MATIMYÁS MATEMATIKA
ISSN 0115-6926

Journal of the Mathematical Society of the Philippines
Vol. 46 No. 2 (2023) pp. 19-35

S-Iwasawa decomposition for $O(p, q)$

Edgar N. Reyes
Southeastern Louisiana University
Hammond, Louisiana 70402
edgar.reyes@selu.edu

Abstract

Let $E_k \in M_{p+q}(\mathbb{R})$ be the matrix where the (k, k)-entry is 1, and all other entries are zero. Let $S = I_{p+q} - 2E_{p+1}$. We consider an involution of $O(p, q)$ defined by inner-conjugation or similarity by S. We define Lie subgroups $\mathcal{K}, \mathcal{A}, \mathcal{N}$ of $O(p, q)$ similar to the Iwasawa decomposition. We state a sufficient condition that if satisfied by any $G \in O(p, q)$ then $G \in \mathcal{KAN}$. Also, we consider inner-conjugation by other Householder-type matrices in $O(p, q)$.

Keywords: Indefinite orthogonal group, Iwasawa decomposition
2020 MSC: 15A23, 22E15

1 Introduction

We present an Iwasawa-like decomposition for $O(p, q)$, $p, q \geq 2$. This article is an extension of an Iwasawa-like decomposition for $O(1, n)$ [6]. The proofs for $O(1, n)$ and $O(p, q)$ are almost identical surprisingly. Prior to these Iwasawa-like decompositions, we studied polar-like decompositions for $O(p, q)$, and several information have been laid as background material for this paper, and we refer the reader to [7].

Let $J_{p,q} = I_p \oplus (-I_q) \in M_{p+q}(\mathbb{R})$ be a diagonal square real matrix where $p, q \geq 2$. The first p diagonal entries of $J_{p,q}$ are ones, and the next q diagonal entries are minus ones. Let $(x, y)_{p,q} = x^T J_{p,q} y$ define an indefinite scalar product on $\mathbb{R}^{p+q}$ where $x, y \in \mathbb{R}^{p+q}$. When $G \in M_{p+q}(\mathbb{R})$, we conventionally say G is orthogonal with respect to $(\cdot, \cdot)_{p,q}$ if $(Gx, Gy)_{p,q} = (x, y)_{p,q}$ for all $x, y \in \mathbb{R}^{p+q}$. The indefinite orthogonal group $O(p, q)$ consists of all matrices G that are orthogonal with respect to $(\cdot, \cdot)_{p,q}$. We easily find

$$O(p, q) = \{G \in GL_{p+q}(\mathbb{R}) : G^T J_{p,q} G = J_{p,q}\}$$

where G^T denotes the transpose of G. The Lie algebra of $O(p, q)$ is given by

$$\mathfrak{so}(p, q) = \{X \in M_{p+q}(\mathbb{R}) : X^T J_{p,q} + J_{p,q} X = 0\}.$$

Let $E_k \in M_{p+q}(\mathbb{R})$ be the matrix where the (k, k)-entry is 1, and the other entries are zero. Let

$$S = I_{p+q} - 2E_{p+1}. \tag{1}$$

Clearly, $S \in M_{p+q}(\mathbb{R})$, $S^{-1} = S$, and it is known that S is a $J_{p,q}$-Householder matrix [5] [7]. Consider an involution ρ of $O(p,q)$ defined by inner-conjugation or similarity by S where $\rho(A) = SAS^{-1}$, $A \in O(p,q)$. We omit in the notation that ρ depends on S for brevity. The differential $d\rho$ of ρ is a Lie algebra involution satisfying $d\rho(X) = SXS^{-1}$, $X \in \mathfrak{so}(p,q)$.

Consider a scalar product on $\mathfrak{so}(p,q)$, namely,

$$(X,Y)_S = -Tr(XSYS^{-1}) \tag{2}$$

$X, Y \in \mathfrak{so}(p,q)$. The bilinear form (2) is indefinite, in general, but similar to the positive-definite bilinear form induced by the Killing form and a Cartan involution in [3, p. 185]. Cartan involutions are ubiquitous in decompositions of simple real Lie algebras and their associated Lie groups. In this paper, even though $d\rho$ is not a Cartan involution in general, we pursue Iwasawa-like decompositions induced by $d\rho$ on $\mathfrak{so}(p,q)$ and $O(p,q)$. There is a wide literature on Iwasawa decompositions of (including infinite-dimensional) Lie groups as early as 1949, and some can be found for instance in [1] [3], [4], [8] and in their listed references.

We easily verify $(X,Y)_S = (Y,X)_S$ and $(d\rho(X), d\rho(Y))_S = (X,Y)_S$. We know $O(p,q)$ acts naturally as a group of linear transformations on $\mathfrak{so}(p,q)$ in the following way. Let $\mathrm{Ad} : O(p,q) \to GL(\mathfrak{so}(p,q))$ be the adjoint representation, i.e., for each $G \in O(p,q)$, $\mathrm{Ad}(G)$ is a linear transformation of $\mathfrak{so}(p,q)$ satisfying $\mathrm{Ad}(G)X = GXG^{-1} \in \mathfrak{so}(p,q)$, $X \in \mathfrak{so}(p,q)$. We say G is S-orthogonal if $(\mathrm{Ad}(G)X, \mathrm{Ad}(G)Y)_S = (X,Y)_S$ for all $X, Y \in \mathfrak{so}(p,q)$. Likewise, G is S-symmetric if $(\mathrm{Ad}(G)X, Y)_S = (X, \mathrm{Ad}(G)Y)_S$ for all $X, Y \in \mathfrak{so}(p,q)$. Let $G^{[S]} = \rho(G^{-1})$. We state $G^{[S]}$ is the S-adjoint of G since

$$(\mathrm{Ad}(G)X, Y)_S = (X, \mathrm{Ad}(G^{[S]})Y)_S, \ \forall X, Y \in \mathfrak{so}(p,q).$$

Definition 1. *The eigenspaces of $d\rho$, and the subgroup of fixed points of ρ are denoted and defined by the following.*

1. $\mathfrak{p} = \{X \in \mathfrak{so}(p,q) : SXS^{-1} = -X\}$
2. $\mathfrak{K} = \{X \in \mathfrak{so}(p,q) : SXS^{-1} = X\}$
3. $\mathcal{K} = \{K \in O(p,q) : SKS^{-1} = K\}$

Clearly, $\mathfrak{p}$ and $\mathfrak{K}$ are S-orthogonal subspaces, i.e., $(X,Y)_S = 0$ for all $X \in \mathfrak{p}$, $Y \in \mathfrak{K}$. We find $K \in \mathcal{K}$ iff $KK^{[S]} = I$ iff K is S-orthogonal. Also, if $X \in \mathfrak{p}$, we easily verify e^X is S-symmetric.

