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Inverse Best Approximation Property of Convex Sets

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Dedicated to the memory of Noli N. Reyes (1963-2020)

Abstract

A pair of nonempty, closed, and convex subsets (C, D) of a real Hilbert space $\mathcal{H}$ is said to have the inverse best approximation property (IBAP) if for every $(p, q) \in C \times D$, there exists $x \in \mathcal{H}$ whose orthogonal projections onto C and D are p and q, respectively. In this paper, we provide several consequences of the IBAP for two nonempty closed convex subsets of $\mathcal{H}$. Denote by $\widehat{C}$ the closed linear subspace parallel to the affine hull of C. We show that at least in the important cases where $\mathcal{H}$ is finite-dimensional or C and D are closed affine subspaces of $\mathcal{H}$, (C, D) has the IBAP if and only if $(\widehat{C}, \widehat{D})$ has the IBAP. We also explore the existence and behavior of approximate solutions to the same system of equations with prescribed orthogonal projections. Results obtained were applied to a general recovery problem.

Keywords: convex sets, convex cones, affine subspaces, orthogonal projections, IBAP
2020 MSC: 41A27, 41A46, 41A50, 47J05, 47J06

1 Introduction and Notations

Throughout, $\mathcal{H}$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. A pair of nonempty, closed, and convex subsets (C, D) of $\mathcal{H}$ is said to have the *inverse best approximation property* (IBAP) if for every $(p, q) \in C \times D$, we can find $x \in \mathcal{H}$ whose orthogonal projections onto C and D are p and q, respectively. This property already appeared in [5, Part One, Chapter 3] as a tool to solve problems in harmonic analysis such as finding a function with prescribed values on subsets of the time and frequency domains [3, Proposition 4.10]. In 2010, Combettes and Reyes deeply investigated the IBAP for closed linear subspaces by providing various characterizations [3, Theorem 2.8] and several applications [3, Section 4]. In 2021, the authors in [4] proposed a block-iterative algorithm which solves a more general framework of a system of equations with convex constraints and prescribed proximal points.

For any pair of subsets X, Y of $\mathcal{H}$ and for any scalars $\alpha, \beta \in \mathbb{R}$, we define

$$\alpha X + \beta Y = \{\alpha x + \beta y \mid x \in X \text{ and } y \in Y\} \quad \text{and} \quad X - Y = X + (-1)Y.$$

Let E be a subset of $\mathcal{H}$. We say that E is an affine subspace of $\mathcal{H}$ if $\alpha E + (1-\alpha)E \subseteq E$ for every $\alpha \in \mathbb{R}$. The smallest affine subspace of $\mathcal{H}$ that contains E is denoted by $\operatorname{aff} E$. If E is affine, then $E - E$ is linear and is called the linear subspace of $\mathcal{H}$ that is parallel to E. Moreover, if E is affine, then $(\forall e \in E)\ E - e = E - E$.

Let K be a subset of $\mathcal{H}$. We say that K is a cone of $\mathcal{H}$ if $\alpha K \subseteq K$ for all $\alpha > 0$. The smallest cone of $\mathcal{H}$ that contains K is denoted by $\operatorname{cone} K$. If K is both a cone and a convex set, it is called a convex cone. The relative interior of K is $\operatorname{ri} K = \{x \in K \mid (\exists \varepsilon > 0)\ \overline{B(x,\varepsilon)} \cap \operatorname{aff} C \subseteq C\}$, where $B(x,\varepsilon) = \{y \in \mathcal{H} \mid \|x - y\| < \varepsilon\}$. It is clear that $\operatorname{int} K \subseteq \operatorname{ri} K \subseteq K$. The polar cone of K is the set $K^{\ominus} = \{x \in \mathcal{H} \mid \sup\langle K, x\rangle \leqslant 0\}$. If K is linear, $K^{\ominus} = K^{\perp}$.

Let C be a nonempty convex subset of $\mathcal{H}$. If $x \in C$, we define the tangent and normal cones of C at x by $\mathrm{T}_C x = \overline{\operatorname{cone}(C - x)}$ and $\mathrm{N}_C x = (C - x)^{\ominus}$, respectively. If $x \notin C$, we define $\mathrm{T}_C x = \varnothing$ and $\mathrm{N}_C x = \varnothing$. The two convex cones satisfy $\mathrm{T}_C^{\ominus} x = \mathrm{N}_C x$ and $\mathrm{N}_C^{\ominus} x = \mathrm{T}_C x$ [1, Proposition 6.44(i)]. Suppose further that C is closed. Then for each $x \in \mathcal{H}$, we denote by $\mathrm{P}_C x$ the unique point in C that is closest to x. If $p \in C$, then $p + \mathrm{N}_C p$ is the set of all vectors in $\mathcal{H}$ whose projection onto C is p [1, Proposition 6.47], i.e.,

$$(\forall x \in \mathcal{H})\ p = \mathrm{P}_C x \Leftrightarrow x - p \in \mathrm{N}_C p. \tag{1}$$

We refer the reader to [1, 2, 8, 10] for a more detailed treatment of these important notions in convex analysis. In this paper, we wish to study the inverse best approximation property of a family of nonempty, closed, and convex subsets of $\mathcal{H}$. Throughout, we let $\mathbb{I}$ be a nonempty finite index set.

Problem 1.1. Let $(C_i)_{i\in\mathbb{I}}$ be a finite family of nonempty, closed, and convex subsets of $\mathcal{H}$. Given $(p_i)_{i\in\mathbb{I}} \in \times_{i\in\mathbb{I}} C_i$, we wish to find conditions for the existence of a vector $x \in \mathcal{H}$ such that

$$(\forall i \in \mathbb{I})\ \mathrm{P}_{C_i} x = p_i. \tag{2}$$

Let us denote the set of solutions to (2) by

$$S\big((p_i)_{i\in\mathbb{I}}\big) = \bigcap_{i\in\mathbb{I}} \{x \in \mathcal{H} \mid \mathrm{P}_{C_i} x = p_i\}.$$

The problem is now equivalent to determining when $S\big((p_i)_{i\in\mathbb{I}}\big)$ is nonempty. In 2010, Combettes and Reyes [3] enumerated several characterizations of when a finite family of nonempty closed linear subspaces $(U_i)_{i\in\mathbb{I}}$ satisfies the IBAP. In particular, for a pair of linear subspaces (U, V), the authors showed that (2) has a solution if and only if $U \cap V = \{0\}$ and $U + V$ is closed. Their results were then applied to harmonic analysis, signal recovery, and consistency of linear systems, among others [3, Section 4]. They also discussed the characterization of the existence of approximate solutions to (2). In [7], the authors investigated the properties of approximate solutions to a special case of (2) and applied them to the behavior of bandlimited approximations to nonsmooth time-limited signals.

The following proposition characterizes $S\big((p_i)_{i\in\mathbb{I}}\big)$. Here, $(C_i)_{i\in\mathbb{I}}$ is a finite family of nonempty, closed, and convex subsets of $\mathcal{H}$.

Proposition 1.2. *Let* $(p_i)_{i\in\mathbb{I}} \in \times_{i\in\mathbb{I}}(C_i)$. *Then* $S\big((p_i)_{i\in\mathbb{I}}\big) = \bigcap_{i\in\mathbb{I}}(p_i + \mathrm{N}_{C_i}p_i)$.

