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S_w-polar decomposition for $O(p,q)$

Edgar N. Reyes[1]
Department of Mathematics
Southeastern Louisiana University
Hammond, Louisiana 70402
edgar.reyes@selu.edu

Abstract

Let $J_{p,q} = I_p \oplus (-I_q)$. Let $w \in \mathbb{R}^{p+q}$ satisfy $w^T J_{p,q} w \neq 0$, and let S_w be the corresponding $J_{p,q}$-Householder matrix. For each $A \in O(p,q)$, we show $A = e^X K$ for some $X \in \mathfrak{so}(p,q)$, $K \in O(p,q)$ satisfying $S_w X S_w^{-1} = -X$ and $S_w K S_w^{-1} = K$.

Key words: Householder matrix, indefinite orthogonal group, polar decomposition

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1 Introduction

Let p, q be integers such that $p, q \geq 2$. Let $\mathbb{R}^{p\times q}$ denote the set of p-by-q real matrices. Let $M_p(\mathbb{R}) = \mathbb{R}^{p\times p}$, and let I_p be the identity matrix in $M_p(\mathbb{R})$. Let $GL_p(\mathbb{R})$ be the multiplicative group of nonsingular matrices in $M_p(\mathbb{R})$. If $X \in M_p(\mathbb{R})$, let X^T denote the transpose of X. Let $O(p)$ be the group of orthogonal matrices $B \in GL_p(\mathbb{R})$ where $BB^T = I_p$. Let $\mathfrak{so}(p)$ denote the Lie algebra of p-by-p skew-symmetric matrices in $M_p(\mathbb{R})$.

Let $J_{p,q} = I_p \oplus (-I_q)$ be a diagonal matrix. Let $x, y \in \mathbb{R}^{p+q}$. Consider an indefinite scalar product in $\mathbb{R}^{p+q}$, namely, $(x, y)_{p,q} = x^T J_{p,q} y$. If $A \in GL_{p+q}(\mathbb{R})$ satisfies $(Ax, Ay)_{p,q} = (x, y)_{p,q}$ for all $x, y \in \mathbb{R}^{p+q}$, we say A is a $J_{p,q}$-orthogonal matrix. The group of all $J_{p,q}$-orthogonal matrices is the indefinite orthogonal group $O(p,q)$. We can easily verify that

$$O(p,q) = \{A \in GL_{p+q}(\mathbb{R}) : A^T J_{p,q} A = J_{p,q}\}.$$

Let $\mathfrak{so}(p,q)$ denote the Lie algebra of $O(p,q)$. If $X \in M_{p+q}(\mathbb{R})$, then $X \in \mathfrak{so}(p,q)$ iff $e^{tX} \in O(p,q)$ for all $t \in \mathbb{R}$. Applying the differentiation operator, $\frac{d}{dt}\big|_{t=0}$, to the equation $e^{tX^T} J_{p,q} e^{tX} = J_{p,q}$, we find

$$\mathfrak{so}(p,q) = \{X \in M_{p+q}(\mathbb{R}) : X^T J_{p,q} + J_{p,q} X = 0\}. \tag{1}$$

Let $\mathfrak{so}(p+q, \mathbb{C})$ be the complex Lie algebra of skew-symmetric matrices in $M_{p+q}(\mathbb{C})$. The Killing form $(,)_K$ for $\mathfrak{so}(p+q, \mathbb{C})$ satisfies $(X', Y')_K = (p+q-2)Tr(X'Y')$, $X', Y' \in \mathfrak{so}(p+q, \mathbb{C})$. We know $\mathfrak{so}(p,q)$ is isomorphic to a real form of $\mathfrak{so}(p+q, \mathbb{C})$. The Killing form for

[1]This article is dedicated to the memory of my brother, Prof. Noli N. Reyes. Noli specialized in approximation theory, and was a faculty member of the Institute of Mathematics at the University of the Philippines when he passed away on April 10, 2020.

$\mathfrak{so}(p,q)$ using the same notation satisfies $(X,Y)_K = (p+q-2)Tr(XY)$ for $X, Y \in \mathfrak{so}(p,q)$. We omit the multiplicative factor, $p+q-2$, and consider $(X,Y)_B = Tr(XY)$ as a scalar product in $\mathfrak{so}(p,q)$.

Let $w \in \mathbb{R}^{p+q}$ satisfy $\theta_w \equiv \frac{1}{2} w^T J_{p,q} w \neq 0$, and let

$$S_w = I_{p+q} - (\theta_w)^{-1} w w^T J_{p,q}. \tag{2}$$

By definition, S_w is a $J_{p,q}$-Householder matrix satisfying $S_w^{-1} = S_w = S_{tw} \in O(p,q)$ for all $t \in \mathbb{R} \setminus \{0\}$ [11]. We induce an involution ρ_w of $O(p,q)$ by setting $\rho_w(A) = S_w A S_w^{-1}$, $A \in O(p,q)$. Let $d\rho_w$ denote the differential of ρ_w at I_{p+q}. By definition, $d\rho_w(X) = \frac{d}{dt}\big|_{t=0} \big[\rho_w(e^{tX})\big]$. Then $d\rho_w$ is an involution of $\mathfrak{so}(p,q)$ satisfying $d\rho_w(X) = S_w X S_w^{-1}$, $X \in \mathfrak{so}(p,q)$. Motivated by the study of Cartan involutions and Killing forms for real semisimple Lie algebras, we consider a S_w-scalar product, namely, $(X,Y)_w = -(X, d\rho_w(Y))_B$, $X, Y \in \mathfrak{so}(p,q)$. Then

$$(X,Y)_w = -Tr(X S_w Y S_w^{-1}). \tag{3}$$

It can easily be verified that $(X,Y)_w = (Y,X)_w$ and $(d\rho_w(X), d\rho_w(Y))_w = (X,Y)_w$. The group $O(p,q)$ acts naturally as a group of linear transformations on $\mathfrak{so}(p,q)$ in the following way. Let $\mathrm{Ad} : O(p,q) \to GL(\mathfrak{so}(p,q))$ be the adjoint representation, i.e., for each $A \in O(p,q)$, $\mathrm{Ad}(A)$ is a well-defined linear transformation of $\mathfrak{so}(p,q)$ satisfying $\mathrm{Ad}(A)X = AXA^{-1} \in \mathfrak{so}(p,q)$, $X \in \mathfrak{so}(p,q)$.

