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A Constructive Approach to Obtaining the Lifespan of Solutions to Cauchy Problems with Small Data

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Abstract

This paper revisits the proof of the lifespan of solutions to second order nonlinear Cauchy problems with small analytic data. The Cauchy problem is solved using the method of successive approximations, thereby constructively obtaining the approximate solutions as well as estimates of their lifespan.

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1 Introduction

The lifespan problem for Cauchy problems has been studied extensively by many authors. In some works, semilinear wave equations are studied in detail and functional analytic methods are used to obtain estimates of the lifespan of their solutions [1, 2, 3]. D'Ancona and Spagnolo stated that the lifespan of solutions of ordinary differential equations becomes larger as the data become smaller, which also holds true in the case of partial differential equations under some hyperbolicity assumption [4].

Gourdin and Mechab [5, 6] also presented several results regarding the lifespan of solutions of certain Cauchy problems. In [5], they considered the generalized Kirchoff equations in the real-analytic category and studied the lifespan of their solutions under certain smallness assumptions. They proved that under some conditions on the equation, or on the Cauchy data, there is an existence-uniqueness result for which the lifespan can also be estimated. In [6], they considered equations of general order without the hyperbolicity condition. They obtained solutions that are holomorphic in t and of Gevrey class with respect to x. Furthermore, in the case where the right-hand side is independent of t, they were able to compute the lifespan of the solution. Lastly, in [6], they considered the case when the Cauchy data are in some Gevrey class and they were also able to show that the lifespan is of some order with respect to $1/\varepsilon$, provided that the right-hand side of the equation is independent of t.

Yamane [7] considered the second-order nonlinear equation

$$\begin{cases} (\partial_t^2 u - P(\partial))u = F\left(\nabla u, \nabla^2 u\right) \\ u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x), \end{cases} \tag{CP}$$

where $(t,x) \in \mathbb{R} \times \mathbb{R}^n$, $\nabla u = (\partial_{x_i} u)_{1 \le i \le n}$, $\nabla^2 u = (\partial_{x_i}\partial_{x_j} u)_{1 \le i,j \le n}$, and $P(\partial)$ is some second-order linear partial differential operator. He formulated an existence-uniqueness result for (CP), together with an estimate for the lifespan of the solution. His result stated that if the Cauchy data satisfy a smallness condition in the form of a Cauchy type estimate, then the lifespan of the solution is of order 1 with respect to $1/\varepsilon$. This result was improved in subsequent papers [8, 9], as the right hand-side of the equation in these works now also depend on the time derivative of the unknown function $u(t,x)$.

In the abovementioned papers, Gourdin and Mechab, and Yamane used the fixed point theorem in proving their results, that is, they constructed a Banach algebra derived from some majorant series and defined a contraction map on the Banach space. A more recent work by Tolentino, Bacani and Tahara [10] that considered a more general Cauchy problem also employed a fixed point approach. The equations considered in these works are non-singular, but uniformly analytic solutions have also been constructed for singular equations [11].

The main goal of this paper is to provide an alternative approach in establishing the unique solvability of second order nonlinear Cauchy problems and constructively obtain estimates of the lifespan of their solutions. In particular, we will show that as the data become smaller in some sense, the lifespan of the solution becomes longer, and so the solution becomes global in t. Furthermore, we will show that the lifespan is indeed of order 1 with respect to $1/\varepsilon$. The Banach Fixed Point Theorem was used in [12] and [7], which required them to choose a predetermined value for T dependent on $1/\varepsilon$ and introduce spaces of functions defined on the interval $(-T,T)$. While we will adopt the same spaces used in [7], we can more concretely acquire the order of the lifespan with respect to $1/\varepsilon$ since our method of proof is constructive. We see merit in this approach because we are able to obtain concretely the sequence of approximate solutions as well as the estimates of their norms, which in turn leads to the estimate of their lifespan.

2 The Lifespan Problem

Let $(t,x) \in \mathbb{R} \times \mathbb{R}$, Ω an open subset of $\mathbb{R}$, and $\mathbb{N} = \{0,1,2,\ldots,\}$.

Definition 2.1. A C^∞ function $\varphi(x)$ is said to be uniformly analytic on Ω if there exists a positive constant C such that for all $\alpha \in \mathbb{N}$,

$$\sup_{x\in\Omega} |\partial_x^\alpha \varphi(x)| \le C^{\alpha+1}\alpha!.$$

We denote by $\mathcal{A}(\Omega)$ the collection of all uniformly analytic functions on Ω.