In Section 2, we let $\mathfrak{h}_\mathfrak{p}$ be a maximal subspace of $\mathfrak{p}$ such that $\mathfrak{h}_\mathfrak{p}$ is an abelian Lie subalgebra of $\mathfrak{so}(p,q)$. We find $\mathfrak{h}_\mathfrak{p}$ is 1-dimensional and spanned by some nonzero $V_1 \in \mathfrak{p}$. Let $\mathfrak{n}$ be the 1-eigenspace of $ad(V_1)$, i.e. , $[V_1, Y] = Y$, $\forall Y \in \mathfrak{n}$. We show $\mathfrak{n}$ is an abelian Lie subalgebra. In Section 3, we let $\mathcal{A}$ and $\mathcal{N}$ be the connected abelian Lie subgroups of $O(p,q)$ with Lie algebras $\mathfrak{h}_\mathfrak{p}$ and $\mathfrak{n}$, respectively. Since $\mathfrak{h}_\mathfrak{p}$ and $\mathfrak{n}$ are abelian, and we find the restriction of the exponential mapping to $\mathfrak{h}_\mathfrak{p}$ and $\mathfrak{n}$ are injective, we obtain $\mathcal{A} = \exp \mathfrak{h}_\mathfrak{p}$ and $\mathcal{N} = \exp \mathfrak{n}$.

If $G \in \mathcal{KAN}$, we say G has an S-Iwasawa decomposition. We state a sufficient condition that if satisfied by G, namely, that the quantity in (6) be nonzero, then $G \in \mathcal{KAN}$. Details and proofs of lemmas and various claims in Section 2 and Section 3 are given in Appendix A and Appendix B, respectively.

The following lemma describes the matrices in the Lie algebra $\mathfrak{so}(p,q)$ [7].

Lemma 1. *Let $X \in M_{p+q}(\mathbb{R})$. Then $X \in \mathfrak{so}(p,q)$ iff there exist $X_1 \in \mathfrak{so}(p)$, $X_3 \in \mathfrak{so}(q)$, and $X_2 \in \mathbb{R}^{p\times q}$ satisfying*

$$X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} \in \mathfrak{so}(p,q).$$

We recall the matrices in $\mathfrak{p}$, i.e., the (-1)-eigenspace of $d\rho$, from [7].

Lemma 2. *Let $X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} \in \mathfrak{so}(p,q)$ be given by Lemma 1. Then $X \in \mathfrak{p}$ iff $X_1 = 0$, and there exist $x \in \mathbb{R}^p$ and $v \in \mathbb{R}^{q-1}$ such that*

$$X = \left[\begin{array}{c|cc} 0 & x & 0 \\ \hline x^T & 0 & -v^T \\ 0 & v & 0 \end{array}\right] \in \mathfrak{p}. \tag{3}$$

Let $A_{i,j}$ denote the (i,j)-entry of $A \in O(p,q)$. We directly verify $A \in \mathcal{K}$ iff $SAS^{-1} = A$ iff $A_{i,j}S_{j,j} = A_{i,j}S_{i,i}$. The next lemma follows directly.

Lemma 3. *Let $A \in O(p,q)$. The following statements are equivalent.*

(a) $A \in \mathcal{K}$

(b) $A_{i,p+1} = 0$ *for all* $i \neq p+1$.

2 S-Iwasawa decomposition of $\mathfrak{so}(p,q)$

For $1 \leq k \leq p$, let $e_k \in \mathbb{R}^p$ be the standard unit vector where the kth entry is 1, and the other entries are zero. Throughout this article, we let

$$V_1 = \left[\begin{array}{c|cc} 0 & e_1 & 0 \\ \hline e_1^T & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \in \mathfrak{p}. \tag{4}$$

If $X \in \mathfrak{so}(p,q)$, let $ad(V_1)(X) = [V_1, X] = V_1X - XV_1$ denote the Lie bracket. For $\lambda \in \mathbb{R}$, denote the λ-eigenspace of $ad(V_1)$ by

$$\mathfrak{so}(p,q)_\lambda = \{X \in \mathfrak{so}(p,q) : ad(V_1)(X) = \lambda X\}.$$

Lemma 4. *The eigenvalues of $ad(V_1)$ are precisely $-1, 0, 1$. Moreover, we have a direct sum of eigenspaces*

$$\mathfrak{so}(p,q) = \mathfrak{so}(p,q)_{-1} \oplus \mathfrak{so}(p,q)_0 \oplus \mathfrak{so}(p,q)_1.$$

The proof of Lemma 4 is computational, and provided in Appendix A.

Definition 2. *We define the following subspaces.*

1. $\mathfrak{h}_\mathfrak{p} = \{tV_1 : t \in \mathbb{R}\}$

2. $\mathfrak{n} = \mathfrak{so}(p,q)_1$

3. $\mathfrak{s} = \mathfrak{h}_\mathfrak{p} \oplus \mathfrak{n}$, *a direct sum of vector spaces*

4. $\mathfrak{m} = \{X \in \mathfrak{K} : [X,H] = 0,\ \forall H \in \mathfrak{h}_\mathfrak{p}\}$, *the centralizer of* $\mathfrak{h}_\mathfrak{p}$ *in* $\mathfrak{K}$.

5. *A subspace* V *of* $\mathfrak{p}$ *is called abelian if* $[X,Y] = 0$, $\forall X, Y \in V$.

Corollary 5. *The following are direct sums of subspaces.*

1. $\mathfrak{so}(p,q) = \mathfrak{K} \oplus \mathfrak{h}_\mathfrak{p} \oplus \mathfrak{n}$, *the* S*-Iwasawa decomposition of* $\mathfrak{so}(p,q)$

2. $\mathfrak{h}_\mathfrak{p}$ *is a maximal abelian subspace of* $\mathfrak{p}$

3. $\mathfrak{so}(p,q)_0 = \mathfrak{m} \oplus \mathfrak{h}_\mathfrak{p}$

Proof Since $\mathfrak{so}(p,q) = \mathfrak{K} \oplus \mathfrak{p}$ is a direct sum of subspaces, Statement 1 of the corollary follows from Lemma 4. Statement 2 is a direct consequence of [7, Lemma 2.3]. Finally, Statement 2 implies Statement 3.

□

Moreover, we have additional properties.

Lemma 6. $\mathfrak{n}$ *is an abelian Lie algebra, and* $\mathfrak{s}$ *is a solvable Lie algebra.*

The proof of the above lemma is also in Appendix A.