We introduce a few more definitions in relation to (3). Let $A \in O(p,q)$. We say A is S_w-orthogonal if $(\mathrm{Ad}(A)X, \mathrm{Ad}(A)Y)_w = (X,Y)_w$ for all $X, Y \in \mathfrak{so}(p,q)$. Likewise, A is S_w-symmetric if $(\mathrm{Ad}(A)X, Y)_w = (X, \mathrm{Ad}(A)Y)_w$ for all $X, Y \in \mathfrak{so}(p,q)$. Let $A^{[w]} = \rho_w(A^{-1})$. The S_w-adjoint $A^{[w]}$ satisfies

$$(\mathrm{Ad}(A)X, Y)_w = (X, \mathrm{Ad}(A^{[w]})Y)_w. \tag{4}$$

Let $\mathfrak{so}(p,q) = \mathfrak{K}_w \oplus \mathfrak{p}_w$ be the eigenspace decomposition induced by $d\rho_w$ corresponding to eigenvalues 1 and -1. Let K_w denote the subgroup of fixed points of ρ_w in $O(p,q)$. That is, we have

1. $\mathfrak{K}_w = \{X \in \mathfrak{so}(p,q) : S_w X S_w^{-1} = X\}$
2. $\mathfrak{p}_w = \{X \in \mathfrak{so}(p,q) : S_w X S_w^{-1} = -X\}$
3. $K_w = \{K \in O(p,q) : S_w K S_w^{-1} = K\}$.

Clearly, $\mathfrak{K}_w$ and $\mathfrak{p}_w$ are S_w-orthogonal subspaces, i.e., $(X,Y)_w = 0$ for all $X \in \mathfrak{K}_w$, $Y \in \mathfrak{p}_w$. We find $K \in K_w$ iff $KK^{[w]} = I$ iff K is S_w-orthogonal. Also, if $X \in \mathfrak{p}_w$, we easily verify e^X is S_w-symmetric. In this paper, given $A \in O(p,q)$, we construct an $X \in \mathfrak{p}_w$ such that $A = e^X K$ for some $K \in K_w$. We say $A = e^X K$ is a S_w-polar decomposition of A.

In an earlier work, the author shows every $A \in O(p,q)$ has a S_w-polar decomposition whenever $p = 1$ and $q \geq 2$ [12]. In this paper, we extend the computations found in the cited paper, and assume through the remainder of this paper that $p, q \geq 2$, see Theorem 3.3. The classic polar decomposition for matrices in $M_n(\mathbb{R})$ or $M_n(\mathbb{C})$ is discussed in [7]. For connected semi-simple Lie groups, the polar decomposition is obtained by Riemannian geometric methods, the adjoint representation, and the classic polar decomposition [3]. The Lie group $O(p,q)$ is not connected but is simple except when $p = q = 2$ and $p = q = 1$; and $\mathfrak{so}(2,2) \simeq \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$ is a product of two simple Lie algebras, and $\mathfrak{so}(1,1)$ is an abelian Lie algebra. In the past twenty-five years, H-scalar products of the form $(x,y)_H = x^* H y$ have been extensively studied where $x, y \in \mathbb{R}^n$ or $\mathbb{C}^n$, and x^* is the conjugate-transpose of

x. Here, H is a nonsingular matrix that is either symmetric, skew-symmetric, hermitian, or satisfies $H^2 = \pm I$. We find several works on H-scalar products and H-polar decompositions in [1] [2] [5] [6] [8] [10]. However, we consider the structure of the Lie algebra $\mathfrak{so}(p,q)$ and group structure of $O(p,q)$ in constructing a S_w-polar decomposition for $A \in O(p,q)$. An overview of generalized polar decompositions of matrices from Lie algebras, Jordan algebras, and automorphism groups can be found in [9].

Consider the situation when $v, w \in \mathbb{R}^{p+q}$ are chosen such that $v^T J_{p,q} v$ and $w^T J_{p,q} w$ are either both positive or both negative. In this case, there is a $J_{p,q}$-Householder matrix D such that $Dw = -v$ or $Dw = v$ [11]. From (2), we obtain $DS_w D^{-1} = S_{Dw} = S_v$. We easily verify $DK_w D^{-1} = K_v$, $D\mathfrak{K}_w D^{-1} = \mathfrak{K}_v$, and $D\mathfrak{p}_w D^{-1} = \mathfrak{p}_v$. Thus, the condition that every $A \in O(p,q)$ has a S_w-polar decomposition is equivalent to the condition that every $A \in O(p,q)$ has a S_v-polar decomposition.

Now, let $e_k \in \mathbb{R}^{p+q}$ be the unit vector with a 1 in the kth-entry, and where the other entries are zero. Clearly, $e_p^T J_{p,q} e_p = 1$ and $e_{p+1}^T J_{p,q} e_{p+1} = -1$. As a consequence of the discussion in the previous paragraph, if each $A \in O(p,q)$ has a S_{e_p}-polar decomposition and a $S_{e_{p+1}}$-polar decomposition, then A has a S_w-polar decomposition for all $w \in \mathbb{R}^{p+q}$ such that $w^T J_{p,q} w \neq 0$. In Section 2, we assume $w = e_{p+1}$, and solve $e^{2X} = AA^{[w]}$ for $X \in \mathfrak{p}_w$ in a rather lengthy calculation that is the bulk of the paper. An advantage of the previous equation, though, is that the eigenvalues and diagonalizabilty of e^{2X} are determined but not exploited further, see Corollary 2.2 below. If $w = e_{p+1}$ and $e^{2X} = AA^{[w]}$ for some $X \in \mathfrak{p}_w$, then we easily verify $A = e^X K$ for some $K \in K_w$.

