^{k,n} + H_2^{k,n} + I_1^{k,n} + I_2^{k,n} + J_1^{k,n} \\ + J_2^{k,n} + L_1^{k,n} + L_2^{k,n} + M_1^{k,n} + M_2^{k,n} + P^{k,n} = 0, \end{aligned} \tag{38}$$

where for $i = 1, 2$,

$$
\begin{aligned}
G_i^{k,n} &= \frac{1}{k}\int_{\Omega_i} (B(x,u_i)\nabla u_i - B(x,v_i)\nabla v_i)\nabla T_k(u_i - v_i)h_n(u_i)h_n(v_i)\,dx \\
H_i^{k,n} &= \frac{1}{k}\int_{\Omega_i} h_n'(u_i)B(x,u_i)\nabla u_i \nabla u_i h_n(v_i)T_k(u_i - v_i)\,dx \\
I_i^{k,n} &= \frac{1}{k}\int_{\Omega_i} h_n'(v_i)B(x,u_i)\nabla u_i \nabla v_i h_n(u_i)T_k(u_i - v_i)\,dx \\
J_i^{k,n} &= -\frac{1}{k}\int_{\Omega_i} h_n'(u_i)B(x,v_i)\nabla v_i \nabla u_i h_n(v_i)T_k(u_i - v_i)\,dx \\
L_i^{k,n} &= -\frac{1}{k}\int_{\Omega_i} h_n'(v_i)B(x,v_i)\nabla v_i \nabla v_i h_n(u_i)T_k(u_i - v_i)\,dx \\
M_i^{k,n} &= \frac{\lambda}{k}\int_{\Omega_i} (u_i - v_i)h_n(u_i)h_n(v_i)T_k(u_i - v_i)\,dx \\
P^{k,n} &= \frac{1}{k}\int_{\Gamma} h(x)[(u_1 - u_2) - (v_1 - v_2)]\big(h_n(u_1)h_n(v_1)T_k(u_1 - v_1) \\
&\qquad\qquad\qquad\qquad - h_n(u_2)h_n(v_2)T_k(u_2 - v_2)\big)\,d\sigma_x.
\end{aligned}
$$

We will study the behavior of each term as k tends to zero and then as n goes to infinity. Starting with $G_i^{k,n}$, $i = 1, 2$, we rewrite the term as

$$
\begin{aligned}
G_i^{k,n} = \frac{1}{k}\int_{\Omega_i} & B(x,u_i)\nabla(u_i - v_i)\nabla T_k(u_i - v_i)h_n(u_i)h_n(v_i)\,dx \\
&+ \frac{1}{k}\int_{\Omega_i} (B(x,u_i) - B(x,v_i))\nabla v_i \nabla T_k(u_i - v_i)h_n(u_i)h_n(v_i)\,dx.
\end{aligned}
$$

Observe that the first integral on the right-hand side can be written as

$$
\begin{aligned}
&\frac{1}{k}\int_{\Omega_i} B(x,u_i)\nabla(u_i - v_i)\nabla T_k(u_i - v_i)h_n(u_i)h_n(v_i)\,dx \\
&\qquad = \frac{1}{k}\int_{\Omega_i} B(x,u_i)\nabla T_k(u_i - v_i)\nabla T_k(u_i - v_i)h_n(u_i)h_n(v_i)\,dx \geq 0.
\end{aligned}
$$

While for the second integral, we will use the technique done by [13, 12]. Then by (2), (31), and (A3.3), for any $0 < k < 1$,