Proof. In view of (1), we obtain

$$(\forall i \in \mathbb{I})\ \mathrm{P}_{C_i}x = p_i \Leftrightarrow (\forall i \in \mathbb{I})\ x \in p_i + \mathrm{N}_{C_i}p_i \Leftrightarrow x \in \bigcap_{i\in\mathbb{I}}(p_i + \mathrm{N}_{C_i}p_i).$$

□

Remark 1.3. The normal cones of special convex sets are known. Let E be a nonempty closed convex subset of $\mathcal{H}$ and let $x \in E$. If E is linear, then $\mathrm{N}_E x = E^\perp$. If E is affine, $\mathrm{N}_E x = (E - E)^\perp$. Finally, if E is a convex cone, then $\mathrm{N}_E x = E^\ominus \cap \{x\}^\perp$. Their proofs can be found in [1, Section 6.4].

2 Consequences of the IBAP

In this section, we will see that the IBAP is more practical to study for affine subspaces or when $\mathcal{H}$ is finite-dimensional. For this section, let us restrict ourselves to the case when $\mathbb{I}$ has only two indices.

In [3, Corollary 2.12(i)⇔(iii)], it is shown that if (C, D) is a pair of closed linear subspaces of $\mathcal{H}$, then (C, D) satisfies the IBAP if and only if $\mathrm{P}_C(D^\perp) = C$. Let (C, D) be a pair of nonempty, closed, and convex subsets of $\mathcal{H}$. We consider the following condition:

$$(\forall d \in D)\ \mathrm{P}_C\big(\mathrm{P}_D^{-1}(\{d\})\big) = C. \tag{3}$$

The condition in (3) is equivalent to $\mathrm{P}_C(D^\perp) = C$ whenever C and D are linear. This is easily seen from the fact that the linearity of C implies the linearity of P_C [1, Corollary 3.24] and the fact that if D is linear and $d \in D$, then (1) implies that $\mathrm{P}_D^{-1}(\{d\}) = d + D^\perp$.

Proposition 2.1. *Let C and D be nonempty, closed, and convex subsets of $\mathcal{H}$. Then (C, D) satisfies the IBAP if and only if Condition* (3) *is satisfied.*

Proof. Assume that (C, D) satisfies the IBAP and let $d \in D$. The inclusion $\mathrm{P}_C\big(\mathrm{P}_D^{-1}(\{d\})\big) \subseteq C$ is clear. Now let $c \in C$. Then there exists $x \in \mathcal{H}$ such that $\mathrm{P}_C x = c$ and $\mathrm{P}_D x = d$. Therefore, $x \in \mathrm{P}_D^{-1}(\{d\})$ and so $c = \mathrm{P}_C x \in \mathrm{P}_C\big(\mathrm{P}_D^{-1}(\{d\})\big)$. This proves the necessity part. Now assume Condition (3) and let $(c, d) \in C \times D$. Then there exists $x \in \mathrm{P}_D^{-1}(d)$ such that $c = \mathrm{P}_C x$. Since $x \in \mathrm{P}_D^{-1}d \Leftrightarrow \mathrm{P}_D x = d$, we see that the pair (C, D) satisfies the IBAP. □

Proposition 2.1 tells us that a pair (C, D) of nonempty, closed, and convex subsets of $\mathcal{H}$ can only satisfy the IBAP if all points of C are projections of points in the pre-image of $\{d\}$ with respect to P_D. If C and D are nonlinear, this is a heavy constraint, which has been pointed out in [3, Remark 1.2]. In there, they investigated a special case of Problem 1.1, where they assumed that each C_i is a closed linear subspace of $\mathcal{H}$. They remarked that the IBAP would be restrictive on general convex sets because the existence of elements that are not support points will cause the IBAP to fail or force the system to be trivial (see Proposition 2.2 below).

Let C be a nonempty, closed, and convex subset of $\mathcal{H}$. Let us recall that the collection $\operatorname{spts} C$ of support points of C is a dense subset of $\operatorname{bdry} C = \overline{C} \backslash \operatorname{int} C$ and satisfies the identity $\operatorname{spts} C = \mathrm{P}_C(\mathcal{H} \backslash C)$ [1, Theorem 7.4]. In other words,

$$c \in C \backslash \operatorname{spts} C \;\Rightarrow\; \big\{(\forall x \in \mathcal{H})\ \mathrm{P}_C x = c \Leftrightarrow x = c\big\}. \tag{4}$$

The next proposition shows that if (C, D) has the IBAP and C contains a point that is not a support point of C, then D is necessarily a singleton and the system $(\mathrm{P}_C x, \mathrm{P}_D x) = (c, d)$ becomes trivial.

Proposition 2.2. *Let C and D be nonempty, closed, and convex subsets of $\mathcal{H}$, and let $c \in C$. Assume that (C, D) satisfies the IBAP. If $c \notin \operatorname{spts} C$, then $D = \{\mathrm{P}_D c\}$.*

Proof. Let $d \in D$. Then there exists $x \in \mathcal{H}$ such that $\mathrm{P}_C x = c$ and $\mathrm{P}_D x = d$. Since $c \notin \operatorname{spts} C$, we obtain from (4) that $x = c$. □

In view of Proposition 2.2, it is necessary to determine conditions so that C coincides with $\operatorname{spts} C$. It is clear that interior points of C are not support points of C. On the other hand, $\operatorname{bdry} C$ may contain points which are not support points of C (see [1, Example 7.7]). The next proposition is an immediate consequence of [1, Corollary 7.6].

Proposition 2.3. *[1, Corollary 7.6] Let C be a nonempty, closed, and convex subset of $\mathcal{H}$ with $\operatorname{int} C = \varnothing$. If C is an affine subspace of $\mathcal{H}$ or if $\mathcal{H}$ is finite-dimensional, then $C = \operatorname{spts} C$.*

The IBAP for nonempty, closed, and convex sets C and D finds more practicability when C and D have empty interior and either C and D are affine (in particular, linear) or $\mathcal{H}$ is finite-dimensional. In both cases, Proposition 2.3 guarantees that $C \backslash \operatorname{spts} C = \varnothing$ which means that the scenario in Proposition 2.2 does not happen. We also emphasize that because Proposition 2.3 is not an if-and-only-if statement, it may still happen that $C = \operatorname{spts} C$ even if the conditions of the proposition are not satisfied.

The next proposition presents natural analogs — in the setting of closed convex cones — of some statements in [3, Corollary 2.12], which states in part that if (U, V) is a pair of linear subspaces of $\mathcal{H}$, then

$$(U, V) \text{ satisfies the IBAP} \Leftrightarrow (\forall u \in U)\ S(u, 0) \neq \varnothing \Leftrightarrow \mathrm{P}_U(V^{\perp}) = U \Leftrightarrow \mathcal{H} = U^{\perp} + V^{\perp}. \tag{5}$$

We recall that if K is a nonempty closed convex cone, then Moreau's decomposition formula [1, Theorem 6.30] provides

$$(\forall x \in \mathcal{H})\ x = \mathrm{P}_K x + \mathrm{P}_{K^{\ominus}} x. \tag{6}$$

Proposition 2.4. *Let K and M be nonempty closed convex cones of $\mathcal{H}$. Consider the following statements:*

(i) (K, M) *satisfies the IBAP.*

(ii) *For any* $p \in K$*, there exists* $x \in \mathcal{H}$ *such that* $\mathrm{P}_K x = p$ *and* $\mathrm{P}_M x = 0$.