In Section 3, we consider $w = e_p$, and apply a similarity between $O(p,q)$ and $O(q,p)$ to show the existence of S_{e_p}-polar decompositions for every $A \in O(p,q)$. As we will see, we can reduce the case of $w = e_p$ to the case of $w = e_{p+1}$.

We complete this section with a lemma that follows directly from (1).

Lemma 1.1. *Let $X \in M_{p+q}(\mathbb{R})$. Then $X \in \mathfrak{so}(p,q)$ iff there exist $X_1 \in \mathfrak{so}(p)$, $X_3 \in \mathfrak{so}(q)$, and $X_2 \in \mathbb{R}^{p\times q}$ satisfying*

$$X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} \in \mathfrak{so}(p,q).$$

2 The case $w = e_{p+1}$

Let $q \geq p \geq 2$. Let $J_q = 1 \oplus (-I_{q-1}) \in M_q(\mathbb{R})$. Let $E_{i,j} \in M_{p+q}(\mathbb{R})$ be a matrix where the (i,j)-entry is 1, and the other entries are zero. If $i = j$, let $E_i = E_{i,i}$. The unit vectors $e_k \in \mathbb{R}^{p+q}$ satisfy $e_i e_j^T = E_{i,j}$.

In this section, we fix and let $w = e_{p+1} \in \mathbb{R}^{p+q}$. Then $w^T J_{p,q} w = -1$ and $ww^T = E_{p+1}$. Applying (2), we find

$$S_w = I_p \oplus (-J_q). \tag{5}$$

Then

$$S_w \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} S_w^{-1} = \begin{bmatrix} X_1 & -X_2 J_q \\ -J_q X_2^T & J_q X_3 J_q \end{bmatrix}. \tag{6}$$

Lemma 2.1. *Let $w = e_{p+1}$. Let $X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} \in \mathfrak{so}(p,q)$ be given by Lemma 1.1.*

Then $X \in \mathfrak{p}_w$ iff $X_1 = 0$, and there exist $x \in \mathbb{R}^p$ and $v \in \mathbb{R}^{q-1}$ such that

$$X = \left[\begin{array}{c|cc} 0 & x & 0 \\ \hline x^T & 0 & -v^T \\ 0 & v & 0 \end{array}\right] \in \mathfrak{p}_w. \tag{7}$$

Given $X \in \mathfrak{p}_w$ as in (7), let $\alpha = x^T x - v^T v$. Applying Schur complements and formulas for determinants in [7, page 21], we obtain that the characteristic polynomial $p(t)$ of X is $p(t) = t^{p+q-2}\left(t^2 - \alpha\right)$. From (7), we directly find $X^3 = \alpha X$. In particular, if $\alpha \neq 0$, then $m(t) = t(t^2 - \alpha)$ is the minimum polynomial of X. Since each of the zeros of $m(t)$ has multiplicity 1, X is diagonalizable when $\alpha \neq 0$.

Corollary 2.2. *Let $w = e_{p+1}$. Let $X \in \mathfrak{p}_w$ be given by (7), and let $\alpha = x^T x - v^T v$. Then the characteristic polynomial $p(t)$ of X satisfies $p(t) = t^{p+q-2}\left(t^2 - \alpha\right)$. If $\alpha \neq 0$, then X is diagonalizable and rank$(X) = 2$.*

Let $X \in \mathfrak{p}_w$ be given by (7). Let $y \in \mathbb{R}^p$, $w \in \mathbb{R}^{q-1}$, and let

$$W = \left[\begin{array}{c|cc} 0 & y & 0 \\ \hline y^T & 0 & -w^T \\ 0 & w & 0 \end{array}\right] \in \mathfrak{p}_w.$$

Recall, $\mathrm{ad}(X)(W) = [X, W]$. A direct calculation of the Lie bracket shows

$$[X, W] = \left[\begin{array}{c|cc} xy^T - yx^T & 0 & yv^T - xw^T \\ \hline 0 & 0 & 0 \\ vy^T - wx^T & 0 & wv^T - vw^T \end{array}\right].$$

Suppose $[X, W] = 0$ and $X \neq 0$. Then $xy^T - yx^T = 0$. If $x \neq 0$, then $y = \gamma x$ for some $\gamma \in \mathbb{R}$. Note, $0 = yv^T - xw^T = x(\gamma v - w)^T$. Then $w = \gamma v$ and $W = \gamma X$. Similarly, if $v \neq 0$, then $W \in \mathbb{R}X$.

Lemma 2.3. *Let $w = e_{p+1}$. Let $X, W \in \mathfrak{p}_w$ satisfy $X \neq 0$ and $[X, W] = 0$. Then $W \in \mathbb{R}X$.*

2.1 Exponential Map

Let $X \in \mathfrak{so}(p, q)$ be given by Lemma 1.1. We use square brackets to denote the block entries of X, e.g., $[X]_{2,1} = X_2^T$. If $X \in \mathfrak{p}_w$ is given by (7), then

$$X^2 = \left[\begin{array}{c|cc} xx^T & 0 & -xv^T \\ \hline 0 & x^T x - v^T v & 0 \\ vx^T & 0 & -vv^T \end{array}\right].$$

Also, we find $X^3 = \alpha X$ where $\alpha = x^T x - v^T v$. Then we obtain e^X as follows.

Lemma 2.4. *Let $w = e_{p+1}$. If $X \in \mathfrak{p}_w$ is given by (7), and $\alpha = x^T x - v^T v$, then*

1. $e^X = I + \frac{\sinh(\sqrt{\alpha})}{\sqrt{\alpha}} X + \frac{\cosh(\sqrt{\alpha}) - 1}{\alpha} X^2$
2. *If $\alpha = 0$, then* $e^X = I + X + \frac{1}{2} X^2$
3. *The block matrix entries of e^X satisfy* $[e^X]_{(1,2)} = ([e^X]_{(2,1)})^T J_q$.