Definition 2.2. Let $k \in \mathbb{N}$. A continuous function function $u(t,x)$ on $\Omega_T = (-T,T) \times \Omega$ is said to belong to the class $C^k(T;\mathcal{A}(\Omega))$ if

a. for all $j \in \{0,1,\ldots,k\}$ and $\alpha \in \mathbb{N}$, $\partial_t^j \partial_x^\alpha u \in C(\Omega_T)$,

b. for all $T' \in (0,T)$, there exists $C = C_{T'} > 0$ such that for all $j \in \{0,1,\ldots,k\}$ and for all $\alpha \in \mathbb{N}$,

$$\sup_{|t|\le T',\, x\in\Omega} |\partial_t^j \partial_x^\alpha u(t,x)| \le C^{\alpha+1}\alpha!.$$

Let $\gamma \in \mathbb{C}$. We consider the second order nonlinear Cauchy problem

$$\begin{cases} (\partial_t^2 - \gamma\partial_x^2)\, u = F(\partial_x u, \partial_x^2 u) \\ u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x), \end{cases} \tag{M}$$

under the following assumptions:

(A1) The function $F(X)$ is analytic in a neighborhood around the origin, and vanishes of second order at $X = 0$;

(A2) The Cauchy data $\varphi(x)$ and $\psi(x)$ are uniformly analytic in an open subset Ω of $\mathbb{R}$.

Yamane was able to prove the following result.

Theorem 2.3 ([7])**.** *Suppose that (A1) and (A2) hold. Then there exists $\mu > 0$ such that the following holds for all $0 < \varepsilon < 1$: if the Cauchy data satisfy $\sup_{x\in\Omega} |\partial_x^\alpha \varphi(x)| \le \varepsilon^{\alpha+1}\alpha!$ and $\sup_{x\in\Omega} |\partial_x^\alpha \psi(x)| \le \varepsilon^{\alpha+1}\alpha!$ for all $\alpha \in \mathbb{N}$, then* (M) *has a unique solution in $C^2(T;\mathcal{A}(\Omega))$, with $T = \mu/\varepsilon$.*

Remark 2.4. We note that the Cauchy-Kowalevsky Theorem (see, e.g., [13]) guarantees the local existence and uniqueness of the analytic solution (in t and x) to (M). The current setup only considers the one-dimensional 'space' variable version of Yamane's problem because our main focus is to illustrate the constructive approach in obtaining the lifespan of solutions.

Yamane remarked in [9] that the best possible order of the lifespan of the solution is of order 1 with respect to $1/\varepsilon$. This claim is illustrated in the following example: Let $\Omega = (\frac{1}{4}, \frac{1}{2})$, and consider the Cauchy problem given by

$$\begin{cases} \partial_t^2 u = -(\partial_x u)^2 \\ u(0,x) = \log(2-\varepsilon x),\ \partial_t u(0,x) = \dfrac{-\varepsilon}{2-\varepsilon x}. \end{cases}$$

The above Cauchy problem has a solution given by $u(t,x) = \log|2 - \varepsilon t - \varepsilon x|$, which is undefined at $t = \frac{1}{\varepsilon}(1 - \frac{\varepsilon}{2}x)$. This gives the lifespan of the solution to be $T_\varepsilon = \frac{1}{\varepsilon}(1-\frac{\varepsilon}{4})$, since $u(t,x)$ will be only be defined on $\Omega_T = (-T,T) \times (\frac{1}{4}, \frac{1}{2})$ when $T \le T_\varepsilon$. Note that $T_\varepsilon \to \infty$ as $\varepsilon \to 0$, and T_ε is of order 1 with respect to $1/\varepsilon$.

3 Preliminaries

Given two formal power series $f(X) = \sum a_k X^k$, $a_k \in \mathbb{C}$, and $g(X) = \sum b_k X^k$, $b_k \in \mathbb{R}$, we say that $f \ll g$ if and only if $|a_k| \leq b_k$ for all k. Also, set the operators D^{-1} and D to be the anti-differentiation and differentiation operators, respectively.

In the following, we recall some of the results of Yamane [7]. We will also provide an improvement to one of his propositions.