(iii) $\mathrm{P}_K(M^{\ominus}) = K$.

(iv) $\mathcal{H} = K^{\ominus} + M^{\ominus} - \mathrm{P}_{K^{\ominus}}(M^{\ominus})$.

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv).

Proof. The implication (i) $\Rightarrow$ (ii) is clear.
(ii) $\Rightarrow$ (iii): The forward inclusion is clear. Let $p \in K$. By (ii), there exists $x \in \mathcal{H}$ such that $\mathrm{P}_K x = p$ and $\mathrm{P}_M x = 0$. It then follows from (1) and Remark 1.3 that $x \in \mathrm{N}_M(0) = M^{\ominus}$. Therefore, $p \in \mathrm{P}_K(M^{\ominus})$.
(iii) $\Rightarrow$ (ii): Let $p \in K$. By (iii), there exists $m \in M^{\ominus}$ such that $p = \mathrm{P}_K m$. Meanwhile, (6) implies $m = \mathrm{P}_M m + \mathrm{P}_{M^{\ominus}} m$. On the other hand, $m = \mathrm{P}_{M^{\ominus}} m$ since $m \in M^{\ominus}$. Therefore, $\mathrm{P}_M m = 0$.
(iii) $\Rightarrow$ (iv): Let $x \in \mathcal{H}$. Then (6) gives $x = \mathrm{P}_{K^{\ominus}} x + \mathrm{P}_K x$. By (iii), there exists $y \in M^{\ominus}$ such that $\mathrm{P}_K x = \mathrm{P}_K y$. But (6) also implies that $\mathrm{P}_K y = y - P_{K^{\ominus}} y$. Therefore, $x = \mathrm{P}_{K^{\ominus}} x + y - \mathrm{P}_{K^{\ominus}} y$ and (iv) follows. ☐

We end this section with examples showing that the reverse implications (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) in Proposition 2.4 are not always true if K and M are nonlinear convex cones. In this respect, the statements in (5) do not generalize naturally to convex cones.

Example 2.5. Take $\mathcal{H} = \mathbb{R}^2$ and consider the closed convex cones $K = [0, +\infty)^2$ and $M_1 = (-\infty, 0] \times \{0\}$. Then for any $p \in K$, $\mathrm{P}_K p = p$ and $\mathrm{P}_{M_1} p = (0, 0)$. Now, suppose that (K, M_1) satisfies the IBAP. Then there exists $x \in \mathcal{H}$ such that $\mathrm{P}_K x = (1, 1)$ and $\mathrm{P}_{M_1} x = (-1, 0)$. But $(1, 1)$ is not a support point of K. Therefore, (4) implies $x = (1, 1)$ and consequently, $\mathrm{P}_{M_1}(1, 1) = (-1, 0)$, which is absurd. Therefore, (K, M_1) has no IBAP and therefore, (ii) $\Rightarrow$ (i) is not always true.

Now, consider the convex cone $M_2 = [0, +\infty) \times \{0\}$. Then $M_2^{\ominus} = (-\infty, 0] \times \mathbb{R}$. Since $K^{\ominus} = (-\infty, 0]^2$, it follows that

$$K^{\ominus} + M_2^{\ominus} - \mathrm{P}_{K^{\ominus}}(M_2^{\ominus}) = (-\infty, 0]^2 + (-\infty, 0] \times \mathbb{R} - (-\infty, 0]^2 = \mathbb{R}^2 = \mathcal{H},$$

but $\mathrm{P}_K(M_2^{\ominus}) = \{0\} \times [0, +\infty) \neq K$. This means that (iv) $\Rightarrow$ (iii) is not always true.

Remark 2.6. A practically useful equivalence found in [3, Corollary 2.12(i)$\Leftrightarrow$(vi)] states that if (U, V) is a pair of closed linear subspaces of $\mathcal{H}$, then

$$(U, V) \text{ satisfies the IBAP } \Leftrightarrow\ U \cap V = \{0\} \text{ and } U + V \text{ is closed.} \tag{7}$$

In the setting of closed convex cones, this equivalence can fail. Indeed, with the same K and M_1 as defined in Example 2.5, we have $K \cap M_1 = \{0\}$ and $K + M_1 = \mathbb{R} \times [0, +\infty)$ is closed. However, we have already seen that the pair (K, M_1) does not satisfy the IBAP.

3 (C, D) has the IBAP if and only if $(\widehat{C}, \widehat{D})$ has the IBAP

For a nonempty subset C of $\mathcal{H}$, we define

$$\widehat{C} = \overline{\operatorname{aff} C - \operatorname{aff} C},$$

i.e., $\widehat{C}$ is the closure of the linear subspace that is parallel to the affine hull of C. In this section, we will show that if (C, D) is a pair of nonempty, closed, and convex subsets of a finite-dimensional Hilbert space $\mathcal{H}$ or if (C, D) is a pair of nonempty, closed, and affine subspaces of $\mathcal{H}$, then (C, D) has the IBAP if and only if $(\widehat{C}, \widehat{D})$ has the IBAP.

The following lemma presents some properties of the linear subspace $\widehat{C}$. In general, $\widehat{C}$ need not equal $\overline{\operatorname{span} C}$.

Lemma 3.1. *Let C be a nonempty convex subset of $\mathcal{H}$. Then the following holds:*

(i) $\overline{\operatorname{span}(C - C)} = \overline{\operatorname{cone}(C - C)} = \widehat{C}$.

(ii) $\overline{C - C} \subseteq \widehat{C} \subseteq \overline{\operatorname{span} C} = \overline{\operatorname{cone} C - \operatorname{cone} C}$.

(iii) *If C is a nonempty convex cone, then $\widehat{C} = \overline{C - C} = \overline{\operatorname{span} C}$.*

Proof. (i): Let $D = C - C$. It can be easily checked that D is a nonempty convex set which satisfies $D = -D$. Applying [1, Proposition 6.4(ii)], we get

$$\operatorname{span}(C - C) = \operatorname{cone}(C - C), \tag{8}$$

and so the first equality follows. Now let $x \in C$. We claim that $\operatorname{aff} C = x + \operatorname{cone}(C - C)$.

Let $y \in \operatorname{aff} C$. Then there exist scalars $\{\lambda_i\}_{1\leqslant i\leqslant m}$ with $\sum_{i=1}^m \lambda_i = 1$ and vectors $\{c_i\}_{1\leqslant i\leqslant m} \subseteq C$ such that $y = \sum_{i=1}^m \lambda_i c_i$. Observe that

$$y - x = \sum_{i=1}^m \lambda_i c_i - x = \sum_{i=1}^m \lambda_i c_i - \sum_{i=1}^m \lambda_i x = \sum_{i=1}^m \lambda_i (c_i - x) \in \operatorname{span}(C - C).$$

In view of (8), $y - x \in \operatorname{cone}(C - C)$ and so, $\operatorname{aff} C \subseteq x + \operatorname{cone}(C - C)$. To show the reverse inclusion, let $z \in x + \operatorname{cone}(C - C)$. Then there exist $\lambda > 0$ and $c_1, c_2 \in C$ such that

$$z = x + \lambda(c_1 - c_2) = 1 \cdot x + \lambda c_1 + (-\lambda) c_2.$$

Hence, $z \in \operatorname{aff} C$. This proves the claim.