3 S-Iwasawa decomposition of $O(p,q)$

Let $\mathcal{A}$, $\mathcal{N}$, and $\mathcal{S}$ be the connected Lie subgroups of $O(p,q)$ with Lie algebras $\mathfrak{h}_\mathfrak{p}$, $\mathfrak{n}$, and $\mathfrak{s}$, respectively. In Appendix B, we show $\mathcal{S} = \mathcal{A}\mathcal{N}$. Using (4) and $t \in \mathbb{R}$, we obtain

$$e^{tV_1} = \left[\begin{array}{cc|cc} \cosh(t) & 0 & \sinh(t) & 0 \\ 0 & I_{p-1} & 0 & 0 \\ \hline \sinh(t) & 0 & \cosh(t) & 0 \\ 0 & 0 & 0 & I_{q-1} \end{array}\right]. \tag{5}$$

We need another definition.

Definition 3. *For* $t \in \mathbb{R}$ *and* $t \neq 0$, *define the following functions that are differentiable on* $\mathbb{R}$.

1. $\alpha_1(t) = \frac{e^t - 1}{t}$, *and* $\alpha_1(0) = 1$

2. $\alpha_2(t) = \frac{1 - e^{-t}}{t}$, *and* $\alpha_2(0) = 1$

3. $\alpha(t) = \frac{\cosh(t) - 1}{t^2}$, *and* $\alpha(0) = \frac{1}{2}$

Let $G \in O(p,q)$. Applying the S-polar decomposition to G^{-1} we obtain $G^{-1} = e^X K_0$ for some $K_0 \in \mathcal{K}$ and $X \in \mathfrak{p}$, see [7, Theorem 2.16]. Let X be given by (3) where $x = (x_1, \ldots, x_p)^T \in \mathbb{R}^p$, and $v \in \mathbb{R}^{q-1}$. Let $\beta = x^T x - v^T v$, and let

$$\xi = \frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}} x_1 - \cosh(\sqrt{\beta}). \tag{6}$$

The condition that guarantees that G has an S-Iwasawa decomposition is that $\xi \neq 0$. In such a case, we let

1. $t_0 = \ln(|\xi|)$
2. $x' = (x_2, \ldots, x_p)^T \in \mathbb{R}^{p-1}$
3. $a = \frac{1}{\xi \alpha_2(t)} \frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}} x' \in \mathbb{R}^{p-1}$
4. $b = -\frac{1}{\xi \alpha_2(t)} \frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}} v \in \mathbb{R}^{q-1}$.

In Appendix B, we show

(1) $Y = \left[\begin{array}{cc|cc} 0 & a^T & 0 & b^T \\ -a & 0 & a & 0 \\ \hline 0 & a^T & 0 & b^T \\ b & 0 & -b & 0 \end{array}\right] \in \mathfrak{n} = \mathfrak{so}(p,q)_1$

(2) $Z \equiv e^{t_0 V_1} e^{\alpha_2(t_0) Y} e^X \in \mathcal{K}$

(3) $(k_0^{-1} Z^{-1}, e^{t_0 V_1}, e^{\alpha_2(t_0) Y}) \in \mathcal{K} \times \mathcal{A} \times \mathcal{N}$

(4) G is the product of the entries in (3), namely,

$$G = (k_0^{-1} Z^{-1})(e^{t_0 V_1})(e^{\alpha_2(t_0) Y}) \in \mathcal{KAN}.$$

The above factorization is called a S-Iwasawa decomposition of G.

Appendix A

Let $e_k \in \mathbb{R}^p$ be the standard unit vector where the kth entry is 1, and the other entries are zero. Following Lemma 1, let

$$X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} \in \mathfrak{so}(p,q). \tag{7}$$

Let $[e_1\ 0] \in \mathbb{R}^{p \times q}$ denote a matrix where the first column is e_1, and the other column vectors are zero vectors. Clearly, $[e_1\ 0]^T = \begin{bmatrix} e_1^T \\ 0 \end{bmatrix} \in \mathbb{R}^{q \times p}$. We recall $V_1 \in \mathfrak{p}$ in (4). Then we obtain the following two identities.

1. $V_1 X = \left[\begin{array}{c|c} [e_1\ 0] X_2^T & [e_1\ 0] X_3 \\ \hline \begin{bmatrix} e_1^T \\ 0 \end{bmatrix} X_1 & \begin{bmatrix} e_1^T \\ 0 \end{bmatrix} X_2 \end{array}\right]$

2. $XV_1 = \left[\begin{array}{c|c} X_2\begin{bmatrix} e_1^T \\ 0 \end{bmatrix} & X_1[e_1\ 0] \\ \hline X_3\begin{bmatrix} e_1^T \\ 0 \end{bmatrix} & X_2^T[e_1\ 0] \end{array}\right]$

Combining the above two identities, we obtain the following lemma.

Lemma 7. *Let V_1 and $X \in \mathfrak{so}(p,q)$ be given by (4) and (7), respectively. Then*

$$[V_1, X] = \left[\begin{array}{c|c} [e_1\ 0]X_2^T - X_2\begin{bmatrix} e_1^T \\ 0 \end{bmatrix} & [e_1\ 0]X_3 - X_1[e_1\ 0] \\ \hline \begin{bmatrix} e_1^T \\ 0 \end{bmatrix} X_1 - X_3\begin{bmatrix} e_1^T \\ 0 \end{bmatrix} & \begin{bmatrix} e_1^T \\ 0 \end{bmatrix} X_2 - X_2^T[e_1\ 0] \end{array}\right] \tag{8}$$

Let $(X_k)_{i,j}$ denote the (i,j)-entry of matrix X_k. The block entries of $[V_1, X]$ in (8) satisfy the next three identities.

1. $[e_1\ 0]X_2^T - X_2\begin{bmatrix} e_1^T \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & (X_2)_{21} & \cdots & (X_2)_{p1} \\ -(X_2)_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(X_2)_{p1} & 0 & \cdots & 0 \end{bmatrix}$

2. $\begin{bmatrix} e_1^T \\ 0 \end{bmatrix} X_2 - X_2^T[e_1\ 0] = \begin{bmatrix} 0 & (X_2)_{12} & \cdots & (X_2)_{1q} \\ -(X_2)_{12} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(X_2)_{1q} & 0 & \cdots & 0 \end{bmatrix}$

3. $[e_1\ 0]X_3 - X_1[e_1\ 0] = \begin{bmatrix} 0 & (X_3)_{12} & \cdots & (X_3)_{1q} \\ -(X_1)_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(X_1)_{p1} & 0 & \cdots & 0 \end{bmatrix}$

Applying the above three identities, we obtain the 0-eigenspace of $ad(V_1)$.