Define the series

$$\phi(X) = \frac{1}{K} \sum_{k \in \mathbb{N}} \frac{X^k}{(k+1)^2}, \quad \text{where } K = 4\pi^2/3. \tag{1}$$

This series is due to Lax [14], except for the constant K, which was later added to simplify calculations. The function ϕ is convergent on the unit disk and satisfies $\phi^2 \ll \phi$ for this choice of K.

Definition 3.1 ([7])**.** Let $\zeta > 0$ and $T > 0$. A function $u(t,x)$ is said to belong to the space $\mathcal{G}_{T,\zeta}(\Omega)$ if it is continuous on Ω_T, infinitely differentiable in x, and there exists a constant $C > 0$ such that for all $\alpha \in \mathbb{N}$ and $t \in (-T,T)$,

$$\sup_{x \in \Omega} |\partial_x^\alpha u(t,x)| \leq C\zeta^\alpha D^\alpha \phi(|t|/T),$$

where ϕ is the one defined in (1).

For $f \in \mathcal{G}_{T,\zeta}(\Omega)$ and $g \in H(\Omega)$, we say that $f(t,x) \lll g(t,x)$ if for all $\alpha \in \mathbb{N}$ and $t \in (-T,T)$

$$\sup_{x \in \Omega} |\partial_x^\alpha u(t,x)| \leq \partial_x^\alpha g(t,0).$$

Set $\phi_{T,\zeta}(t,x) = \phi\left(|t|/T + \zeta x\right)$. Clearly, for any $\alpha \in \mathbb{N}$, $\partial_x^\alpha \phi_{T,\zeta}(t,0) = \zeta^\alpha D^\alpha \phi(|t|/T)$. Therefore, $u(t,x) \in \mathcal{G}_{T,\zeta}(\Omega)$ if there exists $C > 0$ such that $u(t,x) \lll C\phi_{T,\zeta}(t,x)$. Define the norm $\|u\|$ as the infimum of such C's. Then from [7], the space $\mathcal{G}_{T,\zeta}(\Omega)$ becomes a Banach algebra with respect to this norm.

Moreover, by equipping the direct sum $\bigoplus \mathcal{G}_{T,\zeta}(\Omega)$ with norm given by $\|\vec{\tau}(t,x)\|_N = \max_j \|\tau_j(t,x)\|$, where $\vec{\tau}(t,x) = (\tau_1(t,x), \ldots, \tau_N(t,x))$, the following estimates can be shown.

Proposition 3.2 ([7])**.** *Let $F(X) = F(X_1, \ldots, X_N) = \sum_{|\alpha| \geq 2} a_\alpha X^\alpha$ be a convergent power series which vanishes of second order at $X = 0$. If $\vec{\tau}(t,x)$, $\vec{\sigma}(t,x) \in \bigoplus \mathcal{G}_{T,\zeta}(\Omega)$ have sufficiently small norms, then $F(\vec{\tau}(t,x))$ and $F(\vec{\sigma}(t,x))$ are well-defined as elements of $\mathcal{G}_{T,\zeta}(\Omega)$. Moreover, there exists a constant $A = A(F) > 0$, depending only on F and independent of $\vec{\tau}, \vec{\sigma}, T, \zeta$ and Ω, such that*

$$\|F(\vec{\tau}(x))\| \leq A \|\vec{\tau}\|_N^2,$$

$$\|F(\vec{\tau}(x)) - F(\vec{\sigma}(x))\| \leq A \|\vec{\tau} - \vec{\sigma}\|_N (\|\vec{\tau}\|_N + \|\vec{\sigma}\|_N).$$

The next proposition will help us deal with derivatives of functions in $\mathcal{G}_{T,\zeta}(\Omega)$.

Proposition 3.3 ([7]). *Let $k, \alpha \in \mathbb{N}$ with $-k+\alpha \le 0$. The operator $\partial_t^{-k}\partial_x^\alpha$ is an endomorphism of the Banach space $\mathcal{G}_{T,\zeta}(\Omega)$. Moreover, there exists $B > 0$ such that*

$$\left\|\partial_t^{-k}\partial_x^\alpha u\right\| \le BT^k\zeta^\alpha \|u\|$$

for any k, α satisfying $\alpha \le k \le 2$.