2.2 Evaluating $AA^{[w]}$ when $w = e_{p+1}$

Let $w = e_{p+1}$. Recall, $J_q = 1 \oplus (-I_{q-1})$. Applying (5), we find $J_0 \equiv S_w J_{p,q} = I_p \oplus J_q = J_{p,q} S_w$. Let $A \in O(p,q)$. Since $AA^{[w]} = AS_w A^{-1} S_w$ and $A^{-1} = J_{p,q} A^T J_{p,q}$, we find $AA^{[w]} = AJ_0 A^T J_0$. As a partitioned matrix, let

$$A = \begin{bmatrix} B_0 & b^T \\ c & A_0 \end{bmatrix} \in O(p,q) \tag{8}$$

for some $b, c \in \mathbb{R}^{q\times p}$, $B_0 \in \mathbb{R}^{p\times p}$, and $A_0 \in \mathbb{R}^{q\times q}$. Then

$$AA^{[w]} = \left[\begin{array}{c|c} B_0 B_0^T + b^T J_q b & B_0 c^T J_q + b^T J_q A_0^T J_q \\ \hline cB_0^T + A_0 J_q b & cc^T J_q + A_0 J_q A_0^T J_q \end{array}\right].$$

Since $A^T J_{p,q} A = J_{p,q}$, we find

$$A^T J_{p,q} A = \begin{bmatrix} B_0^T B_0 - c^T c & B_0^T b^T - c^T A_0 \\ bB_0 - A_0^T c & bb^T - A_0^T A_0 \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}. \tag{9}$$

Since $A^T \in O(p,q)$, we obtain

$$AJ_{p,q}A^T = \begin{bmatrix} B_0 B_0^T - b^T b & B_0 c^T - b^T A_0^T \\ cB_0^T - A_0 b & cc^T - A_0 A_0^T \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}. \tag{10}$$

From (10), we find a block entry where $[AJ_{p,q}A^T]_{1,1} = B_0 B_0^T - b^T b = I_p$. From which we obtain $[AA^{[w]}]_{1,1} = I_p + b^T b + b^T J_q b \in \mathbb{R}^{p\times p}$. Let $b_{k,l}$ be the (k,l)-entry of $b \in \mathbb{R}^{q\times p}$. Notice, $(b^T b + b^T J_q b)_{k,l} = 2b_{1,k} b_{1,l}$ where $1 \le k, l \le p$.

Lemma 2.5. *Let $w = e_{p+1}$, and let $k_1, k_2 \in \{1,\dots,p\}$. Then the (k_1,k_2)-entry of $[AA^{[w]}]_{1,1}$ is $\delta_{k_1,k_2} + 2b_{1,k_1} b_{1,k_2}$. In particular, $[AA^{[w]}]_{1,1}$ is a symmetric matrix and*

$$[AA^{[w]}]_{1,1} = \begin{bmatrix} 1 + 2b_{1,1}^2 & 2b_{1,1}b_{1,2} & 2b_{1,1}b_{1,3} & \cdots & 2b_{1,1}b_{1,p} \\ 2b_{1,2}b_{1,1} & 1 + 2b_{1,2}^2 & 2b_{1,2}b_{1,3} & \cdots & 2b_{1,2}b_{1,p} \\ 2b_{1,3}b_{1,1} & 2b_{1,3}b_{1,2} & 1 + 2b_{1,3}^2 & \cdots & 2b_{1,3}b_{1,p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2b_{1,p}b_{1,1} & 2b_{1,p}b_{1,2} & 2b_{1,p}b_{1,3} & \cdots & 1 + 2b_{1,p}^2 \end{bmatrix}$$

From (10), we have a block entry where $[AJ_{p,q}A^T]_{2,1} = 0$. From which, we find $cB_0^T = A_0 b$. Then $[AA^{[w]}]_{2,1} = A_0 b + A_0 J_q b \in \mathbb{R}^{q\times p}$. Let $a_{i,j}$ be the (i,j)-entry of $A_0 \in \mathbb{R}^{q\times q}$. Notice, $(A_0 b + A_0 J_q b)_{k,l} = 2a_{k,1} b_{1,l}$ where $k \in \{1,\dots,q\}$ and $l \in \{1,\dots,p\}$.

Lemma 2.6. *Let $w = e_{p+1}$. Let $k \in \{1,\dots,q\}$ and $l \in \{1,\dots,p\}$. Then the (k,l)-entry of $[AA^{[w]}]_{2,1}$ is $2a_{k,1} b_{1,l}$. In particular,*

$$[AA^{[w]}]_{2,1} = \begin{bmatrix} 2a_{1,1}b_{1,1} & 2a_{1,1}b_{1,2} & \cdots & 2a_{1,1}b_{1,p} \\ 2a_{2,1}b_{1,1} & 2a_{2,1}b_{1,2} & \cdots & 2a_{2,1}b_{1,p} \\ \vdots & \vdots & \ddots & \vdots \\ 2a_{q,1}b_{1,1} & 2a_{q,1}b_{1,2} & \cdots & 2a_{q,1}b_{1,p} \end{bmatrix}.$$

Notice, we have block entries satisfying $[AA^{[w]}]_{1,2} = \left([AA^{[w]}]_{2,1}\right)^T J_q \in \mathbb{R}^{p\times q}$. Let $k \in \{1,\dots,p\}$ and $l \in \{1,\dots,q\}$. Let $P_l = 1$ if $l = 1$, and let $P_l = -1$ if $l \neq 1$. Then the (k,l)-entry of $[AA^{[w]}]_{1,2}$ is $2b_{1,k}a_{l,1}P_l$.

