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Asymptotic behavior of a quasilinear problem with Neumann boundary condition in domains with small holes

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Abstract

In a domain in $\mathbb{R}^N$ with $N \geq 3$, periodically perforated with small holes, the asymptotic behavior of a quasilinear elliptic problem is studied in this work, via unfolding method. On the boundary of the holes, a nonhomogeneous Neumann boundary condition is prescribed; while a Dirichlet boundary condition is imposed in the exterior boundary. This homogenization process reveals a strange term at the limit depending on the capacity of the holes and the limit function. A corrector result is also presented to complete the homogenization process of the problem.

Keywords: correctors, homogenization, Neumann condition, unfolding method, small holes
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1 Introduction

Let $N \geq 3$ and consider a domain Ω in $\mathbb{R}^N$. For positive sequences ε and $\delta = \delta(\varepsilon)$ tending to zero, we denote by $\Omega_{\varepsilon\delta}$ the domain with small holes which is obtained by removing ε-periodically distributed holes $B_{\varepsilon\delta}$ of size $\varepsilon\delta$ in Ω. We denote by $\partial B_{\varepsilon\delta}$ the boundary of the small holes which are fully contained in Ω but do not intersect the outer boundary $\partial\Omega$. This paper is devoted to the multiscale analysis of the following quasilinear elliptic problem:

$$\begin{cases} -\operatorname{div}\left[A^\varepsilon(x, u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\right] = f & \text{in } \Omega_{\varepsilon\delta}, \\ A^\varepsilon(x, u_{\varepsilon\delta})\nabla u_{\varepsilon\delta} \cdot n_{\varepsilon\delta} = g_{\varepsilon\delta} & \text{on } \partial B_{\varepsilon\delta}, \\ u_{\varepsilon\delta} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $n_{\varepsilon\delta}$ is the unit exterior normal vector to $B_{\varepsilon\delta}$. In this problem, we assume that f is a square integrable function in Ω, $g_{\varepsilon\delta}$ is a function defined on ∂B, and that the quasilinear highly oscillating coefficient A^ε is bounded and uniformly elliptic.

The homogenization of elliptic partial differential equations in perforated domains can be traced back to the seminal works of E.Ja. Hrouslov in [22, 23] and with V.A. Marcenko in [25] for the Dirichlet problem. Later on, Neumann variants were studied by E.Ja. Hrouslov in [24] while mixed-type problems were examined in [15, 16] by D. Cioranescu and J. Saint Jean Paulin.

We highlight the important contributions of D. Cioranescu and F. Murat in [13] via the energy method of L. Tartar (see [28, 29]) for the Poisson equation with Dirichlet boundary condition, and to the work of C. Conca and P. Donato in [18] for the Laplacian with nonhomogeneous Neumann boundary condition. These works emphasize the existence of a critical size for the small holes which is significant to the limit behavior. In fact, for the Dirichlet problem in [13], the size $\varepsilon^{N/(N-2)}$ for $N \geq 3$ is "critical" in the sense that this leads to the appearance of a "strange term" in the limit problem which corresponds to the capacity of the holes as ε approaches 0. We call this size of holes here as "Dirichlet critical". A similar behavior was observed for size $\varepsilon^{N/(N-1)}$ for $N \geq 2$ on the Neumann variant in [18] For this critical size, we say that it is "Neumann critical". For further readings on optimal controls involved in limit problems with strange terms, one can also check [20] by J.I. Diaz, A.V. Podolskiy, and T.A. Shaposhnikova, and for approximate controllability in [19] by C. Conca, E. Jose, and I. Mishra.

A linear problem with Neumann boundary condition related to (1.1) was considered by A. Ould-Hammouda in [26] wherein the size of the holes is Neumann critical as in [18]. For the heat and wave equations with Dirichlet boundary conditions, the reader is referred the work of B. Cabarrubias and P. Donato in [5] and to [27] by A. Ould-Hammouda and R. Zaki for a linear elliptic problem with nonlinear Robin boundary condition posed in a setting with two sets of small holes. One can also see the work of the authors in [1] for the quasilinear case with nonlinear Robin boundary condition in a domain with two different sets of small holes. In the present work, we deal with quasilinear matrix coefficients in perforated domains where the size of the holes is Dirichlet critical as considered in [13].

It may seem at first that this work is a special case [1] but there is a subtle difference. In [1], the two set of holes have different critical sizes; Dirichlet critical for the Dirichlet boundary condition on one set of the holes while Neumann critical for the Robin boundary condition on the other set of holes. The present work investigates what happens if one has a Neumann boundary condition on holes whose size is Dirichlet critical. One more difference of this work not just with [1] but also with the other works mentioned above, is the presence of corrector result.

Physically, problem (1.1) can model complex interactions involved in a heat diffusion process for periodic heterogeneous media. For instance, $u_{\varepsilon\delta}$ represents the temperature of the material, $A^\varepsilon(x, u_{\varepsilon\delta})\nabla u_{\varepsilon\delta} \cdot n_{\varepsilon\delta}$ the heat flux which is also determined by $g_{\varepsilon\delta}$, and the function f acting as an external heat source. Some actual applications related to this problem involve the chemical reactions happening on the wall of a reactor with periodically distributed grains as in the Fruendlich kinetics model and the Langmuir kinetics model (see, for instance, [17]) Furthermore, it is known that certain composite materials such as ceramics or semiconductors exhibit a nonlinear dependence on its thermal conductivity. This makes our work relevant as an addition to the roster of models of such phenomena especially in situations where the flux of the temperature on the boundary of the material behaves in a particular way as specified by the Neumann boundary condition.

The upscaling process for (1.1) is done using the periodic unfolding method. This homogenization technique is originally conceptualized for fixed domains by D. Cioranescu, A. Damlamian, and G. Griso in [7]. Subsequently, several extensions of the method were established. Among these are by D. Cioranescu, P. Donato and R. Zaki in [11] for periodically perforated materials, the work of P. Donato, K.H. Le Nguyen and R. Tardieu [21] for two-component domains, by D. Cioranescu, A. Damlamian, G. Griso, and D. Onofrei in [9] for domains with small holes, and that of B. Cabarrubias and P. Donato [5] for time-dependent functions in small holes. One can also refer to [8] by D. Cioranescu, A. Damlamian, and G. Griso for a comprehensive survey of this method.

One of the difficulties addressed in this work is the passage to the limit in the quasilinear matrix coefficient. In addition, due to the size of the holes, a suitable class of test functions is needed in order to reveal the contribution of the small holes. Another difficulty relies on obtaining the corrector results which have not been explored in the literature for the Neumann case. In order to resolve these obstacles, we first show in Proposition 4.1 the convergence of the quasilinear matrix by exploiting the properties of the unfolding operators. Then, we prove the homogenization results in Theorem 4.2 and the classical formulation in Corollary 4.3 by adapting some arguments from [9] to our case. Finally, by suitable modifications in the ideas given in [8], the corrector results for this work are obtained by showing first the convergence of the energy in Theorem 5.2 leading to the strong convergence of the solution in Corollary 5.3.

The paper is organized as follows: Section 2 provides the geometric framework, data assumptions, and functional setting of problem (1.1). Next, Section 3 recalls the suitable version of the unfolding method for our case. Section 4 presents the asymptotic behavior of the problem while, lastly, Section 5 gives the corrector result.

2 Framework of The Problem

Let $N \geq 3$ and consider two positive sequences ε and δ such that $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. First, we introduce the geometric framework as presented in [9] (see also [8]).