Now, this claim and (8) imply that $\operatorname{cone}(C - C)$ is the linear subspace of $\mathcal{H}$ that is parallel to $\operatorname{aff} C$. Therefore, $\operatorname{aff} C - \operatorname{aff} C = \operatorname{cone}(C - C)$ and the second equality follows.
(ii): The first inclusion follows from the fact that $C \subseteq \operatorname{aff} C$. For the second inclusion, since $\operatorname{aff} C \subseteq \operatorname{span} C$, we get $\widehat{C} \subseteq \overline{\operatorname{span} C - \operatorname{span} C} = \overline{\operatorname{span} C}$. The last equality follows from [1, Proposition 6.4(i)].
(iii): Suppose C is a cone. Then $\operatorname{cone} C = C$. Therefore, by (ii),

$$\overline{C - C} \subseteq \widehat{C} \subseteq \overline{\operatorname{span} C} = \overline{\operatorname{cone} C - \operatorname{cone} C} = \overline{C - C}.$$

□

The next example shows that strict inclusions in Lemma 3.1(ii) may occur.

Example 3.2. Consider the convex subset $C = \{1\} \times [0,1]$ of $\mathbb{R}^2$. Then $\overline{C - C} = \{0\} \times [-1,1]$, $\widehat{C} = \{0\} \times \mathbb{R}$, and $\operatorname{span} C = \mathbb{R}^2$. We see that $\overline{C-C} \subset \widehat{C} \subset \operatorname{span} C$.

The next lemma is useful to show the sufficiency part of our main result. If C is a nonempty subset of $\mathcal{H}$, the orthogonal complement of $\widehat{C}$ will be denoted by $\widehat{C}^{\perp}$.

Lemma 3.3. *Let C be a nonempty convex subset of $\mathcal{H}$, and let $p \in C$. Then*

$$\widehat{C}^{\perp} \subseteq \mathrm{N}_C p.$$

Proof. By Lemma 3.1(i), $\overline{\operatorname{cone}(C-p)} \subseteq \overline{\operatorname{cone}(C-C)} = \widehat{C}$. Thus, $\mathrm{T}_C p \subseteq \widehat{C}$. Finally, in view of [1, Propositions 6.23 and 6.24(i)],

$$\widehat{C}^{\perp} = \big(\widehat{C}\big)^{\ominus} \subseteq \ (\mathrm{T}_C p)^{\ominus} = \mathrm{N}_C p.$$

□

We are now ready to present our main results for this section. In the next proposition, we show that if C and D are nonempty closed convex subsets of $\mathcal{H}$, then (C, D) satisfies the IBAP whenever $(\widehat{C}, \widehat{D})$ satisfies the IBAP.

Proposition 3.4. *Let C and D be nonempty, closed, and convex subsets of $\mathcal{H}$. If $(\widehat{C}, \widehat{D})$ has the IBAP then (C, D) has the IBAP.*

Proof. Suppose that $(\widehat{C}, \widehat{D})$ has the IBAP but (C, D) has no IBAP. In view of Proposition 1.2, there exists $(p, q) \in C \times D$ such that $(p + \mathrm{N}_C p) \cap (q + \mathrm{N}_D q) = \varnothing$. But by Lemma 3.3,

$$(p + \widehat{C}^{\perp}) \cap (q + \widehat{D}^{\perp}) \subseteq (p + \mathrm{N}_C p) \cap (q + \mathrm{N}_D q) = \varnothing.$$

We therefore obtain $(p + \widehat{C}^{\perp}) \cap (q + \widehat{D}^{\perp}) = \varnothing$. Since $\widehat{C}$ and $\widehat{D}$ are linear and $(\widehat{C}, \widehat{D})$ has the IBAP, it follows from (5) that $\widehat{C}^{\perp} + \widehat{D}^{\perp} = \mathcal{H}$. Hence there exist $p_C, q_C \in \widehat{C}^{\perp}$ and $p_D, q_D \in \widehat{D}^{\perp}$ such that

$$p = p_C + p_D \quad \text{ and } \quad q = q_C + q_D.$$

Take $z = q_C + p_D$. Then $z - p = q_C - p_C \in \widehat{C}^{\perp}$ and $z - q = p_D - q_D \in \widehat{D}^{\perp}$. Consequently, we have $z \in (p + \widehat{C}^{\perp}) \cap (q + \widehat{D}^{\perp})$, a contradiction. □

We now turn our interest to the converse statement of Proposition 3.4. We will need the next two lemmas.

Lemma 3.5. *Let A and B be nonempty subsets of $\mathcal{H}$ and suppose that $A \subseteq B$. Then for any $x \in A$, $\mathrm{N}_B x \subseteq \mathrm{N}_A x$.*

Proof. Let $x \in A$. Then $A - x \subseteq B - x$. Thus in view of [1, Proposition 6.24(i)], we obtain $\mathrm{N}_B x = (B - x)^{\ominus} \subseteq (A - x)^{\ominus} = \mathrm{N}_A x$. □

In the next lemma, we show that if a segment in a closed convex set C and a segment in a closed convex set D are parallel, then the pair (C, D) will not satisfy the IBAP. In Lemma

3.6, we will use the notation:

$$\text{if } \Lambda \subseteq \mathbb{R} \text{ and } e \in \mathcal{H}, \text{ then } \Lambda e = \{\lambda e \mid \lambda \in \Lambda\}$$

and the following well-known fact: if $S \subseteq \mathcal{H}$ and $a \in \mathcal{H}$, then for any $x \in a + S$,

$$\mathrm{N}_{a+S}x = ((a+S)-x)^{\ominus} = (S-(x-a))^{\ominus} = \mathrm{N}_S(x-a). \tag{9}$$

Lemma 3.6. *Let C and D be nonempty, closed, and convex subsets of $\mathcal{H}$. Let $a, b, e \in \mathcal{H}$ with $e \neq 0$ and suppose that $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$. If*

$$S = a + [\alpha_1, \alpha_2]e \subseteq C \text{ and } T = b + [\beta_1, \beta_2]e \subseteq D$$

then (C, D) has no IBAP.

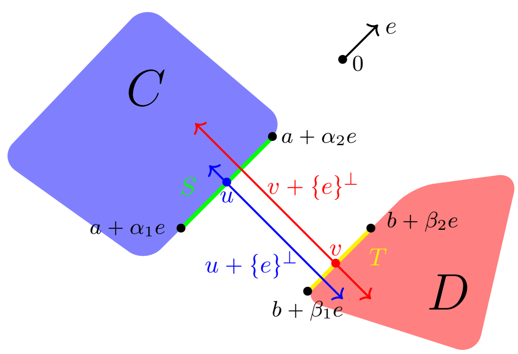

Figure 1: The segments S and T are parallel so a translate of the normal cone of an element of S is parallel to that of an element of T.