Lemma 2.7. *Let $w = e_{p+1}$. Then $[AA^{[w]}]_{1,2} = \left([AA^{[w]}]_{2,1}\right)^T J_q$. Also, if $k \in \{1,\dots,p\}$ and $l \in \{1,\dots,q\}$, the (k,l)-entry of $[AA^{[w]}]_{1,2}$ is $2b_{1,k}a_{l,1}P_l$. In particular,*

$$[AA^{[w]}]_{1,2} = \begin{bmatrix} 2a_{1,1}b_{1,1} & -2a_{2,1}b_{1,1} & \cdots & -2a_{q,1}b_{1,1} \\ 2a_{1,1}b_{1,2} & -2a_{2,1}b_{1,2} & \cdots & -2a_{q,1}b_{1,2} \\ \vdots & \vdots & \ddots & \vdots \\ 2a_{1,1}b_{1,p} & -2a_{2,1}b_{1,p} & \cdots & -2a_{q,1}b_{1,p} \end{bmatrix}.$$

Since $[AA^{[w]}]_{2,2} = cc^T J_q + A_0 J_q A_0^T J_q$ and $[AJ_{p,q}A^T]_{2,2} = -I_q$, we obtain

$$[AA^{[w]}]_{2,2} = (A_0A_0^T + A_0J_qA_0^T)J_q - J_q \in M_q(\mathbb{R}).$$

Let $k,l \in \{1,\dots,q\}$. Recall, $a_{k,l}$ is the (k,l)-entry of A_0. We find the (k,l)-entry of $(A_0A_0^T + A_0J_qA_0^T)J_q$ is given by $2a_{k,1}a_{l,1}P_l$.

Lemma 2.8. *Let $w = e_{p+1}$. Let $k,l \in \{1,\dots,q\}$. Then the (k,l)-entry of $[AA^{[w]}]_{2,2}$ is given by $2a_{k,1}a_{l,1}P_l - (J_q)_{k,l}$. In particular, we have*

$$[AA^{[w]}]_{2,2} = \begin{bmatrix} 2a_{1,1}^2 - 1 & -2a_{1,1}a_{2,1} & -2a_{1,1}a_{3,1} & \cdots & -2a_{1,1}a_{q,1} \\ 2a_{1,1}a_{2,1} & 1 - 2a_{2,1}^2 & -2a_{2,1}a_{3,1} & \cdots & -2a_{2,1}a_{q,1} \\ 2a_{1,1}a_{3,1} & -2a_{3,1}a_{2,1} & 1 - 2a_{3,1}^2 & \cdots & -2a_{3,1}a_{q,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & -2a_{q-1,1}a_{q,1} \\ 2a_{1,1}a_{q,1} & -2a_{q,1}a_{2,1} & -2a_{q,1}a_{3,1} & -2a_{q,1}a_{q-1,1} & 1 - 2a_{q,1}^2 \end{bmatrix}.$$

2.3 S_w-Polar decomposition when $w = e_{p+1}$

Let $w = e_{p+1}$, and let $A \in O(p,q)$ be partitioned as in (8). Recall, the (i,j)-entry of $A_0 \in \mathbb{R}^{q\times q}$ is $a_{i,j}$, and the (k,l)-entry of $b \in \mathbb{R}^{q\times p}$ is $b_{k,l}$.

We know $\cosh(it) = \cos(t)$ for $t \in \mathbb{R}$, and $2a_{1,1}^2 - 1 \geq -1$. Then there is a unique $\beta \geq -\pi^2$ such that $\cosh(\sqrt{\beta}\,) = 2a_{1,1}^2 - 1$. Now, choose some $x = (x_1,\dots,x_p)^T \in \mathbb{R}^p$ and $v = (v_2,\dots,v_q)^T \in \mathbb{R}^{q-1}$ satisfying

1. $\frac{\cosh(\sqrt{\beta}\,)-1}{\beta}x_k^2 = 2b_{1,k}^2$, $k \in \{1,\dots,p\}$
2. $\frac{\cosh(\sqrt{\beta}\,)-1}{\beta}v_l^2 = 2a_{l,1}^2$, $l \in \{2,\dots,q\}$.

We refer to the above equations as Statements 1 and 2. After choosing some $x \in \mathbb{R}^p$ and $v \in \mathbb{R}^{q-1}$, let $X \in \mathfrak{p}_w$ be defined by (7). If we replaced some entries x_k and v_l by $-x_k$ and

$-v_l$, respectively, then Statements 1 and 2 would remain true, and X may change but still belong to $\mathfrak{p}_w$. Except for possible changes in the sign of some components of $x \in \mathbb{R}^p$ and $v \in \mathbb{R}^{q-1}$, we intend to show $AA^{[w]} = e^X$ in this section. The proof will depend on whether $\beta > -\pi^2$ or $\beta = -\pi^2$, see Corollaries 2.13, 2.14 and 2.15. Recall, we are assuming $w = e_{p+1}$ throughout Section 2.

From (9), we find $[A^T J_{p,q} A]_{2,2} = bb^T - A_0^T A_0 = -I_q$. From the (1,1)-entry of $[A^T J_{p,q} A]_{2,2}$, we find $\sum_{k=1}^p b_{1,k}^2 - \sum_{l=1}^q a_{l,1}^2 = -1$. Then $\sum_{k=1}^p b_{1,k}^2 - \sum_{l=2}^q a_{l,1}^2 = a_{1,1}^2 - 1$. Combining Statements 1 and 2, we obtain

$$\frac{\cosh(\sqrt{\beta}\,) - 1}{\beta}(x^T x - v^T v) = 2(a_{1,1}^2 - 1) = \cosh(\sqrt{\beta}) - 1.$$

Then $\beta = x^T x - v^T v$. From Lemma 2.4, we find $\beta = \alpha$. Also, for $k \in \{1, \ldots p\}$, the first p diagonal entries of e^X satisfy

$$(e^X)_{k,k} = 1 + \frac{\cosh(\sqrt{\beta}\,) - 1}{\beta} x_k^2.$$

From Lemma 2.5, we see $([AA^{[w]}]_{1,1})_{k,k} = 1 + 2b_{1,k}^2$. Applying Statement 1, we find $(e^X)_{k,k} = (AA^{[w]})_{k,k}$ for $k \in \{1, \ldots, p\}$. Also,

$$\begin{aligned} (e^X)_{p+1,p+1} &= 1 + \frac{\cosh(\sqrt{\beta}\,) - 1}{\beta}\beta \\ &= \cosh(\sqrt{\beta}\,). \end{aligned}$$