Yamane showed in [7] that for all $T > 0$ and $\zeta > 0$, $\mathcal{G}_{T,\zeta}(\Omega) \subseteq C(T; \mathcal{A}(\Omega))$. He further noted that for certain values of ζ, the converse inclusion also holds. Since $\varphi(x) \in \mathcal{A}(\Omega)$, there exist positive constants $p(\varphi)$ and $q(\varphi)$, not necessarily unique, such that for all $\alpha \in \mathbb{N}$,

$$\sup_{x\in\Omega} |\partial_x^\alpha \varphi(x)| \le p(\varphi)q(\varphi)^\alpha \alpha!\,. \tag{2}$$

The almost converse inclusion is as follows:

Proposition 3.4 ([7]). *If $\psi(x) \in \mathcal{A}(\Omega)$, then for all $T > 0$ and $\zeta \ge e^2 q(\psi)$, we have $\psi(x) \in \mathcal{G}_{T,\zeta}(\Omega)$ and $\|\psi\| \le Kp(\psi)$, where K is the one in (1). Here, e is the Euler's number.*

We now state a slightly improved version of a proposition in [7], in the case when the estimate in (2) takes a particular form.

Proposition 3.5. *Assume $\varphi(x) \in \mathcal{A}(\Omega)$ satisfies (2) with $p(\varphi) = q(\varphi) = \varepsilon$, and let m be a positive integer. Then for $j = 1, 2, \ldots, m$:*

$$p(\partial_x^j \varphi) = m!\,\varepsilon^{j+1} \quad \text{and} \quad q(\partial_x^j\varphi) = (m+1)!\,\varepsilon.$$

Proof. Fix $j \in \{1, 2, \ldots, m\}$. Since $\varphi(x) \in \mathcal{A}(\Omega)$, so are its higher-order derivatives. Moreover, by (2),

$$\begin{aligned}\sup_{x\in\Omega} |\partial_x^\alpha(\partial_x^j\varphi(x))| &\le \varepsilon^{\alpha+j+1}(\alpha+j)! \\ &= \varepsilon^{j+1}\big((m+1)!\,\varepsilon\big)^\alpha \frac{(\alpha+j)(\alpha+j-1)\cdots(\alpha+1)}{\big[(m+1)!\big]^\alpha}\,\alpha! \\ &\le m!\,\varepsilon^{j+1}\big((m+1)!\,\varepsilon\big)^\alpha \alpha!\end{aligned}$$

from which we obtain the desired result. Here, we used the fact that if $f_j(\alpha) := (\alpha+j)(\alpha+j-1)\cdots(\alpha+1)[(m+1)!]^{-\alpha}$, then $f_j(\alpha) \le f_m(\alpha)$, for any $j = 1, \ldots, m$ and $\alpha \in \mathbb{N}$. Moreover, the sequence $\{f_m(\alpha)\}_{\alpha\in\mathbb{N}}$ is decreasing on $\mathbb{N}$, and attains the maximum value of $m!$ when $\alpha = 0$. □

Remark 3.6. Note that the coefficient of ε^{j+1} in our estimate of $p(\partial_x^j\varphi)$ is always $m!$ for any $j \le m$, as opposed to the one in [7] where the coefficient depends on the order of the derivative. Although this is a slight improvement, it will result to simpler computations if we are to consider equations of general order.

4 Proof of Theorem 2.3

We now prove Theorem 2.3 through the method of successive approximations. For the preliminary step, we first reduce the problem into an initial-value problem as in [7] and afterwards define our approximate solutions. Throughout the proof, we let the constants A and B, defined in Propositions 3.2 and 3.3 be sufficiently large. For brevity of notations, set $D^{1,2}u = (\partial_x u, \partial_x^2 u)$.

Let $v(t,x) = u(t,x) - \varphi(x) - t\psi(x)$. This implies that $v(0,x) = 0 = \partial_t v(0,x)$. Hence, (M) becomes

$$\partial_t^2 v = \gamma \partial_x^2 (v + \varphi + t\psi) + F\big(D^{1,2}\left(v + \varphi + t\psi\right)\big).$$

Furthermore, set $w(t,x) = \partial_t^2 v(t,x)$ and so $v(t,x) = \partial_t^{-2} w(t,x)$. The original equation (M) is then reduced to an equation of the form $w = \mathcal{L}(w)$, where the operator $\mathcal{L}$ is defined as

$$\mathcal{L}(w) := \gamma \partial_x^2 (\partial_t^{-2} w + \varphi + t\psi) + F\big(D^{1,2}\left(\partial_t^{-2} w + \varphi + t\psi\right)\big).$$