$$
\begin{aligned}
&\left|\frac{1}{k}\int_{\Omega_i}(B(x,u_i)-B(x,v_i))\nabla v_i \nabla T_k(u_i-v_i)h_n(u_i)h_n(v_i)\,dx\right| \\
&= \left|\frac{1}{k}\int_{\{|u_i-v_i|<k\}}(B(x,u_i)-B(x,v_i))\nabla v_i \nabla T_k(u_i-v_i)h_n(u_i)h_n(v_i)\,dx\right| \\
&\le \frac{1}{k}\int_{\substack{\{|u_i-v_i|<k\}\\ \cap\{|u_i|<2n+1\}\\ \cap\{|v_i|<2n+1\}}} |B(x,u_i)-B(x,v_i)\nabla T_{2n+1}(v_i)\nabla T_k(u_i-v_i)|\,dx \\
&\le \frac{M}{k}\int_{\substack{\{|u_i-v_i|<k\}\\ \cap\{|u_i|<2n+1\}\\ \cap\{|v_i|<2n+1\}}} |u_i-v_i||\nabla T_{2n+1}(v_i)\nabla T_k(u_i-v_i)|\,dx \\
&\le M\int_{\{u_i-v_i|<k\}} |\nabla T_{2n+1}(v_i)\nabla T_k(u_i-v_i)|\,dx.
\end{aligned}
$$

By the regularity condition (4a) of Definition 2.3, we have

$$|\nabla T_{2n+1}(v_i)\nabla T_k(u_i-v_i)| \le |\nabla T_{2n+1}(v_i)\nabla T_1(u_i-v_i)| \in L^1(\Omega_i).$$

Moreover,

$$\nabla T_k(u_i-v_i)\chi_{\{|u_i-v_i|<k\}} \longrightarrow 0 \quad \text{a.e. in } \Omega_i.$$

Thus, by LDCT,

$$\lim_{k\to 0} M\int_{\{u_i-v_i|<k\}} |\nabla T_{2n+1}(v_i)\nabla T_k(u_i-v_i)|\,dx = 0.$$

It then follows that

$$\lim_{n\to\infty}\lim_{k\to 0} G_i^{k,n} \ge 0, \quad i=1,2. \tag{39}$$

Now, we look at $H_i^{k,n}$, $i=1,2$. Note that by (2) and (31), we have

$$
\begin{aligned}
|H_i^{k,n}| &= \left|\frac{1}{k}\int_{\Omega_i} h_n'(u_i)B(x,u_i)\nabla u_i\nabla u_i h_n(v_i)T_k(u_i-v_i)\,dx\right| \\
&\le \frac{1}{k}\int_{\{|u_i|\le 2n\}} |h_n'(u_i)|)|B(x,u_i)\nabla u_i\nabla u_i||h_n(v_i)||T_k(u_i-v_i)|\,dx \\
&\le \frac{1}{n}\int_{\{|u_i|\le 2n\}} B(x,u_i)\nabla u_i\nabla u_i\,dx.
\end{aligned}
$$

Observe that this last integral is independent of k and we take the limit as n goes to infinity, we get by (5a),

$$\lim_{n\to\infty}\lim_{k\to 0} H_i^{k,n} = 0, \quad i=1,2. \tag{40}$$

Similarly, we can also show that

$$|L_i^{k,n}| \le \frac{1}{n}\int_{\{|v_i|\le 2n\}} B(x,v_i)\nabla v_i\nabla v_i\,dx, \quad i=1,2.$$

Thus, again using (5a)

$$\lim_{n\to\infty}\lim_{k\to 0} L_i^{k,n} = 0, \quad i = 1, 2. \tag{41}$$

For the terms $I_i^{k,n}$ and $J_i^{k,n}$, $i = 1, 2$, we again adapt the arguments in [13, 12], this time, utilizing Proposition 3.1. Note that by (2), (31), and Young's inequality, we have

$$\begin{aligned} |I_i^{k,n}| &= \left| \frac{1}{k} \int_{\Omega_i} h_n'(v_i) B(x, u_i) \nabla u_i \nabla v_i h_n(u_i) T_k(u_i - v_i)\, dx \right| \\ &\le \frac{1}{n} \int_{\substack{\{|u_i|<2n\} \\ \cap\{|v_i|<2n\}}} |B(x, u_i) \nabla u_i \nabla v_i|\, dx \\ &\le \frac{1}{2n} \int_{\substack{\{|u_i|<2n\} \\ \cap\{|v_i|<2n\}}} B(x, u_i) \nabla u_i \nabla u_i\, dx + \frac{1}{2n} \int_{\substack{\{|u_i|<2n\} \\ \cap\{|v_i|<2n\}}} B(x, u_i) \nabla v_i \nabla v_i\, dx. \end{aligned}$$