From Lemma 2.8, we see $(AA^{[w]})_{p+1,p+1} = ([AA^{[w]}]_{2,2})_{1,1} = 2a_{1,1}^2 - 1 = \cosh(\sqrt{\beta}\,)$. Thus, by Lemma 2.4, we obtain $(e^X)_{p+1,p+1} = (AA^{[w]})_{p+1,p+1}$. For $l \in \{2, \ldots, q\}$, applying Lemma 2.4 and Statement 2, we find

$$\begin{aligned} (e^X)_{p+l,p+l} &= 1 + \frac{\cosh(\sqrt{\beta}\,) - 1}{\beta}(-v_l^2) \\ &= 1 - 2a_{l,1}^2. \end{aligned}$$

Applying Lemma 2.8, we obtain $(AA^{[w]})_{p+l,p+l} = ([AA^{[w]}]_{2,2})_{l,l} = 1 - 2a_{l,1}^2$.

Lemma 2.9. *Let $w = e_{p+1}$. Let $x = (x_1, \ldots, x_p)^T \in \mathbb{R}^p$ and $v = (v_2, \ldots, v_q)^T \in \mathbb{R}^{q-1}$ satisfy Statements 1 and 2. Let $X \in \mathfrak{p}_w$ be defined by (7). The respective diagonal entries of e^X and $AA^{[w]}$ are equal.*

Notice, by replacing x_k by $-x_k$ or v_l by $-v_l$, if needed, Lemma 2.9 would still hold. From Statements 1 and 2, we obtain $4b_{1,k}^2 a_{11}^2 = \frac{\sinh^2(\sqrt{\beta})}{\beta} x_k^2$ and $4b_{1,k}^2 a_{l,1}^2 = \frac{(\cosh(\sqrt{\beta})-1)^2}{\beta^2} x_k^2 v_l^2$. We choose the signs of x_k and v_l according to (a) and (b) in the next corollary.

Corollary 2.10. *Let $w = e_{p+1}$. Let $x = (x_1, \ldots, x_p)^T \in \mathbb{R}^p$ and $v = (v_2, \ldots, v_q)^T \in \mathbb{R}^{q-1}$ satisfy Statements 1 and 2, and the following*

(a) $2b_{1,k}a_{11} = \frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}} x_k$

(b) $2b_{1,k}a_{l,1} = \frac{\cosh(\sqrt{\beta})-1}{\beta} x_k v_l$

for all $k \in \{1, \ldots, p\}$ and $l \in \{2, \ldots, q\}$. Let $X \in \mathfrak{p}_w$ be defined by (7). Then the respective diagonal entries of e^X and $AA^{[w]}$ are equal. Moreover, the following block-matrices are equal, namely,

1. $[e^X]_{(1,2)} = [AA^{[w]}]_{(1,2)}$
2. $[e^X]_{(2,1)} = [AA^{[w]}]_{(2,1)}$
3. *if* $\beta \neq -\pi^2$, *then* $[e^X]_{(1,1)} = [AA^{[w]}]_{(1,1)}$.

Proof The discussion before the corollary shows the respective diagonal entries of e^X and $AA^{[w]}$ are equal. Recall, $\beta = \alpha$. Let $k \in \{1, \ldots, p\}$ and $l \in \{2, \ldots, q\}$. Applying Lemma 2.4, we obtain the following matrix entries.

- $([e^X]_{(1,2)})_{(k,1)} = \frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}} x_k$
- $([e^X]_{(1,2)})_{(k,l)} = \frac{\cosh(\sqrt{\beta})-1}{\beta}(-x_k v_l)$

Also, by Lemma 2.7, we obtain

- $([AA^{[w]}]_{(1,2)})_{(k,1)} = 2b_{1,k}a_{1,1}$
- $([AA^{[w]}]_{(1,2)})_{(k,l)} = -2b_{1,k}a_{l,1}$.

Applying *(a)* and *(b)* of the corollary, we obtain $[e^X]_{(1,2)} = [AA^{[w]}]_{(1,2)}$. Recall, $[e^X]_{(2,1)} = J_q([e^X]_{(1,2)})^T$ and $[AA^{[w]}]_{(2,1)} = J_q([AA^{[w]}]_{(1,2)})^T$ by Lemma 2.7. Then $[e^X]_{(2,1)} = [AA^{[w]}]_{(2,1)}$.

Applying part *(a)*, for $k_1 \neq k_2$ and $k_1, k_2 \in \{1, \ldots, p\}$, we find

$$\begin{aligned} 2b_{1,k_1}b_{1,k_2}\left(\cosh(\sqrt{\beta})+1\right) &= \frac{\cosh^2(\sqrt{\beta})-1}{\beta} x_{k_1}x_{k_2} \\ 2b_{1,k_1}b_{1,k_2} &= \frac{\cosh(\sqrt{\beta})-1}{\beta} x_{k_1}x_{k_2} \qquad (11) \end{aligned}$$

where in the last line we used $\cosh(\sqrt{\beta}) + 1 \neq 0$ since $\beta \neq -\pi^2$. Thus, if $\beta \neq -\pi^2$, and by applying identity (11), Lemma 2.4, Lemma 2.5, we find $[e^X]_{1,1} = [AA^{[w]}]_{1,1}$.

□

Lemma 2.11. *Let $w = e_{p+1}$. Choose $x = (x_1, \ldots, x_p)^T \in \mathbb{R}^p$ and $v = (v_2, \ldots, v_q)^T \in \mathbb{R}^{q-1}$ as in Corollary 2.10. Suppose there exists $k \in \{1, \ldots, p\}$ satisfying $b_{1,k} \neq 0$.*

(a) If $l \in \{2, \ldots, q\}$, then $2a_{1,1}a_{l,1} = \frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}} v_l$.