In a bounded domain $\Omega \subseteq \mathbb{R}^N$, we use $Y = \big(-\frac{1}{2}, \frac{1}{2}\big)^N$ as the reference cell. For $\Xi_\varepsilon = \{\xi \in \mathbb{Z}^N \mid \varepsilon(\xi + Y) \subset \Omega\}$, denote by $\widehat{\Omega}_\varepsilon = \operatorname{int}\big\{ \cup_{\xi\in\Xi_\varepsilon} \varepsilon(\xi + \overline{Y})\big\}$ and $\Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon$. To obtain the perforations, we begin with an open set $B \subseteq Y$ and distribute the rescaled version δB with period ε. Then, $Y_\delta = Y \setminus \delta\overline{B}$ is the perforated reference cell with $B_{\varepsilon\delta} = \cup_{\xi\in\Xi_\varepsilon} \varepsilon(\xi + \delta B)$ the copies of the holes inside Ω such that $\partial B_{\varepsilon\delta} \cap \partial\Omega = \varnothing$. Therefore, the domain with small holes of size $\varepsilon\delta$ is defined as $\Omega_{\varepsilon\delta} = \big\{x \in \Omega \big| \, \big\{\frac{x}{\varepsilon}\big\}_Y \in Y_\delta\big\}$ as in Figure 1.

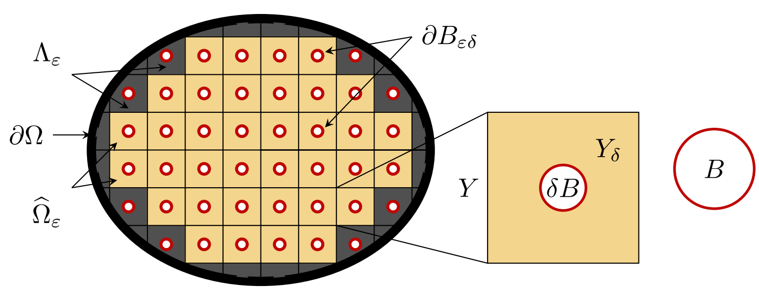

Figure 1: The perforated domain $\Omega_{\varepsilon\delta}$.

Moreover, for an open set $\mathcal{O} \subseteq \mathbb{R}^N$ and $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < \beta$, we denote by $\mathcal{M}(\alpha, \beta, \mathcal{O})$ the set of matrix fields $A \in L^\infty(\mathcal{O})^{N\times N}$ satisfying

$$(A(y)\xi, \xi) \geq \alpha|\xi|^2 \quad \text{and} \quad |A(y)\xi| \leq \beta|\xi|, \quad \forall \xi \in \mathbb{R}^N, \forall y \in \mathcal{O}.$$

Also, the notation $\mathcal{M}_{\mathcal{O}}(v) = \dfrac{1}{|\mathcal{O}|}\displaystyle\int_{\mathcal{O}} v\,dx$ is the average in $\mathcal{O}$ of a function $v \in L^1(\mathcal{O})$.

To proceed, let us now give the data hypotheses in problem (1.1) for any ε and δ :

($\mathcal{H}1$) The matrix field $A^\varepsilon(x,s) = A\left(\frac{x}{\varepsilon}, s\right)$ such that $A : Y \times \mathbb{R} \to \mathbb{R}$ satisfies:

- (i) A is a Carathéodory function,
- (ii) $A(\cdot, s) \in \mathcal{M}(\alpha, \beta, Y)$ is a Y-periodic function for all $s \in \mathbb{R}$,
- (iii) there exists a function $\omega : \mathbb{R} \to \mathbb{R}$ such that
  - (a) ω is continuous and nondecreasing with $\omega(s) > 0$ for any $s > 0$,
  - (b) $|A(y,s_1) - A(y,s_2)| \leq \omega(|s_1 - s_2|)$ for a.e. $y \in Y$ and for any $s_1 \neq s_2$,
  - (c) for any $t > 0$, $\displaystyle\lim_{y\to 0^+}\int_y^t \frac{ds}{\omega(s)} = +\infty$.

($\mathcal{H}2$) The function f is in $L^2(\Omega)$.

($\mathcal{H}3$) The function $g_{\varepsilon\delta}(x) = g\left(\frac{1}{\delta}\left\{\frac{x}{\varepsilon}\right\}\right)$ is such that $g \in L^2(\partial B)$.

($\mathcal{H}4$) The parameters ε and δ satisfy

$$0 \leq \lambda = \lim_{\varepsilon\to 0}\frac{\delta^{\frac{N}{2}-1}}{\varepsilon} < +\infty.$$

Remark 2.1. The number λ corresponds to the critical size of Dirichlet small holes from [13].

Remark 2.2. When $f \in (V_{\varepsilon\delta})'$ and $g \in H^{-\frac{1}{2}}(\partial B)$ instead of the data assumptions in ($\mathcal{H}2$) and ($\mathcal{H}3$), the results obtained here are still valid and the proofs are similar.

Next, define the space

$$V_{\varepsilon\delta} = \{\varphi \in H^1(\Omega_{\varepsilon\delta}) \mid \varphi = 0 \text{ on } \partial\Omega\},$$

equipped with the norm

$$\|u\|_{V_{\varepsilon\delta}} = \|\nabla u\|_{L^2(\Omega_{\varepsilon\delta})}, \qquad \forall u \in V_{\varepsilon\delta}. \tag{2.1}$$

Remark 2.3. A Poincaré inequality holds in $V_{\varepsilon\delta}$. Moreover, the norms in $V_{\varepsilon\delta}$ and $H^1(\Omega_{\varepsilon\delta})$ are equivalent.

In the sequel, we still denote by $\varphi \in V_{\varepsilon\delta}$, its extension by zero in $B_{\varepsilon\delta}$. The variational formulation of (1.1) reads as

$$\begin{cases} \text{Find } u_{\varepsilon\delta} \in V_{\varepsilon\delta} \text{ such that} \\ \displaystyle\int_{\Omega_{\varepsilon\delta}} A^\varepsilon(x, u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla v\,dx = \int_{\Omega_{\varepsilon\delta}} fv\,dx + \int_{\partial B_{\varepsilon\delta}} g_{\varepsilon\delta} v\,d\sigma, \\ \text{for every } v \in V_{\varepsilon\delta}. \end{cases} \tag{2.2}$$

The following result which implies the well-posedness of problem (1.1) follows from [3] for the case $\gamma = 0$. One makes use of the Lax-Milgram Theorem together with the Schauder's Fixed Point Theorem for the existence of the solution and some technique introduced in [6] to deal with the uniqueness part. The boundedness of the solution invokes the assumptions as well as Cauchy-Schwarz and Poincaré inequalities, and (2.1).

Theorem 2.4. *For every fixed $\varepsilon, \delta > 0$ and under $(\mathcal{H}1)$ - $(\mathcal{H}3)$, problem (2.2) admits a unique solution $u_{\varepsilon\delta} \in V_{\varepsilon\delta}$. Moreover, this solution satisfies the estimate*

$$\|u_{\varepsilon\delta}\|_{V_{\varepsilon\delta}} \leq C, \tag{2.3}$$

for some constant $C > 0$.

3 The Periodic Unfolding Method

In this section, we present the operators and some properties under the unfolding method necessary for this work as given in [7] for fixed domains and in [9] for small holes. The reader is referred to these references for the details. In what follows, we assume that $p \in [1, +\infty)$.

Definition 3.1 ([7])**.** The unfolding operator $\mathcal{T}_\varepsilon : \varphi \in L^p(\Omega) \mapsto L^p(\Omega \times Y)$ is given by

$$\mathcal{T}_\varepsilon(\varphi)(x,y) = \begin{cases} \varphi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon y\right), & \text{a.e. in } \widehat{\Omega}_\varepsilon \times Y, \\ 0, & \text{a.e. in } \Lambda_\varepsilon \times Y. \end{cases}$$

Proposition 3.2 ([7])**.** *The operator $\mathcal{T}_\varepsilon$ is linear and continuous. Moreover, for $\varphi \in L^1(\Omega)$, and $v, w \in L^p(\Omega)$, one has*

(i) $\mathcal{T}_\varepsilon(vw)(x,y) = \mathcal{T}_\varepsilon(v)(x,y)\,\mathcal{T}_\varepsilon(w)(x,y)$;

(ii) $\displaystyle\int_{\Omega\times Y} \mathcal{T}_\varepsilon(\varphi)(x,y)\,dxdy = \int_\Omega \varphi(x)\,dx - \int_{\Lambda_\varepsilon} \varphi(x)\,dx = \int_{\widehat{\Omega}_\varepsilon} \varphi(x)\,dx$;