Lemma 8. *Let $X \in \mathfrak{so}(p,q)$ be given by (7). Then $[V_1, X] = 0$ iff each identity below holds.*

1. $X_2 = \begin{bmatrix} (X_2)_{11} & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{bmatrix}$

2. $X_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{bmatrix}$, *and* $X_3 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{bmatrix}$

where $*$ denotes arbitrary and not necessarily the same entries, and X_1 and X_3 are real skew-symmetric matrices. That is, $[V_1, X] = 0$ iff

$$X = \left[\begin{array}{c|c} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{bmatrix} & \begin{bmatrix} (X_2)_{11} & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{bmatrix} \\ \hline \begin{bmatrix} (X_2)_{11} & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{bmatrix}^T & \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{bmatrix} \end{array}\right].$$

Next, the 1-eigenspace of $ad(V_1)$ is described below

Lemma 9. *Let $X \in \mathfrak{so}(p,q)$ be given by (7). Then $[V_1, X] = X$ iff each identity below holds.*

1. $X_2 = \begin{bmatrix} 0 & (X_2)_{12} & \cdots & (X_2)_{1q} \\ (X_2)_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (X_2)_{p1} & 0 & \cdots & 0 \end{bmatrix}$
2. $X_1 = \begin{bmatrix} 0 & (X_2)_{21} & \cdots & (X_2)_{p1} \\ -(X_2)_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(X_2)_{p1} & 0 & \cdots & 0 \end{bmatrix}$, $X_3 = \begin{bmatrix} 0 & (X_2)_{12} & \cdots & (X_2)_{1q} \\ -(X_2)_{12} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(X_2)_{1q} & 0 & \cdots & 0 \end{bmatrix}$

That is, $[V_1, X] = X$ iff

$$X = \left[\begin{array}{c|c} \begin{bmatrix} 0 & (X_2)_{21} & \cdots & (X_2)_{p1} \\ -(X_2)_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(X_2)_{p1} & 0 & \cdots & 0 \end{bmatrix} & \begin{bmatrix} 0 & (X_2)_{12} & \cdots & (X_2)_{1q} \\ (X_2)_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (X_2)_{p1} & 0 & \cdots & 0 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & (X_2)_{12} & \cdots & (X_2)_{1q} \\ (X_2)_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (X_2)_{p1} & 0 & \cdots & 0 \end{bmatrix}^T & \begin{bmatrix} 0 & (X_2)_{12} & \cdots & (X_2)_{1q} \\ -(X_2)_{12} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(X_2)_{1q} & 0 & \cdots & 0 \end{bmatrix} \end{array}\right].$$

Likewise, the (-1)-eigenspace of $ad(V_1)$ is described below

Lemma 10. *Let $X \in \mathfrak{so}(p,q)$ be given by (7). Then $[V_1, X] = -X$ iff each identity below holds.*

1. $X_2 = \begin{bmatrix} 0 & (X_2)_{12} & \cdots & (X_2)_{1q} \\ (X_2)_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (X_2)_{p1} & 0 & \cdots & 0 \end{bmatrix}$

2. $X_1 = \begin{bmatrix} 0 & -(X_2)_{21} & \cdots & -(X_2)_{p1} \\ (X_2)_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (X_2)_{p1} & 0 & \cdots & 0 \end{bmatrix}, X_3 = \begin{bmatrix} 0 & -(X_2)_{12} & \cdots & -(X_2)_{1q} \\ (X_2)_{12} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (X_2)_{1q} & 0 & \cdots & 0 \end{bmatrix}$

That is, $[V_1, X] = -X$ *iff*

$$X = \left[\begin{array}{c|c} \begin{bmatrix} 0 & -(X_2)_{21} & \cdots & -(X_2)_{p1} \\ (X_2)_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (X_2)_{p1} & 0 & \cdots & 0 \end{bmatrix} & \begin{bmatrix} 0 & (X_2)_{12} & \cdots & (X_2)_{1q} \\ (X_2)_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (X_2)_{p1} & 0 & \cdots & 0 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & (X_2)_{12} & \cdots & (X_2)_{1q} \\ (X_2)_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (X_2)_{p1} & 0 & \cdots & 0 \end{bmatrix}^T & \begin{bmatrix} 0 & -(X_2)_{12} & \cdots & -(X_2)_{1q} \\ (X_2)_{12} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (X_2)_{1q} & 0 & \cdots & 0 \end{bmatrix} \end{array}\right].$$

If $\lambda \in \mathbb{R}$, we denote the λ-eigenspace of $ad(V_1)$ by

$$\mathfrak{so}(p,q)_\lambda = \{X \in \mathfrak{so}(p,q) : ad(V_1)(X) = \lambda X\}.$$

Combining Lemma 1, 8, 9, and 10, we find that the eigenvalues of $ad(V_1)$ are precisely $-1, 0, 1$. Moreover, $\mathfrak{so}(p,q)$ is a direct sum of its eigenspaces. This proves Lemma 4.

Next, we apply Lemma 9. If $a \in \mathbb{R}^{p-1}$ and $b \in \mathbb{R}^{q-1}$, then

$$Y = \left[\begin{array}{cc|cc} 0 & a^T & 0 & b^T \\ -a & 0 & a & 0 \\ \hline 0 & a^T & 0 & b^T \\ b & 0 & -b & 0 \end{array}\right] \in \mathfrak{n} = \mathfrak{so}(p,q)_1. \tag{9}$$

A direct calculation shows $[X, Y] = 0$ for all $X, Y \in \mathfrak{n}$. That is, $\mathfrak{n}$ is an abelian Lie subalgebra of $\mathfrak{so}(p,q)$. Let

$$\mathfrak{s} = \mathfrak{h}_\mathfrak{p} \oplus \mathfrak{n} \tag{10}$$

be a direct sum of subspaces. If $t_1, t_2 \in \mathbb{R}$, and $X_1, X_2 \in \mathfrak{n}$, then $[t_1V_1 + X_1, t_2V_1 + X_2] = t_1X_2 - t_2X_1 \in \mathfrak{n}$. Then $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{so}(p,q)$. Let $\mathcal{D}\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}]$ denote the derived Lie algebra of $\mathfrak{s}$, i.e., $\mathcal{D}\mathfrak{s}$ is the real linear span of all $[X, Y]$ where $X, Y \in \mathfrak{s}$. Then $\mathcal{D}\mathfrak{s} = \mathfrak{n}$. Since $\mathfrak{n}$ is an abelian Lie algebra, the second derived algebra of $\mathfrak{s}$ satisfies $\mathcal{D}^2\mathfrak{s} = \mathcal{D}\mathfrak{n} = [\mathfrak{n}, \mathfrak{n}] = 0$. Thus, $\mathfrak{s}$ is a solvable Lie algebra. This proves Lemma 6.