(b) If $l_1, l_2 \in \{2, \ldots, q\}$ and $l_1 \neq l_2$, then $2a_{l_1,1}a_{l_2,1} = \frac{\cosh(\sqrt{\beta})-1}{\beta} v_{l_1}v_{l_2}$.

Proof Suppose $b_{1,k} \neq 0$ for some $k \in \{1, \ldots, p\}$. From Statement 1 in the beginning of this section, we find $x_k \neq 0$. From Corollary 2.10, we find $a_{1,1} = 0$ iff $\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}} = 0$. We may assume $a_{1,1} \neq 0$ since part (a) clearly holds when $a_{11} = 0$. Applying Corollary 2.10 twice, for any $l \in \{2, \ldots, q\}$, we find

$$v_l = \left(\frac{\cosh(\sqrt{\beta}) - 1}{\beta}\right)^{-1} (2b_{1,k}a_{l,1})x_k^{-1} \tag{12}$$
$$v_l = \left(\frac{\cosh(\sqrt{\beta}) - 1}{\beta}\right)^{-1} (2b_{1,k}a_{l,1})(2b_{1,k}a_{1,1})^{-1}\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}.$$

Multiplying by $\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}$, we obtain

$$\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}v_l = 2a_{l,1}a_{1,1}^{-1}\frac{\cosh(\sqrt{\beta}) + 1}{2}.$$

Since $\cosh(\sqrt{\beta}) = 2a_{1,1}^2 - 1$, we find $\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}v_l = 2a_{1,1}a_{l,1}$. This proves part 1.

Applying (12) and Statement 1, we find

$$\begin{aligned} v_{l_1}v_{l_2} &= \left(\frac{\cosh(\sqrt{\beta}) - 1}{\beta}\right)^{-2} (x_k)^{-2}(2b_{1,k}a_{l_1,1})(2b_{1,k}a_{l_2,1}) \\ &= \left(\frac{\cosh(\sqrt{\beta}) - 1}{\beta}\right)^{-1} (2a_{l_1,1}a_{l_2,1}). \end{aligned}$$

Thus, part (b) of the lemma is proved. □

Corollary 2.12. *Let $w = e_{p+1}$. Choose $x = (x_1, \ldots, x_p)^T \in \mathbb{R}^p$ and $v = (v_2, \ldots, v_q)^T \in \mathbb{R}^{q-1}$ as in Corollary 2.10. Let $X \in \mathfrak{p}_w$ be defined by (7). Suppose there exists $k \in \{1, \ldots, p\}$ satisfying $b_{1,k} \neq 0$. Then $[e^X]_{(2,2)} = [AA^{[w]}]_{(2,2)}$.*

Proof From Corollary 2.10, we have seen that the respective diagonal entries of $[e^X]_{(2,2)}$ and $[AA^{[w]}]_{(2,2)}$ are the same. Recall, $\alpha = \beta$. Combining Lemma 2.4, Lemma 2.8, and Lemma 2.11, we find $[e^X]_{(2,2)} = [AA^{[w]}]_{(2,2)}$.

□

Now, we combine Corollary 2.10 and Corollary 2.12.

Corollary 2.13. *Let $w = e_{p+1}$. Choose $x = (x_1, \ldots, x_p)^T \in \mathbb{R}^p$ and $v = (v_2, \ldots, v_q)^T \in \mathbb{R}^{q-1}$ as in Corollary 2.10. Let $X \in \mathfrak{p}_w$ be defined by (7). Suppose $\beta \neq -\pi^2$, and there exists some $k \in \{1, \ldots, p\}$ satisfying $b_{1,k} \neq 0$. Then $e^X = AA^{[w]}$.*

Next, assume $b_{1,k} = 0$ for all $k \in \{1, \ldots, p\}$. From Lemmas 2.5, 2.6, and 2.7, we find $[AA^{[w]}]_{1,1} = I_p$, $[AA^{[w]}]_{1,2} = 0$, and $[AA^{[w]}]_{2,1} = 0$. Choose $x = (x_1, \ldots, x_p)^T \in \mathbb{R}^p$ and $v = (v_2, \ldots, v_q)^T \in \mathbb{R}^{q-1}$ as in Statements 1 and 2. Then $x_k = 0$ for all $k \in \{1, \ldots, p\}$, and $x = 0 \in \mathbb{R}^p$. Let $X \in \mathfrak{p}_w$ be defined by (7). From Lemma 2.4, we obtain $[e^X]_{(1,1)} =$

$I_p = [AA^{[w]}]_{(1,1)}$, $[e^X]_{(1,2)} = 0 = [AA^{[w]}]_{(1,2)}$, and $[e^X]_{(2,1)} = 0 = [AA^{[w]}]_{(2,1)}$. Multiplying $(\cosh(\sqrt{\beta})+1)$ in Statement 2, we obtain

$$\frac{\sinh^2(\sqrt{\beta})}{\beta} v_l^2 \quad = \quad 4a_{1,1}^2 a_{l,1}^2.$$

Recall, if we replace v_l by $-v_l$, if needed, then Statement 2 will still hold.

Corollary 2.14. *Let $w = e_{p+1}$. Suppose $\beta \neq -\pi^2$, and $b_{1,k} = 0$ for all $k \in \{1, \dots, p\}$. Choose $x = (x_1, \dots, x_p)^T \in \mathbb{R}^p$ and $v = (v_2, \dots, v_q)^T \in \mathbb{R}^{q-1}$ as in Statements 1 and 2, and satisfying $2a_{1,1}a_{l,1} = \frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}} v_l$ for all $l \in \{2, \dots, q\}$. Let $X \in \mathfrak{p}_w$ be defined by (7). Then $e^X = AA^{[w]}$.*

Proof Since $\beta \neq -\pi^2$, we have $\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}} \neq 0$. From the hypothesis on v_l, we find

$$\begin{aligned} \frac{\cosh(\sqrt{\beta})-1}{\beta} v_{l_1} v_{l_2} &= \left(\frac{\cosh(\sqrt{\beta})-1}{\beta}\right)\left(\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}\right)^{-2} 4a_{1,1}^2 a_{l_1,1} a_{l_2,1} \\ &= 2a_{l_1,1} a_{l_2,1}. \end{aligned}$$

Thus, from Lemmas 2.4 and 2.8, we find $[e^X]_{(2,2)} = [AA^{[w]}]_{(2,2)}$. Hence, by applying the discussion before the lemma, we obtain $e^X = AA^{[w]}$.