Proof. In view of Proposition 1.2, it suffices to show that there exists $u \in C$ and $v \in D$ such that

$$(u + \mathrm{N}_C u) \cap (v + \mathrm{N}_D v) = \varnothing. \tag{10}$$

Take $u = a + \dfrac{\alpha_1+\alpha_2}{2}e$, which is the midpoint of S. In view of (9), we obtain

$$\mathrm{N}_S u = \mathrm{N}_{[\alpha_1,\alpha_2]e}(u-a) = \left\{ x \in \mathcal{H} \;\middle|\; \sup_{\alpha' \in \left[-\frac{\alpha_2-\alpha_1}{2}, \frac{\alpha_2-\alpha_1}{2}\right]} \alpha'\langle e, x\rangle \leqslant 0 \right\} = \{e\}^{\perp}.$$

On the other hand, if we take $v = b + \dfrac{\beta_1+\beta_2}{2}e \in T$, we similarly get

$$\mathrm{N}_T v = \mathrm{N}_{[\beta_1,\beta_2]e}(v-b) = \{e\}^{\perp}.$$

Since $S \subseteq C$ and $T \subseteq D$, it follows from Lemma 3.5 that

$$(u + \mathrm{N}_C u) \cap (v + \mathrm{N}_D v) \subseteq (u + \mathrm{N}_S u) \cap (v + \mathrm{N}_T v) = (u + \{e\}^\perp) \cap (v + \{e\}^\perp).$$

Thus, in view of (10), it suffices to show that the affine sets $u + \{e\}^\perp$ and $v + \{e\}^\perp$ are disjoint.

Claim 1: For $p, q \in \mathcal{H}$, either $p + \{e\}^\perp = q + \{e\}^\perp$ or $\big(p + \{e\}^\perp\big) \cap \big(q + \{e\}^\perp\big) = \varnothing$.
Assume that $\big(p + \{e\}^\perp\big) \cap \big(q + \{e\}^\perp\big) \neq \varnothing$. We will show that

$$p + \{e\}^\perp = q + \{e\}^\perp.$$

Let $x \in \big(p + \{e\}^\perp\big) \cap \big(q + \{e\}^\perp\big)$. Then $x - p$ and $x - q$ are vectors in $\{e\}^\perp$. It follows that $q-p = (x-p)-(x-q) \in \{e\}^\perp$. By linearity of $\{e\}^\perp$, $p+\{e\}^\perp = p+\big((q-p)+\{e\}^\perp\big) = q+\{e\}^\perp$. This proves Claim 1.

If $u + \{e\}^\perp \neq v + \{e\}^\perp$, then by Claim 1, $\big(u + \{e\}^\perp\big) \cap \big(v + \{e\}^\perp\big) = \varnothing$, and we are done. On the other hand, if $u + \{e\}^\perp = v + \{e\}^\perp$, we consider the midpoint w of $(a + \alpha_1 e)$ and u, i.e.,

$$w = \frac{a + \alpha_1 e + u}{2} = a + \frac{3\alpha_1 + \alpha_2}{4} e \in S.$$

It is clear that $w \neq u$ because otherwise, we will obtain $\alpha_1 = \alpha_2$, a contradiction. Moreover, we can use the same argument as above to show that $\mathrm{N}_S w = \{e\}^\perp$.

Claim 2: Suppose that $u + \{e\}^\perp = v + \{e\}^\perp$. Then $(w + \{e\}^\perp) \cap (v + \{e\}^\perp) = \varnothing$.
We proceed by contradiction. Assume that $(w + \{e\}^\perp) \cap (v + \{e\}^\perp) \neq \varnothing$. Then by Claim 1, $u + \{e\}^\perp = v + \{e\}^\perp = w + \{e\}^\perp$. Therefore, $w - u \in \{e\}^\perp$. It then follows that

$$0 = \langle w - u, e \rangle = \left\langle \left(a + \frac{3\alpha_1 + \alpha_2}{4} e\right) - \left(a + \frac{\alpha_1 + \alpha_2}{2} e\right), e \right\rangle = \frac{\alpha_1 - \alpha_2}{4} \|e\|^2$$

This implies $\alpha_1 = \alpha_2$, a contradiction. This proves Claim 2.

In any case, there exists $z \in S$ (either $z = u$ or $z = w$) such that

$$(z + \mathrm{N}_S z) \cap (v + \mathrm{N}_T v) = \big(z + \{e\}^\perp\big) \cap \big(v + \{e\}^\perp\big) = \varnothing.$$

Since $S \subseteq C$ and $T \subseteq D$, we have $z \in C$ and $v \in D$. Moreover, in view of Lemma 3.5, we obtain

$$(z + \mathrm{N}_C z) \cap (v + \mathrm{N}_D v) \subseteq (z + \mathrm{N}_S z) \cap (v + \mathrm{N}_T v) = \varnothing.$$

The desired conclusion then follows from Proposition 1.2. □

The next proposition shows that for a pair (C, D) of nonempty closed convex subsets of $\mathcal{H}$ to satisfy the IBAP, it is necessary that $\widehat{C} \cap \widehat{D} = \{0\}$. This extends [3, Proposition 2.2], which implies the same result for two closed linear subspaces of $\mathcal{H}$.

Proposition 3.7. *Let C and D be nonempty closed convex subsets of $\mathcal{H}$. If $\widehat{C} \cap \widehat{D} \neq \{0\}$ then (C, D) has no IBAP.*

Proof. Let $e \in \widehat{C} \cap \widehat{D}$ with $e \neq 0$. Since $\widehat{C} \cap \widehat{D}$ is linear, we may assume without loss of generality that $\|e\| = 1$. Note that if C or D is a singleton, then $\widehat{C}$ or $\widehat{D}$ equals $\{0\}$ which implies $\widehat{C} \cap \widehat{D} = \{0\}$, a contradiction. Therefore, C and D each contains at least two elements. Since C is convex and not a singleton, $\operatorname{ri} C$ contains at least an open line segment. Fix $u \in \operatorname{ri} C$. Then there exists $\varepsilon > 0$ such that

$$\overline{B(u, 2\varepsilon)} \cap \operatorname{aff} C \subseteq C. \tag{11}$$

Now, since $\widehat{C}$ is linear, $[-\varepsilon, \varepsilon]e \subseteq \widehat{C}$. But

$$\widehat{C} = \overline{\operatorname{aff} C - \operatorname{aff} C} = \overline{\operatorname{aff} C - u} = \overline{\operatorname{aff} C} - u.$$

Therefore, $u + [-\varepsilon, \varepsilon]e \subseteq \overline{\operatorname{aff} C}$. Now, let us show that $S = u + [-\varepsilon, \varepsilon]\, e \subseteq C$. Fix $\lambda \in [-\varepsilon, \varepsilon]$. Since $u + \lambda e \in \overline{\operatorname{aff} C}$, we can find a sequence $\{y_n^\lambda\} \subseteq \operatorname{aff} C$ such that

$$y_n^\lambda \to u + \lambda e. \tag{12}$$

Therefore, we can find $N \in \mathbb{N}$ such that

$$\|u - y_n^\lambda\| \leqslant \|\lambda e\| + \varepsilon \leqslant \varepsilon + \varepsilon = 2\varepsilon,$$

whenever $n \geqslant N$. Therefore, it follows that $\{y_n^\lambda\}_{n=N}^{+\infty} \subseteq \overline{B(u, 2\varepsilon)} \cap \operatorname{aff} C$. In view of (11), $\{y_n^\lambda\}_{n=N}^{+\infty} \subseteq C$. Since C is closed, it follows from (12) that $u + \lambda e \in C$. This proves $S \subseteq C$.