Appendix B

Let $\mathcal{A}$, $\mathcal{N}$, and $\mathcal{S}$ be the connected Lie subgroups of $O(p,q)$ with Lie algebras $\mathfrak{h}_\mathfrak{p}$, $\mathfrak{n}$, and $\mathfrak{s}$, respectively. Let $\mathcal{AN} = \{an : a \in \mathcal{A}, n \in \mathcal{N}\}$. By definition, if $Y \in \mathfrak{n}$, then Y is an eigenvector of $ad(V_1)$ with eigenvalue 1. Recall, if ϕ is a differentiable homomorphism between Lie groups and $d\phi$ is the differential at the identity, then $\phi \circ \exp = \exp \circ d\phi$ [3, p.110]. In particular, $Ad(e^{tV_1})(Y) = e^{t\,ad(V_1)}(Y) = e^t Y$. Consequently, $e^{tV_1}e^Y e^{-tV_1} = \exp(Ad(e^{tV_1})(Y)) \in \mathcal{N}$. Thus, $\mathcal{AN}$ is a group, and $\mathcal{N}$ is a normal subgroup of $\mathcal{AN}$.

Clearly, $\mathcal{AN} \subseteq \mathcal{S}$. Moreover, since we have a direct sum in (10), and $\mathcal{AN}$ and $\mathcal{S}$ are connected Lie subgroups, and there is a one-to-one correspondence between Lie subalgebras and connected Lie subgroups, we obtain $\mathcal{AN} = \mathcal{S}$. Due to (10), the mapping $\beta : \mathcal{A} \times \mathcal{N} \to \mathcal{S}$ given by $\beta(a,n) = an$ is everywhere regular [3, p. 271, Lemma 5.2]. If $Y \in \mathfrak{n}$ is given by (9), we find

$$Y^2 = (b^T b - a^T a) \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and $Y^3 = 0$. Then

$$e^Y = I_{p+q} + Y + \frac{1}{2}Y^2. \tag{11}$$

Consequently, the restriction of exp to $\mathfrak{n}$ is a bijection. Since exp maps $\mathfrak{h}_\mathfrak{p}$ bijectively onto $\mathcal{A}$, the mapping $\beta : \mathcal{A} \times \mathcal{N} \to \mathcal{S}$ is one-to-one. Applying the inverse function theorem, we obtain β is a diffeomorphism onto $\mathcal{S}$.

Corollary 11. *Let $\mathcal{A}$, $\mathcal{N}$, and $\mathcal{S}$ be the connected Lie subgroups of $O(p,q)$ with Lie algebras $\mathfrak{h}_\mathfrak{p}$, $\mathfrak{n}$, and $\mathfrak{s}$, respectively. Then*

1. *$\mathcal{S} = \mathcal{AN}$*
2. *$\mathcal{N}$ is a normal subgroup of $\mathcal{S}$, and*
3. *The mapping $\beta : \mathcal{A} \times \mathcal{N} \to \mathcal{S}$ is a diffeomorphism where $\beta(a,n) = an$.*

From Definition 3, the next lemma can be proved easily.

Lemma 12. *For all $t \in \mathbb{R}$, we find*

1. *$\alpha_1(t)\alpha_2(t) = 2\alpha(t)$*
2. *$\dfrac{\alpha_1(t)}{\alpha_2(t)} = e^t$.*

Let $\delta = b^T b - a^T a$. Applying the exponential identity (11), we find

$$e^{\alpha_2(t)Y} = \left[\begin{array}{cc|cc} 1 + \frac{1}{2}\delta\alpha_2(t)^2 & \alpha_2(t)a^T & -\frac{1}{2}\delta\alpha_2(t)^2 & \alpha_2(t)b^T \\ -\alpha_2(t)a & I_{p-1} & \alpha_2(t)a & 0 \\ \hline \frac{1}{2}\delta\alpha_2(t)^2 & \alpha_2(t)a^T & 1 - \frac{1}{2}\delta\alpha_2(t)^2 & \alpha_2(t)b^T \\ \alpha_2(t)b & 0 & -\alpha_2(t)b & I_{q-1} \end{array}\right]. \tag{12}$$

From Lemma 12, we directly obtain the following identities.

(a) $\frac{1}{2}\delta\alpha_2(t)^2(\cosh(t)+\sinh(t)) = \delta\alpha(t)$, and

(b) $\alpha_2(t)(\cosh(t)+\sinh(t)) = \alpha_1(t)$.

Multiplying the matrices in (5) and (12), we obtain the next lemma. The matrix product is rather elaborate.

Lemma 13. *Let $t \in \mathbb{R}$. Let $Y \in \mathfrak{n}$ be given by (9), and using the entries of Y, let $\delta = b^T b - a^T a$. Then*

$$e^{tV_1}e^{\alpha_2(t)Y} = \left[\begin{array}{cc|cc} \cosh(t)+\delta\alpha(t) & \alpha_1(t)a^T & \sinh(t)-\delta\alpha(t) & \alpha_1(t)b^T \\ -\alpha_2(t)a & I_{p-1} & \alpha_2(t)a & 0 \\ \hline \sinh(t)+\delta\alpha(t) & \alpha_1(t)a^T & \cosh(t)-\delta\alpha(t) & \alpha_1(t)b^T \\ \alpha_2(t)b & 0 & -\alpha_2(t)b & I_{q-1} \end{array}\right]. \tag{13}$$

We point out an interesting result in [2, p. 136, Exercise 5] that we can combine with Lemma 13. Since $Y \in \mathfrak{n}$ is a 1-eigenvalue for $ad(V_1)$, we obtain the following.

Lemma 14. *If $Y \in \mathfrak{n}$, then $e^{tV_1}e^{\alpha_2(t)Y} = e^{tV_1+Y}$. Moreover, the exponential mapping is a diffeomorphism from $\mathfrak{s}$ onto $\mathcal{S}$.*

Proof The second claim of the above lemma follows since we have a composition of the following diffeomorphisms.