The sum above is independent of k and hence, taking the limit as k tends to zero will have no effect. Moreover, by (5a), the limit of the first integral is zero as n goes to infinity. While for the second integral, using arguments used in [13, 12] and Proposition 3.1, we can prove that

$$\lim_{n\to\infty} \frac{1}{2n} \int_{\substack{\{|u_i|<2n\} \\ \cap\{|v_i|<2n\}}} B(x, u_i) \nabla v_i \nabla v_i\, dx = 0.$$

It then follows that

$$\lim_{n\to\infty}\lim_{k\to 0} I_i^{k,n} = 0, \quad i = 1, 2. \tag{42}$$

Likewise, we can show that

$$|J_i^{k,n}| \le \frac{1}{2n} \int_{\substack{\{|u_i|<2n\} \\ \cap\{|v_i|<2n\}}} B(x, v_i) \nabla v_i \nabla v_i\, dx + \frac{1}{2n} \int_{\substack{\{|u_i|<2n\} \\ \cap\{|v_i|<2n\}}} B(x, v_i) \nabla u_i \nabla u_i\, dx,$$

and thus,

$$\lim_{n\to\infty}\lim_{k\to 0} J_i^{k,n} = 0, \quad i = 1, 2. \tag{43}$$

For the last three terms, using (36) and (37),

$$u_i - v_i \in L^1(\Omega_i) \quad \text{and} \quad \gamma_i(u_i) - \gamma_i(v_i) \in L^1(\Gamma), \quad i = 1, 2. \tag{44}$$

Now, we study the term $M_i^{k,n}$, $i = 1, 2$. Observe that by (31) and (44), for any $k > 0$,

$$\frac{1}{k} |(u_i - v_i) h_n(u_i) h_n(v_i) T_k(u_i - v_i)| \le |u_i - v_i| \in L^1(\Omega_i).$$

In addition,

$$\frac{T_k(u_i - v_i)}{k} \longrightarrow \operatorname{sgn}(u_i - v_i), \quad \text{a.e. in } \Omega_i,\ i = 1, 2.$$

Then, by LDCT,

$$\lim_{k\to 0} M_i^{k,n} = \lambda \int_{\Omega_i} (u_i - v_i) h_n(u_i) h_n(v_i) \operatorname{sgn}(u_i - v_i)\, dx.$$

Looking now at the behavior as n goes to infinity, note that by (31) and (44), we have

$$|(u_i - v_i)h_n(u_i)h_n(v_i)\,\mathrm{sgn}(u_i - v_i)| \le |u_i - v_i| \in L^1(\Omega_i),$$

and

$$h_n(u_i) \longrightarrow 1 \quad \text{and} \quad h_n(v_i) \longrightarrow 1 \quad \text{a.e. in } \Omega_i.$$

Hence, we again apply LDCT to conclude that

$$\begin{aligned}\lim_{n\to\infty}\lim_{k\to 0} M_i^{k,n} &= \lambda \int_{\Omega_i} (u_i - v_i)\,\mathrm{sgn}(u_i - v_i)\,dx \\ &= \lambda \|u_i - v_i\|_{L^1(\Omega_i)}, \quad i = 1, 2.\end{aligned} \tag{45}$$

Finally, for $P^{k,n}$, we divide it into two terms, that is, we first look at the term

$$\frac{1}{k}\int_\Gamma h(x)[(u_1 - u_2) - (v_1 - v_2)]h_n(u_1)h_n(v_1)T_k(u_1 - v_1)\,d\sigma_x.$$