Similarly, we can find $v \in \operatorname{ri} D$ and $\delta > 0$ such that $T = v + [-\delta, \delta]\, e \subseteq D$. By Lemma 3.6, (C, D) has no IBAP. □

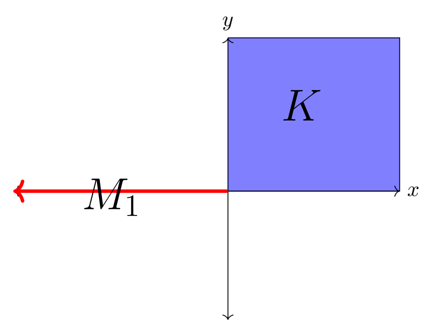

Figure 2: The convex cones $K = (0, +\infty]^2$ and $M_1 = (-\infty, 0] \times \{0\}$.

Example 3.8. Using the setting in Example 2.5, let us illustrate that the pair of convex subsets (K, M_1), as shown in Figure 2, has no IBAP in view of Proposition 3.7. We see in Figure 3 that

$$\widehat{K} \cap \widehat{M_1} = \mathbb{R} \times \{0\} \neq \{(0, 0)\}.$$

Therefore, we can conclude from Proposition 3.7 that (K, M_1) has no IBAP.

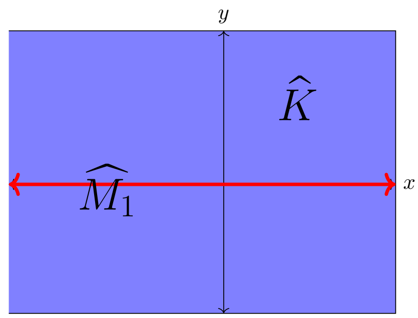

Figure 3: The linear subspaces $\widehat{K} = \mathbb{R}^2$ and $\widehat{M_1} = \mathbb{R} \times \{0\}$.

By taking the contrapositive of the previous proposition and by making use of the characterization (7) of the IBAP for linear subspaces, we immediately obtain the next result.

Proposition 3.9. *Let C and D be nonempty closed convex subsets of $\mathcal{H}$ such that $\widehat{C} + \widehat{D}$ is closed. If (C, D) has the IBAP, then $(\widehat{C}, \widehat{D})$ has the IBAP.*

The following is our main theorem.

Theorem 3.10. *Let C and D be nonempty closed convex subsets of $\mathcal{H}$. Consider the following statements:*

(i) *$(\widehat{C}, \widehat{D})$ has the IBAP.*

(ii) *(C, D) has the IBAP.*

Then (i) $\Rightarrow$ (ii). *Moreover, if we assume that $\widehat{C} + \widehat{D}$ is closed, then* (ii) $\Rightarrow$ (i).

Proof. Combine Proposition 3.4 and Proposition 3.9. ☐

The following example illustrates Theorem 3.10.

Example 3.11. Set $\mathcal{H} = \mathbb{R}^3$. Consider the two closed convex sets

$$C = \{0\} \times \{0\} \times [1, +\infty) \text{ and } D = \{(x, y, 0) \in \mathcal{H} \,|\, (x-1)^2 + (y-2)^2 \leqslant 1,\ y \leqslant 2\}.$$

Observe that $\widehat{C}$ equals the $z-$axis while $\widehat{D}$ coincides with the $xy-$plane. Clearly, $\widehat{C} \cap \widehat{D} = \{0\}$ and $\mathbb{R}^3 = \widehat{C} + \widehat{D}$. By the characterization (7) of the IBAP for linear subspaces, $(\widehat{C}, \widehat{D})$ has the IBAP. Invoking Theorem 3.10, (C, D) has the IBAP.

The sum of two closed subspaces of an infinite-dimensional Hilbert space need not be closed (see [1, Example 3.41]). Classical results which investigate conditions for closedness of a direct sum can be found in [6, Chapter 4, Section 4] and [9, Chapter 7, Section 4].

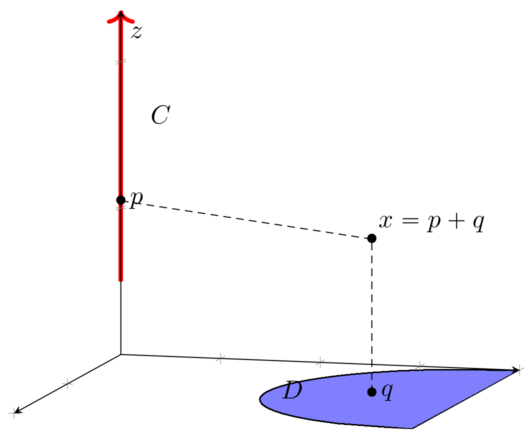

Figure 4: For all $p \in C$ and $q \in D$, the solution to the system $(\mathrm{P}_C x, \mathrm{P}_D x) = (p, q)$ is $x = p + q$.

In Theorem 3.10, we still don't know whether the closedness of $\widehat{C} + \widehat{D}$ is necessary to prove (ii)⇒(i). However, in view of Propositions 2.2 and 2.3, we are contented that in the important cases where $\mathcal{H}$ is finite-dimensional or when C and D are closed affine subspaces, we are able to drop the condition that $\widehat{C} + \widehat{D}$ must be closed (see Corollary 3.12 and Proposition 3.14).

Corollary 3.12. *Let C and D be two nonempty, closed, and convex subsets of $\mathcal{H}$. Suppose that either $\widehat{C}$ or $\widehat{D}$ is finite-dimensional. Then (C, D) has the IBAP if and only if $(\widehat{C}, \widehat{D})$ has the IBAP.*

Proof. Since $\widehat{C}$ or $\widehat{D}$ is finite-dimensional, it follows from [9, Corollary 7-4.9] that $\widehat{C} + \widehat{D}$ is closed. The desired conclusion now follows from Theorem 3.10. □

The next proposition will use the following well-known fact about projections onto translates of closed convex sets.

Lemma 3.13. *[1, Proposition 3.19] Let C be a nonempty, closed, and convex subset of $\mathcal{H}$ and let $x, y \in \mathcal{H}$. Then $\mathrm{P}_{y+C} x = y + \mathrm{P}_C(x - y)$.*

Proposition 3.14. *Let X and Y be nonempty, closed, and affine subspaces of $\mathcal{H}$. Then (X, Y) has the IBAP if and only if $(\widehat{X}, \widehat{Y})$ has the IBAP.*

Proof. The sufficiency part is true by Theorem 3.4. Now, suppose that (X, Y) has the IBAP. Let $(p, q) \in \widehat{X} \times \widehat{Y}$. We write $X = a + \widehat{X}$ and $Y = b + \widehat{Y}$ for some $a \in X$ and for some $b \in Y$. Since $\widehat{X}$ and $\widehat{Y}$ are linear, then $a + (p + \mathrm{P}_{\widehat{X}} b) \in X$ and $b + (q + \mathrm{P}_{\widehat{Y}} a) \in Y$. Because (X, Y) has the IBAP, there exists $z \in \mathcal{H}$ such that

$$\mathrm{P}_X z = a + p + \mathrm{P}_{\widehat{X}} b \text{ and } \mathrm{P}_Y z = b + q + \mathrm{P}_{\widehat{Y}} a. \tag{13}$$

Meanwhile, the linearity of $P_{\widehat{X}}$ and $P_{\widehat{Y}}$ implies

$$P_{\widehat{X}}(z-a-b) = P_{\widehat{X}}(z-a) - P_{\widehat{X}}b \text{ and } P_{\widehat{Y}}(z-a-b) = P_{\widehat{Y}}(z-b) - P_{\widehat{Y}}a. \tag{14}$$

Lemma 3.13 implies that

$$P_{\widehat{X}}(z-a) = P_{-a+X}(z-a) = -a + P_X((z-a)-(-a)) = -a + P_X z. \tag{15}$$

Therefore, in view of (14), (15), and (13), we obtain

$$P_{\widehat{X}}(z-a-b) = -a + P_X z - P_{\widehat{X}}b = p.$$

Similarly, $P_{\widehat{Y}}(z-a-b) = -b + P_Y z - P_{\widehat{Y}}a = q$. This shows that $(\widehat{X}, \widehat{Y})$ has the IBAP. □

4 Approximate Solutions to $(P_C x, P_D x) = (p, q)$

It may happen that the system (2) in Problem 1.1 does not admit a solution but admits approximate solutions. In this section, we investigate the approximate solution to the system and provide a characterization for the existence of approximate solutions.