(a) $(tV_1, Y) \in \mathfrak{s} \mapsto (tV_1, \alpha_2(t)Y) \in \mathfrak{s}$

(b) $(tV_1, Y) \in \mathfrak{s} \mapsto (e^{tV_1}, e^Y) \in \mathcal{A} \times \mathcal{N}$

(c) $(a, n) \in \mathcal{A} \times \mathcal{N} \mapsto \beta(a, n) = an \in \mathcal{S}$

□

Let $G \in O(p,q)$, and let $G^{-1} = e^X K_0$ be the S-polar decomposition of G^{-1} where $K_0 \in \mathcal{K}$, $X \in \mathfrak{p}$. Let $x = (x_1, \ldots, x_p)^T \in \mathbb{R}^p$ and $v \in \mathbb{R}^{q-1}$ be chosen such that $X \in \mathfrak{p}$ is given by (3). From [7, Lemma 2.4], we recall the evaluation of e^X. Namely,

$$X^2 = \left[\begin{array}{c|cc} xx^T & 0 & -xv^T \\ \hline 0 & x^T x - v^T v & 0 \\ vx^T & 0 & -vv^T \end{array}\right].$$

If we let $\beta = x^T x - v^T v$, we find $X^3 = \beta X$. Then we obtain e^X as follows.

Lemma 15. *If $X \in \mathfrak{p}$ is given by (3), and $\beta = x^T x - v^T v$, then*

1. *If $\beta \neq 0$, then $e^X = I + \frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}X + \frac{\cosh(\sqrt{\beta})-1}{\beta}X^2$.*

2. *If $\beta = 0$, then $e^X = I + X + \frac{1}{2}X^2$.*

Let $Y \in \mathfrak{n}$ be given by (9) where $a \in \mathbb{R}^{p-1}$ and $b \in \mathbb{R}^{q-1}$. Let $\delta = b^T b - a^T a$. Given $t \in \mathbb{R}$, consider the matrix product

$$Z \equiv e^{tV_1}e^{\alpha_2(t)Y}e^X. \tag{14}$$

Let $Z_{k,l}$ denote the (k,l)-entry of Z. Recall, the ξ in (6) is a function of X. We evaluate the entries in the $(p+1)$st column of Z by multiplying $e^{tV_1}e^{\alpha_2(t)Y}$ in (13) to e^X in Lemma 15. Recall, $X \in \mathfrak{p}$ is given by (3) where $x = (x_1, \ldots, x_p)^T \in \mathbb{R}^p$ and $v \in \mathbb{R}^{q-1}$. Let $x' = (x_2, \ldots, x_p)^T \in \mathbb{R}^{p-1}$.

Lemma 16. *If $k \neq p+1$, we have the following entries $Z_{k,p+1}$ of Z.*

(a) $Z_{1,p+1} = \frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}\left[\cosh(t) + \delta\alpha(t)\right] x_1 + \alpha_1(t)\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}} a^T x' +$

$$[\sinh(t) - \delta\alpha(t)]\cosh(\sqrt{\beta}) + \alpha_1(t)\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}} b^T v$$

(b) If $2 \leq i \leq p$, *then* $Z_{i,p+1} = -\xi\alpha_2(t)a_{i-1} + \frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}} x_i$.

(c) If $1 \leq i \leq q-1$, *then* $Z_{p+1+i,p+1} = \xi\alpha_2(t)b_i + \frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}} v_i$.

Rewriting (a) in Lemma 16, we obtain

$$\begin{aligned} Z_{1,p+1} &= \frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}\cosh(t)x_1 + \delta\xi\alpha(t) + \\ &\quad \frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}\alpha_1(t)\left[a^T x' + b^T v\right] + \sinh(t)\cosh(\sqrt{\beta}). \end{aligned} \tag{15}$$

Definition 4. *Let $X \in \mathfrak{p}$ be given by (3) where $x = (x_1, \ldots, x_p)^T \in \mathbb{R}^p$ and $v \in \mathbb{R}^{q-1}$. In reference to (6), suppose $\xi \neq 0$. Let $x' = (x_2, \ldots, x_p)^T \in \mathbb{R}^{p-1}$, and let $t \in \mathbb{R}$. Consider the following vectors.*

(a) $a_{t,X} = \frac{1}{\xi\alpha_2(t)}\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}} x' \in \mathbb{R}^{p-1}$

(b) $b_{t,X} = -\frac{1}{\xi\alpha_2(t)}\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}} v \in \mathbb{R}^{q-1}$

The following lemma follows directly from parts (b) and (c) of Lemma 16.

Lemma 17. *We assume the given in Definition 4. Let $Y \in \mathfrak{n}$ be defined by (9) where $a = a_{t,X} \in \mathbb{R}^{p-1}$ and $b = b_{t,X} \in \mathbb{R}^{q-1}$. For any $t \in \mathbb{R}$, the entries of Z in (14) satisfy the following.*

(B) If $2 \leq i \leq p$, *then* $Z_{i,p+1} = 0$.

(C) If $1 \leq i \leq q-1$, *then* $Z_{p+1+i,p+1} = 0$.

We substitute $a = a_{t,X}$ and $b = b_{t,X}$ into $\delta = b^T b - a^T a$ and $Z_{1,p+1}$ in (15). Then

$$\begin{aligned} Z_{1,p+1} &= \frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}\cosh(t)x_1 + \frac{1}{\xi}\left(\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}\right)^2 \left(v^T v - (x')^T x'\right)\left[\frac{\alpha(t)}{\alpha_2(t)^2} - \frac{\alpha_1(t)}{\alpha_2(t)}\right] + \\ &\quad \sinh(t)\cosh(\sqrt{\beta}). \end{aligned}$$

Notice,

$$\frac{\alpha(t)}{\alpha_2(t)^2} - \frac{\alpha_1(t)}{\alpha_2(t)} = -\frac{1}{2}e^t.$$

Then

$$Z_{1,p+1} = \frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}\cosh(t)x_1 - \frac{e^t}{2\xi}\left(\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}\right)^2 (v^Tv - (x')^Tx') + \sinh(t)\cosh(\sqrt{\beta}).$$

Multiplying both sides by 2ξ, we find

$$\begin{aligned} 2\xi Z_{1,p+1} &= \xi\left[\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}(e^t + e^{-t})x_1 + (e^t - e^{-t})\cosh(\sqrt{\beta})\right] - \\ &\left[\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}\right]^2 e^t\left[v^Tv - (x')^Tx'\right]. \end{aligned}$$

Apply the definition of ξ.

$$\begin{aligned} 2\xi Z_{1,p+1} &= \left[\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}x_1\right]^2 e^{-t} + \left[\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}x_1\right](e^t - e^{-t})\cosh(\sqrt{\beta}) - \\ &\cosh(\sqrt{\beta})\left[\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}(e^t + e^{-t})x_1 + (e^t - e^{-t})\cosh(\sqrt{\beta})\right] - \\ &\left[\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}\right]^2 e^t(v^Tv - (x')^Tx' - x_1^2). \end{aligned}$$

Observe,

$$\left[\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}\right]^2 (v^Tv - x^Tx) = -\left[\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}\right]^2 \beta = 1 - \cosh^2(\sqrt{\beta}).$$

Substituting, we find

$$\begin{aligned} 2\xi Z_{1,p+1} &= \left[\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}x_1\right]^2 e^{-t} - \left[\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}x_1\right]e^{-t}\cosh(\sqrt{\beta}) - \\ &\cosh(\sqrt{\beta})\left[\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}e^{-t}x_1 - e^{-t}\cosh(\sqrt{\beta})\right] - e^t. \end{aligned}$$

Thus,

$$2\xi Z_{1,p+1} = e^{-t}\left[\xi^2 - e^{2t}\right] \tag{16}$$

Clearly, if $t_0 = \ln(|\xi|)$ then $Z_{1,p+1} = 0$. Now, we combine Lemma 3 and 17.