By (31) and (44), we have

$$\begin{aligned}&\frac{1}{k}|h(x)[(u_1 - u_2) - (v_1 - v_2)]h_n(u_1)h_n(v_1)T_k(u_1 - v_1)| \\ &\qquad\qquad \le \|h\|_{L^\infty(\Gamma)}|(u_1 - v_1) - (u_2 - v_2)| \in L^1(\Gamma).\end{aligned}$$

Furthermore,

$$\frac{T_k(u_1 - v_1)}{k} \longrightarrow \mathrm{sgn}(u_1 - v_1) \quad \text{a.e. on } \Gamma.$$

As a consequence, by LDCT,

$$\begin{aligned}&\lim_{k\to 0}\frac{1}{k}\int_\Gamma h(x)[(u_1 - u_2) - (v_1 - v_2)]h_n(u_1)h_n(v_1)T_k(u_1 - v_1)\,d\sigma_x \\ &\qquad = \int_\Gamma h(x)[(u_1 - v_1) - (u_2 - v_2)]h_n(u_1)h_n(v_1)\,\mathrm{sgn}(u_1 - v_1)\,d\sigma_x.\end{aligned}$$

Still using (31) and (44), we obtain

$$\begin{aligned}&|h(x)[(u_1 - v_1) - (u_2 - v_2)]h_n(u_1)h_n(v_1)\,\mathrm{sgn}(u_1 - v_1)| \\ &\qquad\qquad \le |(u_1 - v_1) - (u_2 - v_2)| \in L^1(\Gamma),\end{aligned}$$

and

$$h_n(u_1) \longrightarrow 1 \quad \text{and} \quad h_n(v_1) \longrightarrow 1 \quad \text{a.e. on } \Gamma.$$

It follows from LDCT that

$$\begin{aligned}&\lim_{n\to\infty}\lim_{k\to 0}\frac{1}{k}\int_\Gamma h(x)[(u_1 - u_2) - (v_1 - v_2)]h_n(u_1)h_n(v_1)T_k(u_1 - v_1)\,d\sigma_x \\ &\qquad = \int_\Gamma h(x)[(u_1 - v_1) - (u_2 - v_2)]\,\mathrm{sgn}(u_1 - v_1)\,d\sigma_x.\end{aligned} \tag{46}$$

Similarly, we have

$$\begin{aligned}\lim_{n\to\infty}\lim_{k\to 0}\frac{1}{k}\int_\Gamma & h(x)[(u_1-u_2)-(v_1-v_2)]h_n(u_2)h_n(v_2)T_k(u_2-v_2)\,d\sigma_x \\ &= \int_\Gamma h(x)[(u_1-v_1)-(u_2-v_2)]\,\mathrm{sgn}(u_2-v_2)\,d\sigma_x.\end{aligned} \tag{47}$$

Combining (46) and (47), we obtain

$$\begin{aligned}&\lim_{n\to\infty}\lim_{k\to 0} P^{k,n} \\ &= \int_\Gamma h(x)[(u_1-v_1)-(u_2-v_2)](\mathrm{sgn}(u_1-v_1)-\mathrm{sgn}(u_2-v_2))\,dx \geq 0.\end{aligned} \tag{48}$$

Combining (38), (39), (40), (41), (42), (43), (45), and (48), we have

$$\lambda\|u_1-v_1\|_{L^1(\Omega_1)}+\lambda\|u_2-v_2\|_{L^1(\Omega_2)}\leq 0.$$

Since $\lambda > 0$, this implies that $u_i - v_i = 0$ a.e. in Ω_i, $i = 1, 2$. As a consequence, we have for $i = 1, 2$, $\gamma_i(u_i) = \gamma_i(v_i)$, a.e. in Γ. Therefore, $u = v$ a.e. in Ω, that is, (P) has a unique renormalized solution. □

Acknowledgements. The author would like to thank the Office of the Vice Chancellor for Research and Development (OVCRD) of the University of the Philippines - Diliman for funding this research through the PhD Incentive Award grant.

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