(iii) $\displaystyle\int_{\Omega\times Y} |\mathcal{T}_\varepsilon(\varphi)(x,y)|\,dxdy \leqslant \int_\Omega |\varphi(x)|\,dx$;

(iv) $\displaystyle\left|\int_\Omega \varphi(x)\,dx - \int_{\Omega\times Y} \mathcal{T}_\varepsilon(\varphi)(x,y)\,dxdy\right| \leqslant \int_{\Lambda_\varepsilon} |\varphi(x)|\,dx$;

(v) if $\varphi_\varepsilon(x) = \varphi\left(\frac{x}{\varepsilon}\right)$ is Y-periodic, then $\mathcal{T}_\varepsilon(\varphi_\varepsilon)(x,y) = \varphi(y)$;

(vi) for a sequence $\{w_\varepsilon\}$ in $L^p(\Omega)$ such that $w_\varepsilon \to w$ strongly in $L^p(\Omega)$, $\mathcal{T}_\varepsilon(w_\varepsilon) \to w$ strongly in $L^p(\Omega \times Y)$;

(vii) for a sequence $\{w_\varepsilon\}$ in $W^{1,p}(\Omega)$ such that $w_\varepsilon \rightharpoonup w$ weakly in $W^{1,p}(\Omega)$, $\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup w$ weakly in $L^p(\Omega; W^{1,p}(Y))$, and that there exists $\widehat{w} \in L^p(\Omega; W^{1,p}_{per}(Y))$ such that up to a subsequence, $\mathcal{T}_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \widehat{w}$ weakly in $L^p(\Omega \times Y)$;

(viii) if a sequence $\{\varphi_\varepsilon\}$ in $L^1(\Omega)$ satisfies $\displaystyle\int_{\Lambda_\varepsilon} |\varphi_\varepsilon(x)|\,dx \to 0$, then we write

$$\int_\Omega \varphi_\varepsilon(x)\,dx \stackrel{\mathcal{T}_\varepsilon}{\simeq} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(\varphi_\varepsilon)(x,y)\,dxdy.$$

Definition 3.3 ([7])**.** The local average operator $\mathcal{M}_Y^\varepsilon : \varphi \in L^p(\Omega) \mapsto L^p(\Omega)$ is given by

$$\mathcal{M}_Y^\varepsilon(\varphi) = \int_Y \mathcal{T}_\varepsilon(\varphi)\,dy.$$

Proposition 3.4 ([7])**.** *If $\{v_\varepsilon\}$ is a bounded sequence in $L^p(\Omega)$ such that $v_\varepsilon \to v$ strongly in $L^p(\Omega)$, then $\mathcal{M}_Y^\varepsilon(v_\varepsilon) \to v$ strongly in $L^p(\Omega)$.*

Definition 3.5 ([12])**.** The averaging operator $\mathcal{U}_\varepsilon : L^p(\Omega\times Y) \to L^p(\Omega)$ is defined as follows:

$$\mathcal{U}_\varepsilon(\Phi)(x) = \begin{cases} \dfrac{1}{|Y|}\displaystyle\int_Y \Phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}_Y\right) dz & \text{a.e. on } \widehat{\Omega}_\varepsilon, \\ 0 & \text{a.e. on } \Lambda_\varepsilon. \end{cases}$$

Proposition 3.6 ([12])**.** *The operator $\mathcal{U}_\varepsilon$ is linear and continuous, and the following holds:*

(i) If $\varphi \in L^p(\Omega)$ is independent of y, then $\mathcal{U}_\varepsilon(\varphi) \to \varphi$ strongly in $L^p(\Omega)$.

(ii) Suppose $\{w_\varepsilon\}$ is a sequence in $L^p(\Omega)$, then the following assertions are equivalent:

$$(a)\quad \mathcal{T}_\varepsilon(w_\varepsilon) \to \widehat{w} \quad \textit{strongly in } L^p(\Omega \times Y) \quad \textit{and} \quad \int_{\Lambda_\varepsilon} |w_\varepsilon|^p\,dx \to 0.$$

$$(b)\quad w_\varepsilon - \mathcal{U}_\varepsilon(\widehat{w}) \to 0 \quad \textit{strongly in } L^p(\Omega).$$

Proposition 3.7 ([12])**.** *For $p \in [1,\infty)$, suppose that α is in $L^p(\Omega)$ and β in $L^\infty(\Omega; L^p(Y))$. Then the product $\mathcal{U}_\varepsilon(\alpha)\mathcal{U}_\varepsilon(\beta)$ belongs to $L^p(\Omega)$ and*

$$\mathcal{U}_\varepsilon(\alpha\beta) - \mathcal{U}_\varepsilon(\alpha)\mathcal{U}_\varepsilon(\beta) \to 0 \quad \textit{strongly in } L^p(\Omega).$$

Next, we consider those for domains with small holes as presented in [9].

Definition 3.8 ([9])**.** The unfolding operator $\mathcal{T}_{\varepsilon,\delta} : \varphi \in L^p(\Omega) \mapsto L^p(\Omega \times \mathbb{R}^N)$, is given by

$$\mathcal{T}_{\varepsilon,\delta}(\varphi)(x,z) = \begin{cases} \mathcal{T}_\varepsilon(\varphi)(x,\delta z) & \text{if } (x,z) \in \widehat{\Omega}_\varepsilon \times \frac{1}{\delta}Y, \\ 0 & \text{otherwise.} \end{cases}$$

Let us now have the properties of the operator $\mathcal{T}_{\varepsilon,\delta}$.

Proposition 3.9 ([9])**.** *The operator $\mathcal{T}_{\varepsilon,\delta}$ is linear and continuous. Moreover, one has the following:*

(i) For any $v, w \in L^p(\Omega)$, $\mathcal{T}_{\varepsilon,\delta}(vw) = \mathcal{T}_{\varepsilon,\delta}(v)\mathcal{T}_{\varepsilon,\delta}(w)$.

(ii) For any $u \in L^1(\Omega)$, $\delta^N \displaystyle\int_{\Omega\times\mathbb{R}^N} |\mathcal{T}_{\varepsilon,\delta}(u)|\,dxdz \leqslant \int_\Omega |u|\,dx$.

(iii) For any $u \in L^2(\Omega)$, $\|\mathcal{T}_{\varepsilon,\delta}(u)\|^2_{L^2(\Omega\times\mathbb{R}^N)} \leqslant \dfrac{1}{\delta^N}\|u\|^2_{L^2(\Omega)}$.

(iv) *For any* $u \in L^1(\Omega)$, $\left| \int_\Omega u\,dx - \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(u)\,dxdz \right| \leqslant \int_{\Lambda_\varepsilon} |u|\,dx.$

(v) *Let* $u \in H^1(\Omega)$. *Then,* $\mathcal{T}_{\varepsilon,\delta}(\nabla_x u) = \frac{1}{\varepsilon\delta}\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u))$ *in* $\Omega \times \frac{1}{\delta}Y$.

(vi) *Suppose* $N \geqslant 3$ *and let* $\omega \subset \mathbb{R}^N$ *be open and bounded. The following estimates hold:*

$$\|\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u))\|^2_{L^2(\Omega\times\frac{1}{\delta}Y)} \leqslant \frac{\varepsilon}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)},$$

$$\|\mathcal{T}_{\varepsilon,\delta}(u - M^\varepsilon_Y(u))\|^2_{L^2(\Omega;L^{2^*}(\mathbb{R}^N))} \leqslant \frac{C\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)},$$

$$\|\mathcal{T}_{\varepsilon,\delta}(u)\|^2_{L^2(\Omega\times\omega)} \leqslant \frac{2C\varepsilon^2}{\delta^{N-2}}|\omega|^{2/N}\|\nabla u\|^2_{L^2(\Omega)} + 2|\omega|\|u\|^2_{L^2(\Omega)},$$

where C *is the Sobolev-Poincaré-Wirtinger constant for* $H^1(Y)$.