Definition 4.1. Let $(C_i)_{i\in\mathbb{I}}$ be a finite family of nonempty closed convex subsets of $\mathcal{H}$ and let $(p_i)_{i\in\mathbb{I}} \in \times_{i\in\mathbb{I}} C_i$. A sequence $\{x_n\}$ is said to be a *sequence of approximate solutions* to $(\forall i \in \mathbb{I})\ P_{C_i}x = p_i$ if $\sum_{i\in\mathbb{I}}\|P_{C_i}x_n - p_i\| \to 0$ as $n \to +\infty$.

In the setting of closed linear subspaces of $\mathcal{H}$ in [3], the authors showed that approximate solutions always exist if and only if the linear subspaces are linearly independent, that is,

$$\left(\forall (u_i)_{i\in\mathbb{I}} \in \bigtimes_{i\in\mathbb{I}} U_i\right) \ \sum_{i\in\mathbb{I}} u_i = 0 \ \Rightarrow \ (\forall i \in \mathbb{I})\ u_i = 0.$$

Note that (U_1, U_2) are linearly independent if and only if $U_1 \cap U_2 = \varnothing$.

Proposition 4.2. *[3, Proposition 2.5] Let $(U_i)_{i\in\mathbb{I}}$ be a finite family of nonempty closed linear subspaces of $\mathcal{H}$. Then the following are equivalent.*

(i) *$(U_i)_{i\in\mathbb{I}}$ are linearly independent.*

(ii) $\big(\forall (u_i)_{i\in\mathbb{I}} \in \times_{i\in\mathbb{I}} U_i\big)(\forall \varepsilon > 0)(\exists x \in \mathcal{H})\ \sum_{i\in\mathbb{I}}\|P_{U_i}x - u_i\| < \varepsilon$.

The next proposition shows that this can be extended to affine subspaces of $\mathcal{H}$.

Proposition 4.3. *Let $(X_i)_{i\in\mathbb{I}}$ be a finite family of nonempty, closed, and affine subspaces of $\mathcal{H}$. Then the following are equivalent:*

(i) $\big(\widehat{X_i}\big)_{i\in\mathbb{I}}$ *are linearly independent.*

(ii) *Given* $(p_i)_{i\in\mathbb{I}} \in \times_{i\in\mathbb{I}} X_i$ *and* $\varepsilon > 0$*, there exists* $x \in \mathcal{H}$ *such that*

$$\sum_{i\in\mathbb{I}} \|\mathrm{P}_{X_i}x - p_i\| < \varepsilon.$$

Proof. For each $i \in \mathbb{I}$, we may write $X_i = a_i + \widehat{X_i}$, for some $a_i \in X_i$.
(i)⇒(ii): Let $(p_i)_{i\in\mathbb{I}} \in \times_{i\in\mathbb{I}} X_i$ and let $\varepsilon > 0$. For each $i \in \mathbb{I}$, the linearity of $\widehat{X_i}$ yields

$$(p_i - a_i) - \mathrm{P}_{\widehat{X_i}}\Big(\sum_{j\neq i} a_j\Big) \in \widehat{X_i}.$$

In view of Proposition 4.2, (i) implies that we can find $x \in \mathcal{H}$ such that

$$\sum_{i\in\mathbb{I}} \left\| \mathrm{P}_{\widehat{X_i}}x - \left[(p_i - a_i) - \mathrm{P}_{\widehat{X_i}}\Big(\sum_{j\neq i} a_j\Big)\right]\right\| < \varepsilon. \tag{16}$$

Set $z = x + \sum_{j\in\mathbb{I}} a_j$. For each $i \in \mathbb{I}$, Lemma 3.13 and the linearlity of $\mathrm{P}_{\widehat{X_i}}$ imply

$$\mathrm{P}_{X_i}z - p_i = \mathrm{P}_{\widehat{X_i}}\Big(x + \sum_{j\neq i} a_j\Big) - (p_i - a_i) = \mathrm{P}_{\widehat{X_i}}x + \mathrm{P}_{\widehat{X_i}}\Big(\sum_{j\neq i} a_j\Big) - (p_i - a_i).$$

In turn, (16) gives $\sum_{i\in\mathbb{I}} \|\mathrm{P}_{X_i}z - p_i\| < \varepsilon$. Thus, (ii) holds.
(ii)⇒(i): Our strategy is to use Proposition 4.2 with $(\forall i \in \mathbb{I})\ U_i = \widehat{X_i}$. Let $(z_i)_{i\in\mathbb{I}} \in \times_{i\in\mathbb{I}} \widehat{X_i}$ and let $\varepsilon > 0$. For each $i \in \mathbb{I}$, $z_i + \mathrm{P}_{\widehat{X_i}}\big(\sum_{j\neq i} a_j\big) \in \widehat{X_i}$, and so, $a_i + z_i + \mathrm{P}_{\widehat{X_i}}\big(\sum_{j\neq i} a_j\big) \in X_i$. By (ii), there exists $y \in \mathcal{H}$ such that

$$\sum_{i\in\mathbb{I}} \left\| \mathrm{P}_{X_i}y - \left[a_i + z_i + \mathrm{P}_{\widehat{X_i}}\Big(\sum_{j\neq i} a_j\Big)\right]\right\| < \varepsilon.$$

In view of Lemma 3.13 and the linearity of each $\mathrm{P}_{\widehat{X_i}}$, this is equivalent to

$$\sum_{i\in\mathbb{I}} \left\| \mathrm{P}_{\widehat{X_i}}\Big(y - \sum_{j\in\mathbb{I}} a_j\Big) - z_i\right\| < \varepsilon.$$

Therefore, Proposition 4.2 implies that $\big(\widehat{X_i}\big)_{i\in\mathbb{I}}$ are linearly independent. □

Now we turn our attention to the behavior of approximate solutions in relation to the solvability of the system in (2). We first recall the notion of nonexpansive operators.

Definition 4.4. An operator $T : \mathcal{H} \to \mathcal{H}$ is said to be nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for every $x, y \in \mathcal{H}$.

It is well-known that if C is nonempty, closed, and convex set, then P_C and $\mathrm{Id} - \mathrm{P}_C$ are nonexpansive. Here, Id is the identity map on $\mathcal{H}$.

The first result shows that if a sequence of approximate solutions converges, then its limit must be a solution to the system.