Define the approximate solutions $\{w_k\}_{k\geq 0}$ such that

$$w_0(t,x) = \mathcal{L}(0) \quad \text{and} \quad w_n(t,x) = \mathcal{L}(w_{n-1}), \text{ for } n \geq 1.$$

Furthermore, define the sequence $\{d_k\}_{k\geq 0}$ as follows:

$$d_0(t,x) = w_0 \quad \text{and} \quad d_n(t,x) = w_n - w_{n-1}, \text{ for } n \geq 1.$$

It then follows that for $n \geq 1$,

$$d_n = \gamma \partial_x^2 (\partial_t^{-2} d_{n-1}) + H_n,$$

where $H_n = F\big(D^{1,2}\left(\partial_t^{-2} w_{n-1} + \varphi + t\psi\right)\big) - F\big(D^{1,2}\left(\partial_t^{-2} w_{n-2} + \varphi + t\psi\right)\big)$ and $w_{-1} \equiv 0$.

Let $0 < \varepsilon < 1$. Set $\zeta = 6e^2\varepsilon$ and $T = \mu/\varepsilon^\sigma$, for some $\sigma > 0$. We will show that the approximate solutions converge to the true solution, say $w(t,x)$, by showing that the partial sums of $\{d_n\}$ converge in $\mathcal{G}_{T,\zeta}(\Omega)$. In the following, we will show that for each $n \in \mathbb{N}$, $\|d_n\| \leq Cr^n$, for some $0 < r < 1$ and $C > 0$, independent of ε.

Let $t \in (-T, T)$. We start with the case $k = 0$. Under the choice $\zeta = 6e^2\varepsilon$, we can use Propositions 3.4 and 3.5 to obtain for $\alpha = 1, 2$,

$$\|\partial_x^\alpha (\varphi + t\psi)\| \leq 2K(1+T)\,\varepsilon^3.$$

Moreover, from Proposition 3.2,

$$\begin{aligned}\left\|F(D^{1,2}(\varphi + t\psi))\right\| &\leq A\left\|D^{1,2}(\varphi + t\psi)\right\|_2^2 \\ &\leq A\left(2K(1+T)\max\left\{\varepsilon^2, \varepsilon^3\right\}\right)^2 \\ &\leq 4AK^2(1+T)^2\varepsilon^4.\end{aligned}$$

Thus, by the triangle inequality, we have

$$\begin{aligned}\|d_0\| = \|w_0\| &= \left\|\gamma\partial_x^2(\varphi + t\psi) + F(D^{1,2}(\varphi + t\psi))\right\| \\ &\leq 2|\gamma|K(1+T)\varepsilon^3 + 4AK^2(1+T)^2\varepsilon^4 \\ &\leq CK^2\varepsilon^3(1+T)\left(1 + \varepsilon(1+T)\right),\end{aligned}$$

where $C = \max\{2|\gamma|, 4A\}$. Finally, since $T = \mu/\varepsilon^\sigma$, if we assume that $\mu < 1$, we have

$$\|d_0\| \leq CK^2(\varepsilon^3 + \mu\varepsilon^{3-\sigma} + \varepsilon^4 + 2\mu\varepsilon^{4-\sigma} + \mu^2\varepsilon^{4-2\sigma})$$

$$\leq 6CK^2\varepsilon^{4-2\sigma}.$$

Since we want the bound to remain finite for any $\varepsilon \in (0,1)$, we must have $\sigma \leq 2$. For the case $n \geq 1$, we have the following proposition.

Proposition 4.1. *Let μ be sufficiently small such that*

$$72e^2BC^2K^2\mu^2\left(3e^2 + 18e^2BCK^2\mu^2 + 4K\right) < \frac{1}{2}, \tag{3}$$

where A and B are the constants defined in Propositions 3.2 and 3.3, $C = \max\{2|\gamma|, 4A\}$, and K is the one in (1)*. Then the following estimate holds for all $n \geq 1$:*

$$\|d_n\| \leq \left(\frac{1}{2}\right)^n \varepsilon^{(6\cdot 2^n-2)-(6\cdot 2^n-4)\sigma}.$$