(vii) *Let* $\{w_{\varepsilon\delta}\}$ *be a sequence in* $H^1(\Omega)$ *which is uniformly bounded as both* ε *and* δ *approach zero. Then there exists* $W \in L^2(\Omega; L^{2^*}(\mathbb{R}^N))$ *with* $\nabla_z W \in L^2(\Omega\times\mathbb{R}^N)$ *such that up to a subsequence,*

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\left(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon\delta}) - M^\varepsilon_Y(w_{\varepsilon\delta})1_{\frac{1}{\delta}Y}\right) \rightharpoonup W \quad \text{weakly in } L^2(\Omega; L^{2^*}(\mathbb{R}^N)),$$

and

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\nabla_z(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon\delta}))1_{\frac{1}{\delta}Y} \rightharpoonup \nabla_z W \quad \text{weakly in } L^2(\Omega\times\mathbb{R}^N).$$

Furthermore, if

$$\limsup_{(\varepsilon,\delta)\to(0^+,0^+)} \frac{\delta^{\frac{N}{2}-1}}{\varepsilon} < +\infty,$$

then one can choose the subsequence and some $U \in L^2_{\mathrm{loc}}(\Omega; L^{2^*}(\mathbb{R}^N))$ *such that*

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon\delta}) \rightharpoonup U \quad \text{weakly in } L^2_{\mathrm{loc}}(\Omega; L^{2^*}(\mathbb{R}^N)),$$

where $2^* = \frac{2N}{N-2}$ *is the associated Sobolev exponent.*

(viii) *If a sequence* $\{\varphi_\varepsilon\}$ *in* $L^1(\Omega)$ *satisfies* $\int_{\Lambda_\varepsilon} |\varphi_\varepsilon(x)|\,dx \to 0$, *then we write*

$$\int_\Omega \varphi_\varepsilon(x)\,dx \overset{\mathcal{T}_{\varepsilon,\delta}}{\simeq} \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(\varphi_\varepsilon)(x,z)\,dxdz.$$

Another operator from [14] is given below to treat the boundary terms.

Definition 3.10 ([14])**.** The boundary unfolding operator $\mathcal{T}^b_{\varepsilon,\delta} : \varphi \in L^p(\partial B_{\varepsilon\delta}) \mapsto L^p(\mathbb{R}^N \times \partial B)$ is given by

$$\mathcal{T}^b_{\varepsilon,\delta}(\varphi)(x,z) = \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon\delta z\right), \quad \forall x \in \mathbb{R}^N, \forall z \in \partial B.$$

Proposition 3.11 ([14]). *Let $v, \varphi \in L^p(\partial B_{\varepsilon\delta})$.*

(i) $\mathcal{T}^b_{\varepsilon,\delta}(v\varphi)(x,z) = \mathcal{T}^b_{\varepsilon,\delta}(v)(x,z)\,\mathcal{T}^b_{\varepsilon,\delta}(\varphi)(x,z)$.

(ii) Set $\varphi_\varepsilon = \varphi\big(\frac{1}{\delta}\left\{\frac{x}{\varepsilon}\right\}\big)$. *Then,* $\mathcal{T}^b_{\varepsilon,\delta}(\varphi_\varepsilon)(x,z) = \varphi(z)$.

(iii) We have the following integration formula:

$$\int_{\partial B_{\varepsilon\delta}} \varphi(x)\, d\sigma_x = \frac{\delta^{N-1}}{\varepsilon}\int_{\mathbb{R}^N\times\partial B} \mathcal{T}^b_{\varepsilon,\delta}(\varphi)(x,z)\, dx d\sigma_z, \quad \forall \varphi \in L^1(\partial B_{\varepsilon\delta}).$$

Proposition 3.12 ([14]). *For $g \in L^2(\partial B)$, set $g_{\varepsilon\delta}(x) = g\left(\frac{1}{\delta}\left\{\frac{x}{\varepsilon}\right\}\right)$ for any $x \in \partial B_{\varepsilon\delta}$. For all $\varphi \in H^1(\Omega)$, as $\varepsilon \to 0$,*

$$\frac{\varepsilon}{\delta^{N-1}}\int_{\partial B_{\varepsilon\delta}} g_{\varepsilon\delta}\varphi\, d\sigma_x \to |\partial B|\mathcal{M}_{\partial B}(g)\int_\Omega \varphi\, dx.$$

To conclude this section, we provide the space in which some of our test functions will be taken. Define the functional space

$$K_B = \{\varphi \in L^{2^*}(\mathbb{R}^N) \mid \nabla\varphi \in L^2(\mathbb{R}^N) \text{ and } \varphi \text{ constant on } B\}.$$

Proposition 3.13 ([9]). *Let $v \in \mathcal{D}(\mathbb{R}^N)\cap K_B$ and set $w_{\varepsilon\delta}(x) = v(B) - v\left(\frac{1}{\delta}\left\{\frac{x}{\varepsilon}\right\}\right)$ for any $x \in \mathbb{R}^N$. Then, $w_{\varepsilon\delta} \rightharpoonup v(B)$ weakly in $H^1(\Omega)$ and $\mathcal{T}_{\varepsilon,\delta}(\nabla w_{\varepsilon\delta}) = -\frac{1}{\varepsilon\delta}\nabla_z v$ in $\widehat{\Omega}_\varepsilon \times \frac{1}{\delta}Y$.*

4 Homogenization Results

At this juncture, we are now in the position to describe the asymptotic behavior of (1.1). In the sequel, we say $\varepsilon \to 0$ to mean that $(\varepsilon,\delta) \to (0+, 0+)$.

Before we proceed, we have a result analogous to that in [1] and [4]. We just provide here the properties used in the proof.

Proposition 4.1. *Let $u_{\varepsilon\delta} \rightharpoonup u_0$ weakly in $H^1(\Omega)$. Then,*

$$\mathcal{T}_\varepsilon\big[A^\varepsilon(x,u_{\varepsilon\delta})\big] \to A(y,u_0) \quad \text{a.e. in } \Omega\times Y, \tag{4.1}$$

$$\mathcal{T}_{\varepsilon,\delta}\big[A^\varepsilon(x,u_{\varepsilon\delta})\big] \to A(z,u_0) \quad \text{a.e. in } \Omega\times(\mathbb{R}^N\setminus\overline{B}). \tag{4.2}$$

Proof. The convergences in (4.1) and (4.2) follow from Proposition 3.2 (*vii*), Definitions 3.1 and 3.8 and assumptions (*i*) and (*ii*) of ($\mathcal{H}1$). □

We now present one of the main results in this work.

Theorem 4.2. *Under ($\mathcal{H}1$) - ($\mathcal{H}4$), let $u_{\varepsilon\delta} \in V_{\varepsilon\delta}$ be the unique solution of (2.2). Then up to a subsequence, there exists $u_0 \in H^1_0(\Omega)$ such that*

$$\widetilde{u_{\varepsilon\delta}} \to u_0 \quad \text{strongly in } L^2(\Omega). \tag{4.3}$$

Moreover, there exist $\widehat{u} \in L^2(\Omega; H^1_{\rm per}(Y))$ *and* $U \in L^2(\Omega; L^2_{\rm loc}(\mathbb{R}^N))$ *vanishing on* $\Omega \times B$ *with* $U - \lambda u_0$ *in* $L^2(\Omega; K_B)$ *such that the ordered triple* $(u_0, \widehat{u}, U)$ *solves the following limit equations:*

$$\int_Y A(y, u_0)(\nabla u_0 + \nabla_y \widehat{u})\nabla_y \varphi(y)\, dy = 0 \quad \text{a.e. in } \Omega \text{ and } \forall \varphi \in H^1_{\rm per}(Y), \tag{4.4}$$

$$\int_{\mathbb{R}^N \setminus \overline{B}} A(z, u_0)\nabla_z U \nabla_z v(z)\, dz = 0 \quad \text{a.e. in } \Omega \text{ and } \forall v \in K_B \text{ with } v(B) = 0, \tag{4.5}$$

and

$$\begin{aligned} &\int_{\Omega \times Y} A(y, u_0)(\nabla u_0 + \nabla_y \widehat{u})\nabla \psi\, dxdy \\ &\qquad -\lambda \int_{\Omega \times \partial B} A(z, u_0)\nabla_z U \nu_B \psi\, dxd\sigma_z = \int_\Omega f\psi\, dx, \quad \text{for all } \psi \in H^1_0(\Omega), \end{aligned} \tag{4.6}$$

where ν_B *is the unit exterior normal to the set* B *and* $d\sigma_z$ *the surface measure.*

Proof. We give the proof in several steps.