Lemma 18. *We assume the given in Definition 4. Let $t_0 = \ln(|\xi|)$, and let $Y \in \mathfrak{n}$ be defined by (9) where $a = a_{t_0,X} \in \mathbb{R}^{p-1}$ and $b = b_{t_0,X} \in \mathbb{R}^{q-1}$. Then $Z_{k,p+1} = 0$ for all $k \neq p+1$. In particular, $Z \in \mathcal{K}$.*

We summarize the following results of this section.

Theorem 19. *Let $p,q \geq 2$, $S = I_{p+q} - 2E_{p+1}$, and let $G \in O(p,q)$. Applying the S-polar decomposition, let $G^{-1} = e^X K_0$ where $K_0 \in \mathcal{K}$,*

$$X = \left[\begin{array}{c|cc} 0 & x & 0 \\ \hline x^T & 0 & -v^T \\ 0 & v & 0 \end{array}\right] \in \mathfrak{p},$$

$x = (x_1, \ldots, x_p)^T \in \mathbb{R}^p$, $v \in \mathbb{R}^{q-1}$. As defined in (6), suppose $\xi \neq 0$, and let $t_0 = \ln(|\xi|)$. Moreover, let $x' = (x_2, \ldots, x_p)^T \in \mathbb{R}^{p-1}$, and let $a = a_{t_0,X} \in \mathbb{R}^{p-1}$ and $b = b_{t_0,X} \in \mathbb{R}^{q-1}$ be given by Definition 4. Let

$$Y = \left[\begin{array}{cc|cc} 0 & a^T & 0 & b^T \\ -a & 0 & a & 0 \\ \hline 0 & a^T & 0 & b^T \\ b & 0 & -b & 0 \end{array}\right] \in \mathfrak{n} = \mathfrak{so}(p,q)_1.$$

Then $Z = e^{t_0 V_1} e^{\alpha_2(t_0)Y} e^X \in \mathcal{K}$ by Lemma 18, and $(K_0^{-1} Z^{-1}, e^{t_0 V_1}, e^{\alpha_2(t_0)Y}) \in \mathcal{K} \times \mathcal{A} \times \mathcal{N}$. The factorization $G = (K_0^{-1} Z^{-1}) e^{t_0 V_1} e^{\alpha_2(t_0)Y} \in \mathcal{KAN}$ is called a S-Iwasawa decomposition of G.

When $\xi = 0$, there are some $G \in O(p,q)$ that do not have a S-Iwasawa decomposition. Consider the case $X \in \mathfrak{p}$ in (3) where $x = (1,1,0,\ldots,0)^T \in \mathbb{R}^p$ and $v = (\sqrt{2}, 0, \ldots, 0)^T \in \mathbb{R}^{q-1}$. Recall, the first and second component of x are denoted by x_1 and x_2, and in this case, $x_1 = x_2 = 1$. Then $\beta = x^T x - v^T v = 0$, $\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}} = 1$, and from (6) we find $\xi = 0$. Let $G = e^{-X}$. Let Z be defined by (14) where $t \in \mathbb{R}$ and $Y \in \mathfrak{n}$. From Lemma 16(b), we find $Z_{2,p+1} = x_2 = 1$. Then $Z \notin \mathcal{K}$ by Lemma 3 for all $t \in \mathbb{R}$ and $Y \in \mathfrak{n}$. Thus, $G = e^{-X}$ does not have a S-Iwasawa decomposition.

4 General Case

The matrix S defined in (1), and the involution ρ of $O(p,q)$ defined by inner-conjugation by S is a special case of the following premise. Let $w \in \mathbb{R}^{p+q}$ satisfy $w^T J_{p,q} w \neq 0$. Then

$$S_w = I_{p+q} - 2(w^T J_{p,q} w)^{-1} w w^T J_{p,q} \tag{17}$$

is a $J_{p,q}$-Householder matrix, and we know $S_w^{-1} = S_w$ for instance see [5]. We define an involution ρ_w of $O(p,q)$ by inner-conjugation by S_w, i.e., $\rho_w(A) = S_w A S_w^{-1}$, $A \in O(p,q)$. Let $d\rho_w$ denote the differential of ρ_w at I_{p+q}. Then $d\rho_w$ is an involution of $\mathfrak{so}(p,q)$ satisfying $d\rho_w(X) = S_w X S_w^{-1}$, $X \in \mathfrak{so}(p,q)$. Consider the following eigenspaces of $d\rho_w$ and the subgroup of fixed points of ρ_w.

(1) $\mathfrak{p}_w = \{X \in \mathfrak{so}(p,q) : S_w X S_w^{-1} = -X\}$

(2) $\mathfrak{K}_w = \{X \in \mathfrak{so}(p,q) : S_w X S_w^{-1} = X\}$

(3) $\mathcal{K}_w = \{K \in O(p,q) : S_w K S_w^{-1} = K\}$.

In particular, if $w = e_{p+1}$, then $e_{p+1}^T J_{p,q} e_{p+1} = -1$ and we easily find $S_{e_{p+1}} = S$. In such a case, the sets in (1)-(3) agree with the sets in Definition 1. If $w^T J_{p,q} w < 0$, then there is a $J_{p,q}$-Householder matrix D such that $DS_w D^{-1} = S$, $D\mathfrak{p}_w D^{-1} = \mathfrak{p}$, $D\mathfrak{K}_w D^{-1} = \mathfrak{K}$, $D\mathcal{K}_w D^{-1} = \mathcal{K}$ [7]. Thus, the S_w-Iwasawa decomposition of $\mathfrak{so}(p,q)$ can be obtained by conjugation from the S-Iwasawa decomposition of $\mathfrak{so}(p,q)$. In fact, let $\mathfrak{h}_{\mathfrak{p}_{\mathfrak{w}}} = D^{-1}\mathfrak{h}_{\mathfrak{p}} D$. Clearly, $\mathfrak{h}_{\mathfrak{p}_{\mathfrak{w}}}$ is the linear span of $D^{-1}V_1 D$. Applying the notation in Corollary 5, we obtain the next lemma.