Proof. Recall that d_1 satisfies the estimate

$$d_1 = \gamma\partial_x^2(\partial_t^{-2}d_0) + H_1.$$

Since $|\gamma| \leq C$, by Proposition 3.3, it is seen that

$$\begin{aligned}\left\|\gamma\partial_x^2\partial_t^{-2}d_0\right\| &\leq |\gamma|BT^2\zeta^2\left\|d_0\right\| = 36|\gamma|e^4B\mu^2\varepsilon^{2-2\sigma}\left\|d_0\right\| \\ &\leq 216e^4BC^2K^2\mu^2\varepsilon^{6-4\sigma}.\end{aligned}$$

From our previous computation, we have $\left\|D^{1,2}(\varphi+t\psi)\right\|_2 \leq 2K(\varepsilon^2+\varepsilon^{2-\sigma}) \leq 4K\varepsilon^{2-\sigma}$. Also, since terms with $T^2\zeta^2$ are smaller than those with $T^2\zeta$, we have $\left\|D^{1,2}\partial_t^{-2}d_0\right\|_2 \leq \left\|\partial_x\partial_t^{-2}d_0\right\| \leq 36e^2BCK^2\mu^2\varepsilon^{5-4\sigma}$. Lastly, since $A \leq C$, by Proposition 3.2 we have

$$\begin{aligned}\|H_1\| &= \left\|F(D^{1,2}(\partial_t^{-2}d_0+\varphi+t\psi)) - F(D^{1,2}(\varphi+t\psi))\right\| \\ &\leq A\left\|D^{1,2}\partial_t^{-2}d_0\right\|_2\left(\left\|D^{1,2}(\partial_t^{-2}d_0+\varphi+t\psi)\right\|_2 + \left\|D^{1,2}(\varphi+t\psi)\right\|_2\right) \\ &\leq 36e^2BC^2K^2\mu^2\varepsilon^{5-4\sigma}\cdot\left(36e^2BCK^2\mu^2\varepsilon^{5-4\sigma} + 8K\varepsilon^{2-\sigma}\right) \\ &= 1296e^4B^2C^3K^4\mu^4\varepsilon^{10-8\sigma} + 288e^2BC^2K^3\mu^2\varepsilon^{7-5\sigma}.\end{aligned}$$

Clearly, for the bound of $\|d_1\|$ to be finite for any $\varepsilon \in (0,1)$, we must have that $\sigma \leq 5/4$. This implies that $\max\{\varepsilon^{6-4\sigma}, \varepsilon^{7-5\sigma}\} \leq \varepsilon^{10-8\sigma}$ and thus we have

$$\begin{aligned}\|d_1\| &\leq \left\|\gamma\partial_x^2\partial_t^{-2}d_0\right\| + \|H_1\| \\ &\leq 216e^4BC^2K^2\mu^2\varepsilon^{10-8\sigma} + 1296e^4B^2C^3K^4\mu^4\varepsilon^{10-8\sigma} + 288e^2BC^2K^3\mu^2\varepsilon^{10-8\sigma} \\ &\leq 72e^2BC^2K^2\mu^2\varepsilon^{10-8\sigma}\left(3e^2 + 18e^2BCK^2\mu^2 + 4K\right) \\ &\leq \left(\frac{1}{2}\right)\varepsilon^{10-8\sigma}\end{aligned}$$

by our choice of μ in (3).

Suppose now that the claim holds for $n = k$. We will show that the claim also holds for $k = n + 1$. The function d_{n+1} satisfies the equation

$$d_{n+1} = \gamma\partial_x^2(\partial_t^{-2}d_n) + F\big(D^{1,2}(\partial_t^{-2}w_n + \varphi + t\psi)\big) - F\big(D^{1,2}(\partial_t^{-2}w_{n-1} + \varphi + t\psi)\big)\,.$$

By the inductive hypothesis,

$$\begin{aligned}\left\|\gamma\partial_x^2(\partial_t^{-2}d_n)\right\| &\leq |\gamma|BT^2\zeta^2 \cdot \left(\frac{1}{2}\right)^n \varepsilon^{(6\cdot 2^n-2)-(6\cdot 2^n-4)\sigma}\\ &= \left(\frac{1}{2}\right)^n \cdot 36e^4BC\mu^2\varepsilon^{(6\cdot 2^n)-(6\cdot 2^n-2)\sigma}.\end{aligned}$$