Step 1. First, note that (4.3) is immediate from the estimate in (2.3) and that the existence of $\widehat{u} \in L^2(\Omega; H^1_{\rm per}(Y))$ is guaranteed by Proposition 3.2 (*vii*).

On the other hand, the existence of $U \in L^2(\Omega; L^2_{\rm loc}(\mathbb{R}^N))$ such that $U - \lambda u_0$ is in $L^2(\Omega; K_B)$ follows by using the same arguments given in [9], so we omit the details here.

Let us now proceed to the limit equations.

Step 2. To show (4.4), let $\psi \in \mathcal{D}(\Omega)$ and $\varphi \in C^1_{\rm per}(Y)$ be vanishing in a neighborhood of the origin. So for ε and δ small enough, one has $\Psi(\cdot) = \varepsilon\psi(\cdot)\varphi\left(\frac{\cdot}{\varepsilon}\right) \in V_{\varepsilon\delta}$.

Take $\Psi \in V_{\varepsilon\delta}$ as a test function in (2.2), and thus,

$$\int_{\Omega_{\varepsilon\delta}} A^\varepsilon(x, u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla \Psi\, dx = \int_{\Omega_{\varepsilon\delta}} f\Psi\, dx + \int_{\partial B_{\varepsilon\delta}} g_{\varepsilon\delta}\Psi\, d\sigma.$$

By the gradient of Ψ and the chain rule applied to φ, we have

$$\begin{aligned} \varepsilon \int_{\Omega_{\varepsilon\delta}} A^\varepsilon(x, u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla\psi\varphi\left(\frac{x}{\varepsilon}\right)\, dx + \int_{\Omega_{\varepsilon\delta}} A^\varepsilon(x, u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\psi\nabla\varphi\left(\frac{x}{\varepsilon}\right)\, dx \\ = \varepsilon \int_{\Omega_{\varepsilon\delta}} f\psi\varphi\left(\frac{x}{\varepsilon}\right)\, dx + \varepsilon \int_{\partial B_{\varepsilon\delta}} g_{\varepsilon\delta}\psi\varphi\left(\frac{x}{\varepsilon}\right)\, d\sigma. \end{aligned}$$

Note that as ε approaches zero, all integrals in this equation approach zero except the second term on the left-hand side. Unfolding this term by $\mathcal{T}_\varepsilon$, in view of Proposition 3.2 (*i*) (*viii*), yields

$$\int_{\Omega_{\varepsilon\delta}} A^\varepsilon(x, u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\psi\nabla\varphi\left(\frac{x}{\varepsilon}\right)\, dx \overset{\mathcal{T}_\varepsilon}{\simeq} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(A^\varepsilon(x, u_{\varepsilon\delta}))\mathcal{T}_\varepsilon(\nabla u_{\varepsilon\delta})\mathcal{T}_\varepsilon(\psi)\mathcal{T}_\varepsilon\left(\nabla\varphi\left(\frac{x}{\varepsilon}\right)\right)\, dxdy.$$

This, together with (4.1), (4.3), and Proposition 3.2 (v) (vi) (vii) allow us to pass to the limit to obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega_{\varepsilon\delta}} A^{\varepsilon}(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\psi\nabla\varphi\left(\frac{x}{\varepsilon}\right)\,dx=\int_{\Omega\times Y}A(y,u_0)(\nabla u_0+\nabla\widehat{u})\psi\nabla_y\varphi(y)\,dxdy,$$

from which (4.4) follows by density.

Step 3. For (4.5) and (4.6), we observe the effect of the perforations in the limit equation by letting $\psi\in\mathcal{D}(\Omega)$ and take $w_{\varepsilon\delta}\psi$ as a test function in (2.2), where $w_{\varepsilon\delta}$ is given in Proposition 3.13. Then, (2.2) is equivalent to

$$\begin{aligned}\int_{\Omega_{\varepsilon\delta}} A^{\varepsilon}(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla w_{\varepsilon\delta}\psi\,dx+\int_{\Omega_{\varepsilon\delta}} A^{\varepsilon}(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}w_{\varepsilon\delta}\nabla\psi\,dx&\\ =\int_{\Omega_{\varepsilon\delta}} fw_{\varepsilon\delta}\psi\,dx+\int_{\partial B_{\varepsilon\delta}} g_{\varepsilon\delta}w_{\varepsilon\delta}\psi\,d\sigma.&\end{aligned}\tag{4.7}$$

This choice of test function implies that the unfolding criterion in Proposition 3.9 $(viii)$ is satisfied.

For the first integral in the left-hand side of this equation, unfolding by $\mathcal{T}_{\varepsilon,\delta}$ together with Propositions 3.9 (i) and 3.13, one has

$$\begin{aligned}&\int_{\Omega_{\varepsilon\delta}} A^{\varepsilon}(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla w_{\varepsilon\delta}\psi\,dx\\
&\overset{\mathcal{T}_{\varepsilon,\delta}}{\simeq}\delta^N\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon,\delta}(A^{\varepsilon}(x,u_{\varepsilon\delta}))\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon\delta})\mathcal{T}_{\varepsilon,\delta}(\nabla w_{\varepsilon\delta})\mathcal{T}_{\varepsilon,\delta}(\psi)\,dxdz\\
&=\delta^N\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon,\delta}(A^{\varepsilon}(x,u_{\varepsilon\delta}))\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon\delta})\left[-\frac{1}{\varepsilon\delta}\nabla_z v\right]\mathcal{T}_{\varepsilon,\delta}(\psi)\,dxdz\\
&=-\frac{\delta^{N-1}}{\varepsilon}\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon,\delta}(A^{\varepsilon}(x,u_{\varepsilon\delta}))\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon\delta})\nabla_z v\mathcal{T}_{\varepsilon,\delta}(\psi)\,dxdz\\
&=-\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon,\delta}(A^{\varepsilon}(x,u_{\varepsilon\delta}))\left[\delta^{\frac{N}{2}}\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon\delta})\right]\mathcal{T}_{\varepsilon,\delta}(\psi)\nabla_z v\,dxdz.\end{aligned}$$

By $(\mathcal{H}4)$, (4.2), (4.3), and the convergences (for more details, see [9]),

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\nabla_z\big(\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon\delta})\big)1_{\frac{1}{\delta}Y}=\delta^{\frac{N}{2}}\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon\delta})\rightharpoonup\nabla_z U\quad\text{weakly in } L^2(\Omega\times\mathbb{R}^N),\tag{4.8}$$

with

$$\mathcal{T}_{\varepsilon,\delta}(\psi)\nabla_z v\to\psi\nabla_z v\quad\text{strongly in } L^2(\Omega\times\mathbb{R}^N),$$

let us pass to the limit so that

$$\lim_{\varepsilon\to 0}\int_{\Omega_{\varepsilon\delta}} A^{\varepsilon}(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla w_{\varepsilon\delta}\psi\,dx=-\lambda\int_{\Omega\times(\mathbb{R}^N\setminus\overline{B})}A(z,u_0)\nabla_z U(x,z)\psi\nabla_z v(z)\,dxdz\tag{4.9}$$

For the second integral in the left-hand side of (4.7), unfolding by $\mathcal{T}_\varepsilon$ gives

$$\int_{\Omega_{\varepsilon\delta}} A^\varepsilon(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta} w_{\varepsilon\delta}\nabla\psi\,dx \overset{\mathcal{T}_\varepsilon}{\simeq} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(A^\varepsilon(x,u_{\varepsilon\delta}))\mathcal{T}_\varepsilon(\nabla u_{\varepsilon\delta})\mathcal{T}_\varepsilon(w_{\varepsilon\delta})\mathcal{T}_\varepsilon(\nabla\psi)\,dxdy.$$