Proposition 4.5. *Let $(C_i)_{i\in\mathbb{I}}$ be a finite family of nonempty, closed, and convex subsets of $\mathcal{H}$, let $(p_i)_{i\in\mathbb{I}} \in \times_{i\in\mathbb{I}} C_i$ and let $z \in \mathcal{H}$. Suppose that there exists a sequence $\{x_n\}$ in $\mathcal{H}$ such that $\sum_{i\in\mathbb{I}} \|\mathrm{P}_{C_i}x_n - p_i\| \to 0$ and that $x_n \to z$. Then $(\forall i \in \mathbb{I})$ $\mathrm{P}_{C_i}z = p_i$.*

Proof. For each $i \in \mathbb{I}$ and for each $n \in \mathbb{N}$, the nonexpansiveness of P_{C_i} implies

$$\|\mathrm{P}_{C_i}z - p_i\| \leqslant \|\mathrm{P}_{C_i}z - \mathrm{P}_{C_i}x_n\| + \|\mathrm{P}_{C_i}x_n - p_i\| \leqslant \|z - x_n\| + \sum_{j\in\mathbb{I}} \|\mathrm{P}_{C_j}x_n - p_j\|.$$

Therefore, the assumptions yield the desired conclusion. □

Remark 4.6. If the system in (2) has both a solution and a sequence of approximate solutions, then the sequence of approximate solutions may not converge. Indeed, if x and y are distinct solutions of the system, then a trivial example of a sequence of approximate solutions that does not converge is $\{x, y, x, y \ldots\}$.

Now suppose that the system in (2) has no solution but we can find a sequence of approximate solutions to the system. Our next result shows that the norm of this sequence of approximate solutions will eventually blow up. We will first need the following lemma which is an immediate consequence of Browder's Demiclosedness Principle [1, Theorem 4.27].

Lemma 4.7. *For each $i \in \mathbb{I}$, let $T_i : \mathcal{H} \to \mathcal{H}$ and suppose that $(\mathrm{Id} - T_i)$ is a nonexpansive operator. Let $\{y_i\}_{i\in\mathbb{I}} \subseteq \mathcal{H}$ and $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}$. Suppose that*

$$(\nexists x \in \mathcal{H})\,(\forall i \in \mathbb{I})\;\; T_i x = y_i \quad \text{but} \quad \sum_{i\in\mathbb{I}} \|T_i x_n - y_i\| \to 0.$$

Then $\|x_n\| \to +\infty$.

Proof. We proceed by contradiction. Suppose otherwise. Then $\{x_n\}$ has a bounded subsequence from which we can further extract a weakly convergent subsequence $\{x_{k_n}\}$ [1, Lemma 2.45]. Let $z \in \mathcal{H}$ be the weak limit of x_{k_n}.

Meanwhile, for each $i \in \mathbb{I}$, we see that

$$\|x_{k_n} - (\mathrm{Id} - T_i)x_{k_n} - y_i\| = \|T_i x_{k_n} - y_i\| \to 0.$$

This means $x_{k_n} - (\mathrm{Id} - T_i)x_{k_n} \to y_i$, for each $i \in \mathbb{I}$. In view of the Demiclosedness Principle, it follows that for each $i \in \mathbb{I}$, $z - (\mathrm{Id} - T_i)z = y_i$, or equivalently, $T_i z = y_i$. This is a contradiction. □

Letting $T_i = \mathrm{P}_{C_i}$, for each $i \in \mathbb{I}$, and noting that each $\mathrm{Id} - \mathrm{P}_{C_i}$ is nonexpansive, we immediately obtain the following result.

Proposition 4.8. *Let $(C_i)_{i\in\mathbb{I}}$ be a finite family of nonempty, closed, and convex subsets of $\mathcal{H}$. Given $(p_i)_{i\in\mathbb{I}} \in \bigtimes_{i\in\mathbb{I}} C_i$, suppose that the system $(\forall i \in \mathbb{I})\ \mathrm{P}_{C_i}x = p_i$ admits no solution. If there exists a sequence of approximate solutions $\{x_n\}$ to the system, then $\|x_n\| \to +\infty$.*

In [7, Proposition 2.1], the authors proved a special case of Proposition 4.8, and posed the problem of relating the rate of convergence of $\sum_{i\in\mathbb{I}} \|\mathrm{P}_{C_i}x_n - p_i\|$ to 0 with the rate of growth of $\|x_n\|$.

5 Application to a Recovery Problem

In this section, we investigate the problem of finding an approximation $x \in \ell^2(\mathbb{N})$ to the original signal $z \in \ell^2(\mathbb{N})$, such that some of the components of x are nonnegative and that its projection onto a fixed closed convex set is known. This problem is related to [4, Equation 1.1].

Problem 5.1. Set $\mathcal{H} = \ell^2(\mathbb{N})$. Let D be a nonempty closed convex subset of $\mathcal{H}$ and let $u \in \mathcal{H}$. Suppose I and J form a partition of $\mathbb{N}$. The problem is to find $x = \{x_n\} \in \mathcal{H}$ such that

$$\min_{x\in\mathcal{H}} \|x - z\| \text{ subject to } (\forall i \in I)\ x_i \geqslant 0 \text{ and } \mathrm{P}_D x = p. \tag{17}$$

Consider the closed convex cone $C = \{\{x_n\} \in \mathcal{H} \mid (\forall i \in I)\ x_i \leqslant 0 \text{ and } (\forall j \in J)\ x_j = 0\}$. Observe that

$$C^{\ominus} = \left\{ \{u_n\} \in \mathcal{H} \,\middle|\, \sup_{x_i \leqslant 0} \sum_{i\in I} x_i u_i \leqslant 0 \right\} = \big\{\{u_n\} \in \mathcal{H} \mid (\forall i \in I)\ u_i \geqslant 0\big\}.$$

Therefore, if $x \in \mathcal{H}$ satisfies the first constraint in (17), then $x \in C^{\ominus}$. By (6), $\mathrm{P}_C x = 0$. Thus, Problem 5.1 is feasible if and only if there exists $x \in \mathcal{H}$ such that

$$\mathrm{P}_C x = 0 \text{ and } \mathrm{P}_D x = p. \tag{18}$$

Propositions 1.2 and 2.4(ii)⇔(iii) give sufficient conditions for the existence of solutions to (18): either $C^{\ominus} \cap (p + \mathrm{N}_D p) \neq \varnothing$ or $D = \mathrm{P}_D(C^{\ominus})$. Now suppose that the solution set $S(0, p)$ of (18) is nonempty. From Proposition 1.2, $S(0, p)$ is an intersection of closed convex sets, so it must also be a closed convex set. We therefore conclude that the solution to Problem 5.1 is $x = \mathrm{P}_{S(0,p)} z$.

To find $x \in \mathcal{H}$ numerically, we may consider the following instantiation of the Dykstra's algorithm [1, Theorem 30.7] applied to $C^{\ominus}$ and $p + \mathrm{N}_D p$:

Data: Set $C_1 := C^{\ominus}$ and $C_2 := p + \mathrm{N}_D p$

Result: Construct a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to \mathrm{P}_{S(0,p)} z$

initialize $x_0 := z$;

set $q_{-1} = q_0 = 0$;

for $n = 1, 2, ...$ **do**

  set $i = n - 2\lfloor (n-1)/2 \rfloor$;

  $x_n = \mathrm{P}_{C_i}(x_{n-1} + q_{n-2})$;

  $q_n = x_{n-1} - x_n + q_{n-2}$;

end

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