Since the function $f(x) = 1+2(6x-4)^{-1}$ is decreasing on $\mathbb{Z}^+$, we see that for the bound of $\|d_n\|$ to be finite for any $\varepsilon \in (0,1)$, σ should satisfy $\sigma \leq (6\cdot 2^n - 2)/(6\cdot 2^n - 4)$. Using this, we can obtain the following estimate for $\|w_i\|$ for any $i \leq n$:

$$\begin{aligned}\|w_i\| \leq \sum_{k=0}^{n}\|d_k\| &\leq \left(6CK^2 + \sum_{k=1}^{n}\left(\frac{1}{2}\right)^k\right) \cdot \varepsilon^{(6\cdot 2^n-2)-(6\cdot 2^n-4)\sigma}\\ &\leq 12CK^2\varepsilon^{(6\cdot 2^n-2)-(6\cdot 2^n-4)\sigma}.\end{aligned}$$

Hence, by Proposition 3.3, we have for any $i \leq n$,

$$\begin{aligned}\left\|D^{1,2}\partial_t^{-2}w_i\right\|_2 &\leq BT^2\zeta \cdot 12CK^2\varepsilon^{(6\cdot 2^n-2)-(6\cdot 2^n-4)\sigma}\\ &\leq 72e^2BCK^2\mu^2\varepsilon^{(6\cdot 2^n-1)-(6\cdot 2^n-2)\sigma}.\end{aligned}$$

Similarly, using Proposition 3.3, we have

$$\left\|D^{1,2}\partial_t^{-2}d_n\right\|_2 \leq \left(\frac{1}{2}\right)^n \cdot 6e^2B\mu^2\varepsilon^{(6\cdot 2^n-1)-(6\cdot 2^n-2)\sigma}.$$

Therefore, by Proposition 3.2 and using again the fact that $\left\|D^{1,2}(\varphi + t\psi)\right\|_2 \leq 4K\varepsilon^{2-\sigma}$, we get

$$\begin{aligned}\|H_{n+1}\| &= \left\|F(D^{1,2}(\partial_t^{-2}w_n + \varphi + t\psi) - F(D^{1,2}(D_t^{-2}(w_{n-1} + \varphi + t\psi))\right\|\\ &\leq A\left\|D^{1,2}\partial_t^{-2}d_n\right\|_2\left(\left\|D^{1,2}\partial_t^{-2}w_n\right\|_2 + \left\|D^{1,2}\partial_t^{-2}w_{n-1}\right\|_2 + 2\left\|D^{1,2}(\varphi + t\psi)\right\|_2\right)\\ &\leq \left(\frac{1}{2}\right)^n \cdot 48e^2BCK\mu^2\left(18e^2BCK\mu^2\varepsilon^{(6\cdot 2^{n+1}-2)-(6\cdot 2^{n+1}-4)\sigma} + \varepsilon^{(6\cdot 2^n+1)-(6\cdot 2^n-1)\sigma}\right).\end{aligned}$$

We again see that for the bound of $\|d_{n+1}\|$ to be finite for any $\varepsilon \in (0,1)$, σ should satisfy $\sigma \leq (6\cdot 2^{n+1} - 2)/(6\cdot 2^{n+1} - 4)$. Finally, by the choice of μ in (3), and the fact that $C, K \geq 1$, we have

$$\|d_{n+1}\| \leq \left\|\gamma\partial_x^2(\partial_t^{-2}d_n)\right\| + \|H_{n+1}\|$$

$$\leq \left(\frac{1}{2}\right)^n \cdot 12e^2BC\mu^2\varepsilon^{(6\cdot 2^{n+1}-2)-(6\cdot 2^{n+1}-4)\sigma}\left(3e^2+72e^2BCK^2\mu^2+4K\right)$$
$$\leq \left(\frac{1}{2}\right)^{n+1} \cdot \varepsilon^{(6\cdot 2^{n+1}-2)-(6\cdot 2^{n+1}-4)\sigma}.$$

□

This proves the existence of a solution in (M); as earlier remarked, uniqueness follows from the Cauchy-Kowalevsky Theorem [13]. Since the sequence $1+2(6\cdot 2^n-2)^{-1}$ is decreasing and tends to 1 as n goes to infinity, we conclude that $\sigma = 1$. This implies furthermore that $T = \mu/\varepsilon$, we have $T \to \infty$ as $\varepsilon \to 0$.

Remark 4.2. In this section, we only considered the real-analytic case. It was stated in [7] that the holomorphic case follows similarly by defining a *uniformly holomorphic* function and using the same machinery discussed in the real-analytic case. Furthermore, the proof for the complex-analytic case is done as in the real case.

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