By (4.1), (4.3), Propositions 3.2 (*vii*) and 3.13 one can pass to the limit to get

$$\lim_{\varepsilon\to 0}\int_{\Omega_{\varepsilon\delta}} A^\varepsilon(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta} w_{\varepsilon\delta}\nabla\psi\,dx = v(B)\int_{\Omega\times Y} A(y,u_0)(\nabla u+\nabla_y\widehat{u})\nabla\psi\,dxdy. \quad (4.10)$$

For the first term in the right-hand side of (4.7), unfolding again by $\mathcal{T}_\varepsilon$ we have

$$\int_{\Omega_{\varepsilon\delta}} f w_{\varepsilon\delta}\psi\,dx \overset{\mathcal{T}_\varepsilon}{\simeq} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(w_{\varepsilon\delta})\mathcal{T}_\varepsilon(\psi)\,dxdy.$$

By $(\mathcal{H}2)$ and Proposition 3.13, one can pass to the limit to obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega_{\varepsilon\delta}} f w_{\varepsilon\delta}\psi\,dx = v(B)\int_\Omega f\psi\,dx. \quad (4.11)$$

For the second term in the right-hand side of (4.7), unfolding by $\mathcal{T}^b_{\varepsilon\delta}$ Proposition 3.11 yields

$$\begin{aligned}\int_{\partial B_{\varepsilon\delta}} g_{\varepsilon\delta} w_{\varepsilon\delta}\psi\,d\sigma &= \frac{\delta^{N-1}}{\varepsilon}\int_{\mathbb{R}^N\times\partial B} \mathcal{T}^b_{\varepsilon\delta}(g_{\varepsilon\delta})\mathcal{T}^b_{\varepsilon\delta}(w_{\varepsilon\delta})\mathcal{T}^b_{\varepsilon\delta}(\psi)\,dxd\sigma_z\\ &= \delta^{\frac{N}{2}}\left(\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\right)\int_{\mathbb{R}^N\times\partial B} \mathcal{T}^b_{\varepsilon\delta}(g_{\varepsilon\delta})\mathcal{T}^b_{\varepsilon\delta}(w_{\varepsilon\delta})\mathcal{T}^b_{\varepsilon\delta}(\psi)\,dxd\sigma_z.\end{aligned}$$

By $(\mathcal{H}4)$, Propositions 3.12 and 3.13 we get the limit

$$\lim_{\varepsilon\to 0}\int_{\partial B_{\varepsilon\delta}} g_{\varepsilon\delta} w_{\varepsilon\delta}\psi\,d\sigma = 0. \quad (4.12)$$

Thus, passing to the limit in (4.7) through (4.9), (4.10), (4.11), and (4.12) we get the limit equation given by

$$\begin{cases} -\lambda\displaystyle\int_{\Omega\times(\mathbb{R}^N\setminus\overline{B})} A(z,u_0)\nabla_z U(x,z)\nabla_z v(z)\psi\,dxdz\\[2ex] +v(B)\displaystyle\int_{\Omega\times Y} A(y,u_0)\nabla\psi(\nabla u_0+\nabla_y\widehat{u})\,dxdy = v(B)\int_\Omega f\psi\,dx.\end{cases} \quad (4.13)$$

which by density holds for all $\psi\in H^1_0(\Omega)$ and $v\in K_B$. When $v(B)=0$, we get (4.5). Moreover, when $v(B)\neq 0$, by integration by parts in the first term in (4.13),

$$\begin{cases} -\lambda v(B)\displaystyle\int_{\Omega\times\partial B} A(z,u_0)\nabla_z U\nu_B\psi\,dxd\sigma_z\\[2ex] +v(B)\displaystyle\int_{\Omega\times Y} A(y,u_0)\nabla\psi(\nabla u_0+\nabla_y\widehat{u})\,dxdy = v(B)\int_\Omega f\psi\,dxdy,\end{cases}$$

from which (4.6) directly follows. □

Let us now obtain the corresponding limit problem which represents the asymptotic behavior of problem (1.1). To this goal, we first introduce the homogenized $N \times N$ matrix $A^{\rm hom} = (a_{ij}^{\rm hom}) \in \mathcal{M}(\alpha, \frac{\beta^2}{\alpha}, \Omega)$ (see [4] for the details of the properties) given by

$$a_{ij}^{\rm hom}(x) = \int_Y \left(a_{ij}(y, u_0(x)) - \sum_{k=1}^N a_{ik}(y, u_0(x)) \frac{\partial \widehat{\chi}_j}{\partial y_k}(y, u_0(x)) \right) dy, \tag{4.14}$$

where the correctors $\widehat{\chi}_j$ for $j = 1, \dots, N$ (for more details, see e.g. [2]) solve the following cell problems:

$$\begin{cases} \widehat{\chi}_j \in L^\infty(\Omega; H^1_{\rm per}(Y)), \\ \displaystyle\int_Y A(y, u_0) \nabla(\widehat{\chi}_j - y_j) \nabla \varphi = 0 \ \text{ a.e. in } \Omega, \\ \forall \varphi \in H^1_{\rm per}(Y). \end{cases}$$

Also, let χ be the solution of the cell problem corresponding to the holes δB given by

$$\begin{cases} \chi \in L^\infty(\Omega; K_B), \quad \chi(x, B) \equiv 1, \\ \displaystyle\int_{\mathbb{R}^N \setminus B} {}^t\!A(z, u_0) \nabla_z \chi(x, z) \nabla_z \Psi(z)\, dz = 0 \text{ a.e. in } \Omega, \\ \forall \Psi \in K_B \text{ with } \Psi(B) = 0. \end{cases}$$

Set the function

$$\Theta(x) = \int_{\mathbb{R}^N \setminus B} {}^t\!A(z, u_0) \nabla_z \chi(x, z) \nabla_z \chi(x, z)\, dz, \tag{4.15}$$

which is nonnegative and can be interpreted as the local capacity of the set B.

The limit problem corresponding to (1.1) is given by the next corollary.

Corollary 4.3. *The limit function $u_0 \in H^1_0(\Omega)$ is the unique solution of the limit problem*

$$\begin{cases} -\mathrm{div}(A^{\rm hom} \nabla u_0) + \lambda^2 \Theta u_0 = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.16}$$

Proof. By using standard arguments (see e.g. [10]), one gets

$$\widehat{u}(x, y) = \sum_{j=1}^N \frac{\partial u_0}{\partial x_j}(x) \widehat{\chi}_j(y, u_0(x)). \tag{4.17}$$

Equipping this in (4.6), in view of (4.14), one gets

$$\int_\Omega A^{\rm hom} \nabla u_0 \nabla \psi\, dx - \lambda \int_{\Omega \times \partial B} A(z, u_0) \nabla_z U \nu_B \psi\, dx d\sigma_z = \int_\Omega f \psi\, dx.$$

By integrating by parts the second term on the left-hand side above, since $\nabla_z U = \nabla_z(U - \lambda u)$, using (4.15), we obtain

$$\int_\Omega A^{\rm hom} \nabla u_0 \nabla \psi\, dx + \lambda^2 \int_\Omega \Theta u_0 \psi\, dx = \int_\Omega f \psi\, dx, \tag{4.18}$$

which is the weak formulation of (4.16). The Lax-Milgram theorem and the fact that $\Theta(x)$ is nonnegative provides the well-posedness of (4.16). $\square$

Remark 4.4. The oscillations in the matrix give rise to the classical homogenized matrix A^{hom} in the limit problem. The set of holes $B_{\varepsilon\delta}$ becomes negligible at the limit. The geometry of the domain gives rise to the zero-order strange term $\lambda^2\Theta u_0$.

We observe that in the limit problems obtained in [1] for domains with two small holes, both a strange term and an average of a Neumann data appear. When one considers the special case of the problem in [1] restricted to the Neumann boundary condition, one gets the asymptotic behavior corresponding to the quasilinear version of the problem from [26], and if it is restricted to the Dirichlet boundary condition, one obtains the homogenization results for the quasilinear version of the problem treated in [9]. Let us compare these two special cases with the present asymptotic analysis.