Lemma 20. *If $w \in \mathbb{R}^{p+q}$ and $w^T J_{p,q} w < 0$, then there exists $D \in O(p,q)$ where*

(a) $DS_w D^{-1} = S$

(b) $\mathfrak{h}_{\mathfrak{p}_{\mathfrak{w}}} \equiv D^{-1}\mathfrak{h}_{\mathfrak{p}} D$ is a maximal subspace of $\mathfrak{p}_w$ such that $\mathfrak{h}_{\mathfrak{p}_{\mathfrak{w}}}$ is an abelian Lie subalgebra

(c) the 1-eigenspace of $ad(D^{-1}V_1 D)$ is $D^{-1}\mathfrak{n}D$

(d) we have a direct sum of subspaces, namely,

$$\mathfrak{so}(p,q) = \mathfrak{K}_w \oplus \mathfrak{h}_{\mathfrak{p}_{\mathfrak{w}}} \oplus D^{-1}\mathfrak{n}D,$$

the S_w-Iwasawa decomposition of $\mathfrak{so}(p,q)$.

We still assume $w^T J_{p,q} w < 0$. Let $\mathcal{A}_w$ and $\mathcal{N}_w$ be the connected abelian Lie subgroups of $O(p,q)$ with Lie algebras $\mathfrak{h}_{\mathfrak{p}_{\mathfrak{w}}}$ and $D^{-1}\mathfrak{n}D$, respectively. Using the notations from Theorem 19, clearly $\mathcal{A}_w = D^{-1}\mathcal{A}D$, $\mathcal{N}_w = D^{-1}\mathcal{N}D$, and $\mathcal{K}_w = D^{-1}\mathcal{K}D$. By definition, DGD^{-1} has a S-Iwasawa decomposition iff $DGD^{-1} \in \mathcal{KAN}$ iff $G \in \mathcal{K}_w\mathcal{A}_w\mathcal{N}_w$, i.e., G has a S_w-Iwasawa decomposition. Let $DG^{-1}D^{-1} = e^X K_0$ where $K_0 \in \mathcal{K}$, $X \in \mathfrak{p}$, and evaluate the corresponding ξ in (6). If $\xi \neq 0$, then by Theorem 19 we find DGD^{-1} has a S-Iwasawa decomposition, and G has a S_w-Iwasawa decomposition.

Now, we consider the case when $w^T J_{p,q} w > 0$. We review some background material in [7, Lemma 18]. Let R be the backward identity matrix. We know $R^{-1} = R$ and $RJ_{p,q}R^{-1} = -J_{q,p}$. Consequently, $RO(p,q)R^{-1} = O(q,p)$ and $R\mathfrak{so}(p,q)R^{-1} = \mathfrak{so}(q,p)$. We denote $J_{q,p}$-Householder matrices with a prime. If $y \in \mathbb{R}^{p+q}$ and $y^T J_{q,p} y \neq 0$, then

$$S'_y = I_{p+q} - 2(y^T J_{q,p} y)^{-1} y y^T J_{q,p}$$

is a $J_{q,p}$-Householder matrix in $O(q,p)$. Let $v = Rw$. Since $w^T J_{p,q} w > 0$, we know $v^T J_{q,p} v < 0$, and $RS_w R^{-1} = S'_v$.

Let ρ'_v and $d\rho'_v$ denote involutions of $O(q,p)$ and $\mathfrak{so}(q,p)$, respectively, given by $\rho'_v(A) = S'_v A (S'_v)^{-1}$, $A \in O(q,p)$ and $d\rho'_v(X) = S'_v X (S'_v)^{-1}$, $X \in \mathfrak{so}(q,p)$. Similarly, define the following eigenspaces and subgroup of fixed points.

(1) $\mathfrak{p}'_v = \{X \in \mathfrak{so}(q,p) : S'_v X (S'_v)^{-1} = -X\}$

(2) $\mathfrak{K}'_v = \{X \in \mathfrak{so}(q,p) : S'_v X (S'_v)^{-1} = X\}$

(3) $\mathcal{K}'_v = \{K \in O(q,p) : S'_v K (S'_v)^{-1} = K\}$.

Also, we have the following identities.

(a) $R\mathfrak{p}_w R^{-1} = \mathfrak{p}'_v$

(b) $R\mathfrak{K}_w R^{-1} = \mathfrak{K}'_v$

(c) $R\mathcal{K}_w R^{-1} = \mathcal{K}'_v$

Applying Lemma 20 where $O(p,q)$ is replaced by $O(q,p)$, we know that any maximal subspace of $\mathfrak{p}'_v$ that is an abelian Lie subalgebra should be 1-dimensional. Let $\mathfrak{h}'_{\mathfrak{p}_v}$ be such a maximal subspace of $\mathfrak{p}'_v$. We let $\mathfrak{h}'_{\mathfrak{p}_v}$ be the real linear span of some nonzero $V'_1 \in \mathfrak{p}'_v$. Let $\mathfrak{n}' \subseteq \mathfrak{so}(q,p)$ denote the 1-eigenspace of $ad(V'_1)$. Then the direct sum

$$\mathfrak{so}(q,p) = \mathfrak{K}'_v \oplus \mathfrak{h}'_{\mathfrak{p}_v} \oplus \mathfrak{n}'$$

is the S'_v-Iwasawa decomposition of $\mathfrak{so}(q,p)$. Since $R\mathfrak{so}(p,q)R^{-1} = \mathfrak{so}(q,p)$, let $\mathfrak{h}_{\mathfrak{p}_w} = R^{-1}\mathfrak{h}'_{\mathfrak{p}_v}R$. Then the direct sum

$$\mathfrak{so}(p,q) = \mathfrak{K}_w \oplus \mathfrak{h}_{\mathfrak{p}_w} \oplus R^{-1}\mathfrak{n}'R$$

is the S_w-Iwasawa decomposition of $\mathfrak{so}(p,q)$. Finally, conjugating by R, the S_w-decomposition of $G \in O(p,q)$ exists iff RGR^{-1} has a S'_v-Iwasawa decomposition in $O(q,p)$.

Acknowledgement

The author is thankful for the valuable comments provided by the referees. I believe the paper is more understandable and formatted better as a result of their reviews. Any errors in the paper is the sole responsibility of the author.

The author gratefully acknowledges the support provided by the Robert and Lydia Hardy Endowed Professorship in Mathematics at Southeastern Louisiana University. The Hardy Professorial Chair helps the author to pursue pure mathematical research which in turn helps me relate to my students in the math classroom.

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