For the first special case, let us point out that the homogenization results for the Neumann critical situation in [26] are different from the present work for the Dirichlet critical case since the former has no strange term in the limit problem but with an average term for the function g, while the latter is in presence of a strange term but with zero contribution from g. The present results are more similar to the second special case (cf. [9]) in a way that both works admit a strange term in the limit problem. This further implies that when in presence of Dirichlet critical holes, the homogenization results obtained for problems with Neumann boundary condition is the same when one initially treats a corresponding version with Dirichlet boundary condition.

5 A Corrector Result

This section is devoted to the convergence of the energy associated to problem (1.1) and consequently, the corrector result.

To proceed, we need the following lemma (see e.g. [8]).

Lemma 5.1. *Let $\{D_\varepsilon\}_\varepsilon$ be a sequence of $N \times N$ matrix fields in $\mathcal{M}(\alpha, \beta, \mathcal{O})$ for some open set $\mathcal{O}$ such that $D_\varepsilon \to D$ almost everywhere on $\mathcal{O}$ (or more generally, in measure in $\mathcal{O}$). If the sequence of vector fields $\{\zeta_\varepsilon\}_\varepsilon$ converges weakly to ζ in $L^2(\mathcal{O})^N$, then*

$$\liminf_{\varepsilon\to 0} \int_{\mathcal{O}} D_\varepsilon \zeta_\varepsilon \zeta_\varepsilon \, dx \geq \int_{\mathcal{O}} D\zeta\zeta \, dx.$$

Furthermore if,

$$\limsup_{\varepsilon\to 0} \int_{\mathcal{O}} D_\varepsilon \zeta_\varepsilon \zeta_\varepsilon \, dx \leq \int_{\mathcal{O}} D\zeta\zeta \, dx,$$

then

$$\lim_{\varepsilon\to 0} \int_{\mathcal{O}} D_\varepsilon \zeta_\varepsilon \zeta_\varepsilon \, dx = \int_{\mathcal{O}} D\zeta\zeta \, dx \quad \text{and} \quad \zeta_\varepsilon \to \zeta \quad \text{strongly in } L^2(\mathcal{O})^N.$$

Next, we introduce the domain $\Omega_{\varepsilon\sqrt{\delta}} = \left\{x \in \Omega \middle| \left\{\frac{x}{\varepsilon}\right\}_Y \in Y_{\sqrt{\delta}}\right\}$ with the perforated reference cell $Y_{\sqrt{\delta}} = Y \setminus \sqrt{\delta}\,\overline{B}$.

The next theorem provides the energy convergence.

Theorem 5.2. *Under the assumptions of Theorem 4.2 and Corollary 4.3, one has*

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega_{\varepsilon\delta}} A^\varepsilon \nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta}\,dx &= \int_\Omega A^{\hom}\nabla u_0 \nabla u_0\,dx + \lambda^2\int_\Omega \Theta u_0^2\,dx \\ &= \frac{1}{|Y|}\int_{\Omega\times Y} A(y,u_0)(\nabla u_0 + \nabla_y \widehat{u})(\nabla u_0 + \nabla_y\widehat{u})\,dx\,dy \\ &\quad + \frac{\lambda^2}{|Y|}\int_{\Omega\times(\mathbb{R}^N\setminus\overline{B})} A(z,u_0)\nabla_z U\,\nabla_z U\,dx\,dz,\end{aligned} \tag{5.1}$$

and

$$\lim_{\varepsilon\to 0}\int_{\Lambda_\varepsilon} |\nabla u_{\varepsilon\delta}|^2\,dx = 0. \tag{5.2}$$

Moreover, we also have the following strong convergences

$$\mathcal{T}_\varepsilon(\nabla u_{\varepsilon\delta})\mathbf{1}_{\Omega\times Y_{\sqrt{\delta}}} \to \nabla u_0 + \nabla_y\widehat{u} \quad \textit{strongly in } L^2(\Omega\times Y)^N, \tag{5.3}$$

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\nabla_z\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon\delta})1_{\frac{1}{\sqrt{\delta}}B} \to \nabla_z U \quad \textit{strongly in } L^2(\Omega\times\mathbb{R}^N)^N. \tag{5.4}$$

Proof. Let $v = u_{\varepsilon\delta}$ be test functions in (2.2). Then,

$$\int_{\Omega_{\varepsilon\delta}} A^\varepsilon\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta}\,dx = \int_{\Omega_{\varepsilon\delta}} f u_{\varepsilon\delta}\,dx + \int_{\partial B_{\varepsilon\delta}} g_{\varepsilon\delta}u_{\varepsilon\delta}\,d\sigma. \tag{5.5}$$

Unfolding the left-hand side of (5.5) using $\mathcal{T}_\varepsilon$, we obtain

$$\begin{aligned}\frac{1}{|Y|}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_{\varepsilon\delta})\mathcal{T}_\varepsilon(\nabla u_{\varepsilon\delta})\,dx + \int_{\Lambda_\varepsilon} A^\varepsilon\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta}\,dx \\ = \int_{\Omega_{\varepsilon\delta}} f u_{\varepsilon\delta}\,dx + \int_{\partial B_{\varepsilon\delta}} g_{\varepsilon\delta}u_{\varepsilon\delta}\,d\sigma.\end{aligned} \tag{5.6}$$

Now, to investigate the convergence of the energy, we transform the first term in the left-hand side of (5.6) with a change of variable $y = \delta z$ and use Proposition 3.9 (v) to obtain

$$\begin{aligned}&\frac{1}{|Y|}\int_{\Omega\times Y_{\sqrt{\delta}}}\mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_{\varepsilon\delta})\mathcal{T}_\varepsilon(\nabla u_{\varepsilon\delta})\,dx\,dy \\ &\quad + \frac{\delta^N}{|Y|}\int_{\Omega\times\frac{1}{\sqrt{\delta}}B}\mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)\left[\frac{1}{\varepsilon\delta}\nabla_z\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon\delta})\right]\left[\frac{1}{\varepsilon\delta}\nabla_z\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon\delta})\right]dx\,dz \\ &\quad + \int_{\Lambda_\varepsilon} A^\varepsilon\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta}\,dx = \int_{\Omega_{\varepsilon\delta}} f u_{\varepsilon\delta}\,dx + \int_{\partial B_{\varepsilon\delta}} g_{\varepsilon\delta}u_{\varepsilon\delta}\,d\sigma.\end{aligned} \tag{5.7}$$

For conciseness in (5.7), we set

$$\begin{aligned}
\mathcal{A}_{\varepsilon\delta} &= \frac{1}{|Y|}\int_{\Omega\times Y_{\sqrt{\delta}}} \mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_{\varepsilon\delta})\mathcal{T}_\varepsilon(\nabla u_{\varepsilon\delta})\,dx\,dy,\\
\mathcal{B}_{\varepsilon\delta} &= \frac{1}{|Y|}\int_{\Omega\times\frac{1}{\sqrt{\delta}}B} \mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)\left[\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\nabla_z\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon\delta})\right]\left[\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\nabla_z\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon\delta})\right]dx\,dz, \qquad (5.8)\\
\mathcal{C}_{\varepsilon\delta} &= \int_{\Lambda_\varepsilon} A^\varepsilon\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta}\,dx.
\end{aligned}$$

By Proposition 3.2 (vii) and (4.8), we have the convergences

$$\begin{aligned}
&\mathcal{T}_\varepsilon(\nabla u_{\varepsilon\delta})\mathbf{1}_{\Omega\times Y_{\sqrt{\delta}}} \to \nabla u_0 + \nabla_y\widehat{u} \quad \text{weakly in } L^2(\Omega\times Y)^N,\\
&\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\nabla_z\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon\delta})1_{\frac{1}{\sqrt{\delta}}B} \to \nabla_z U \quad \text{weakly in } L^2(\Omega\times\mathbb{R}^N)^N.
\end{aligned} \qquad (5.9)$$

Taking in $\psi = u_0$ in (4.18), using Theorem 5.1 together with (5.5), (5.6), (5.8), and (5.9) yield

$$\begin{aligned}
\int_\Omega f u_0\,dx &= \frac{1}{|Y|}\int_{\Omega\times Y} A(y,u_0)(\nabla u_0+\nabla_y\widehat{u})(\nabla u_0+\nabla_y\widehat{u})\,dx\,dy\\
&\quad + \frac{1}{|Y|}\int_{\Omega\times(\mathbb{R}^N\setminus\overline{B})} A(z,u_0)\nabla_z U(x,z)\nabla_z U(x,z)\,dx\,dz\\
&\le \liminf_{\varepsilon\to 0}\mathcal{A}_{\varepsilon\delta} + \liminf_{\varepsilon\to 0}\mathcal{B}_{\varepsilon\delta} \le \liminf_{\varepsilon\to 0}(\mathcal{A}_{\varepsilon\delta}+\mathcal{B}_{\varepsilon\delta})\\
&= \liminf_{\varepsilon\to 0}\left(\int_{\Omega_{\varepsilon\delta}} A^\varepsilon\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta}\,dx - \mathcal{C}_{\varepsilon\delta}\right) \qquad (5.10)\\
&\le \limsup_{\varepsilon\to 0}\left(\int_{\Omega_{\varepsilon\delta}} A^\varepsilon\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta}\,dx - \mathcal{C}_{\varepsilon\delta}\right) \le \limsup_{\varepsilon\to 0}\int_{\Omega_{\varepsilon\delta}} A^\varepsilon\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta}\,dx\\
&= \lim_{\varepsilon\to 0}\left(\int_{\Omega_{\varepsilon\delta}} f u_{\varepsilon\delta}\,dx - \varepsilon^\gamma\int_{\Gamma^{\delta,\varepsilon}} h^{\delta,\varepsilon}(u_{\varepsilon\delta}-u_{\varepsilon\delta})(u_{\varepsilon\delta}-u_{\varepsilon\delta})\,d\sigma_x\right) = \int_\Omega f u_0\,dx,
\end{aligned}$$

which implies that these inequalities are actually equalities. Hence, equations (5.1) and (5.2) hold true. Finally, the convergences in (5.3) and (5.4) follow from (5.10) and with the application of some properties of limits as well as lim sup and lim inf . □

Corollary 5.3. *Under the assumption of Theorem 5.2, we have the corrector result.*

$$\left\|\nabla u_{\varepsilon\delta}\mathbf{1}_{\Omega_{\varepsilon\sqrt{\delta}}} - \nabla u_0 - \sum_{j=1}^N \mathcal{U}_\varepsilon\left(\frac{\partial u_0}{\partial x_j}\right)\mathcal{U}_\varepsilon\left(\nabla_y\widehat{\chi}_j(x,y)\right)\right\|_{L^2(\Omega_{\varepsilon\delta})} \to 0, \qquad (5.11)$$

and

$$\|\nabla u_{\varepsilon\delta}\|_{L^2(\Omega_{\varepsilon\delta}\setminus\Omega_{\varepsilon\sqrt{\delta}})} \to \frac{1}{|Y|^{1/2}} \|\nabla_z U\|_{L^2(\Omega\times\mathbb{R}^N)} . \tag{5.12}$$

Furthermore,

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon\delta}) \to U \quad \text{strongly in } L^2(\Omega; L^2_{\mathrm{loc}}(\mathbb{R}^N)). \tag{5.13}$$

Proof. In view of (4.17), one has

$$\nabla_y \widehat{u} = \nabla_y \sum_{j=1}^N \frac{\partial u_0}{\partial x_j}\widehat{\chi}_j(x,y) = \sum_{j=1}^N \frac{\partial u_0}{\partial x_j}\nabla_y\widehat{\chi}_j(x,y).$$

This along with (5.2), (5.3), Propositions 3.6 and 3.7, and triangle inequality yield

$$\begin{aligned}
&\left\|\nabla u_{\varepsilon\delta}\mathbf{1}_{\Omega_{\varepsilon\sqrt{\delta}}} - \nabla u_0 - \sum_{j=1}^N \mathcal{U}_\varepsilon\left(\frac{\partial u_0}{\partial x_j}\right)\mathcal{U}_\varepsilon\left(\nabla_y\widehat{\chi}_j(x,y)\right)\right\|_{L^2(\Omega_{\varepsilon\delta})} \\
&= \left\|\nabla u_{\varepsilon\delta}\mathbf{1}_{\Omega_{\varepsilon\sqrt{\delta}}} - \nabla u_0 - \mathcal{U}_\varepsilon(\nabla_y\widehat{u})\right\|_{L^2(\Omega_{\varepsilon\delta})} \\
&\le \left\|\nabla u_{\varepsilon\delta}\mathbf{1}_{\Omega_{\varepsilon\sqrt{\delta}}} - \mathcal{U}_\varepsilon(\nabla u_0) - \mathcal{U}_\varepsilon(\nabla_y\widehat{u})\right\|_{L^2(\Omega_{\varepsilon\delta})} + \|\mathcal{U}_\varepsilon(\nabla u_0) - \nabla u_0\|_{L^2(\Omega_{\varepsilon\delta})} \to 0,
\end{aligned}$$

which implies (5.11). Let us prove (5.12). Indeed, from Proposition 3.9 (v), $(\mathcal{H}4)$, and (5.4) we have by unfolding

$$\begin{aligned}
\lim_{\varepsilon\to 0}\|\nabla u_{\varepsilon\delta}\|^2_{L^2(\Omega_{\varepsilon\delta}\setminus\Omega_{\varepsilon\sqrt{\delta}})} &= \lim_{\varepsilon\to 0}\frac{\delta^N}{|Y|}\int_{\Omega\times\frac{1}{\sqrt{\delta}}B} \mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon\delta})\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon\delta})\,dx\,dz \\
&= \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_{\Omega\times\frac{1}{\sqrt{\delta}}B}\left[\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\nabla_z\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon\delta})\right]\left[\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\nabla_z\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon\delta})\right]dx\,dz \\
&= \frac{1}{|Y|}\int_{\Omega\times\mathbb{R}^N}\nabla_z U\nabla_z U\,dx\,dz = \frac{1}{|Y|}\|\nabla_z U\|^2_{L^2(\Omega\times\mathbb{R}^N)} .
\end{aligned}$$

Finally, we prove (5.13). Let ω be an open and bounded set and choose $R > 0$ such that $\omega \cup B \subset \mathbf{B}(O,R)$, the ball in $\mathbb{R}^N$ with center at O of radius R. Also, a Poincaré inequality holds on the space $\mathbf{B}(O,R)$. By Definition 3.8 and since $\widehat{\Omega}_\varepsilon = \Omega\setminus\Lambda_\varepsilon$, we obtain

$$\begin{aligned}
&\left\|\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon\delta}) - U\right\|^2_{L^2(\widehat{\Omega}_\varepsilon\times\mathbf{B}(O,R))} \\
&= \left\|\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon\delta}) - U\right\|^2_{L^2(\Omega\times\mathbf{B}(O,R))} - \left\|\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon\delta}) - U\right\|^2_{L^2(\Lambda_\varepsilon\times\mathbf{B}(O,R))} \\
&\le C\left(\left\|\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\nabla_z\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon\delta}) - \nabla_z U\right\|^2_{L^2(\Omega\times\mathbf{B}(O,R))} + \|\nabla_z U\|^2_{L^2(\Lambda_\varepsilon\times\mathbf{B}(O,R))} + \|U\|^2_{L^2(\Lambda_\varepsilon\times\mathbf{B}(O,R))}\right),
\end{aligned}$$

where C is a generic constant.

For δ small enough, $\omega \subset \mathbf{B}(O,R) \subset \frac{1}{\sqrt{\delta}}B$. This and when (5.4) is applied to the first term in the right-hand side, and since $U = 0$ in $\Omega \times B$, then the left-hand side above approaches zero and so we obtain

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon\delta}) \to U \quad \text{strongly in } L^2(\Omega \times \omega),$$

which yields (5.13). □

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