□

To conclude this subsection, we address the case when $\beta = -\pi^2$ and $w = e_{p+1}$. Then $a_{1,1} = 0$ and $\frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}} = 0$. Choose $x = (x_1, \dots, x_p)$ such that

$$\sqrt{\frac{\cosh(\sqrt{\beta})-1}{\beta}}\, x_k = \sqrt{2} b_{1,k}, \forall k \in \{1, \dots, p\}. \tag{13}$$

Then Statement 1 is satisfied necessarily. Choose $v \in \mathbb{R}^{q-1}$ satisfying Statement 2 and Corollary 2.10(b). Let $X \in \mathfrak{p}_w$ be defined by (7). Then $\alpha = \beta$, similarly. Applying (13), we find $[e^X]_{1,1} = [AA^{[w]}]_{1,1}$. By Corollary 2.10, we find $[e^X]_{1,2} = [AA^{[w]}]_{1,2}$, and $[e^X]_{2,1} = [AA^{[w]}]_{2,1}$. If $b_{1,k} \neq 0$ for some $k \in \{1, \dots, p\}$, then Lemma 2.11 is satisfied, and we find $[e^X]_{2,2} = [AA^{[w]}]_{2,2}$ by Corollary 2.12. However, if $b_{1,k} = 0$ for all $k \in \{1, \dots, p\}$, then $x_k = 0$ and identity (13) is trivially satisfied. If needed, change the sign of v_l such that

$$\sqrt{\frac{\cosh(\sqrt{\beta})-1}{\beta}}\, v_l = \sqrt{2} a_{l,1}, \forall l \in \{2, \dots, q\}. \tag{14}$$

Then Statement 2 and Lemma 2.11(b) are satisfied by (14). Thus, $[e^X]_{2,2} = [AA^{[w]}]_{2,2}$.

Corollary 2.15. *Let $w = e_{p+1}$, and let $\beta = -\pi^2$.*

(a) Suppose $b_{1,k} \neq 0$ for some $k \in \{1, \dots, p\}$. Choose $x = (x_1, \dots, x_p)^T \in \mathbb{R}^p$ satisfying Statement 1 and (13), and choose $v = (v_2, \dots, v_q)^T \in \mathbb{R}^{q-1}$ satisfying Statement 2, and Corollary 2.10(b). Let $X \in \mathfrak{p}_w$ be defined by (7).

(b) *Suppose* $b_{1,k} = 0$ *for all* $k \in \{1,\ldots,p\}$. *Choose* $x = (x_1,\ldots,x_p)^T \in \mathbb{R}^p$ *satisfying Statement 1, and* $v = (v_2,\ldots,v_q)^T \in \mathbb{R}^{q-1}$ *satisfying Statement 2, and (14). Then* $x = 0$. *Let* $X \in \mathfrak{p}_w$ *be defined by (7).*

Then $e^X = AA^{[w]}$.

Theorem 2.16. *Let* $w = e_{p+1}$, *and let* $A \in O(p,q)$. *Then there exists* $(X,K) \in \mathfrak{p}_w \times K_w$ *satisfying* $A = e^X K$. *That is,* A *has a* S_w*-polar decomposition.*

Proof We apply Corollaries 2.13, 2.14, and 2.15. Since $\mathfrak{p}_w$ is a subspace, we find $AA^{[w]} = e^{2X}$ for some $X \in \mathfrak{p}_w$. Then $A = e^X K$ for some $K \in O(p,q)$. Since $(e^{-X})^{[w]} = e^{-X}$, we find $KK^{[w]} = e^{-X}AA^{[w]}(e^{-X})^{[w]} = I$. Thus, $K \in K_w$.

□

We conclude this section by considering the special case when $\beta = 0$. Recall, $\beta = 0$ iff $a_{1,1}^2 = 1$. In (9), the $(p+1,p+1)$-entry of $A^T J_{p,q} A$ satisfies $a_{1,1}^2 = 1$ iff $\sum_{k=1}^p b_{1,k}^2 - \sum_{l=2}^q a_{l,1}^2 = 0$. Thus, $\beta = 0$ iff $A_{p+1,p+1}^2 = 1$. Also, we know $\beta = x^T x - v^T v$. Then $\beta = 0$ iff $X^3 = 0$ iff the minimum polynomial $p(t)$ of X is either t, t^2, or t^3 (see discussion before Corollary 2.2) iff the eigenvalues of $AA^{[w]} = e^{2X}$ are all 1's. In particular, if $A \in K_w$ then $AA^{[w]} = I$ and $\beta = 0$.

3 The case $w = e_p$

As described in the introduction, we consider a group isomorphism defined by an inner-conjugation or similarity between $O(p,q)$ and $O(q,p)$ as follows. We know $J_{q,p} = I_q \oplus (-I_p)$, and consider $(x,y)_{q,p} = x^T J_{q,p} y$ where $x, y \in \mathbb{R}^{p+q}$. Whenever $G \in GL_{p+q}(\mathbb{R})$, $G \in O(q,p)$ iff $G^T J_{q,p} G = J_{q,p}$. Let R be the backward identity matrix. Then $R^{-1} = R$ and $RJ_{p,q}R^{-1} = -J_{q,p}$. Recall, if $y \in \mathbb{R}^{p+q}$ and $\theta'_y = \frac{1}{2} y^T J_{q,p} y \neq 0$, then

$$S'_y \equiv I_{p+q} - (\theta'_y)^{-1} y y^T J_{q,p}$$

is a $J_{q,p}$-Householder matrix. The next lemma can be easily verified.

Lemma 3.1. *Let* $A \in O(p,q)$, $X \in \mathfrak{so}(p,q)$, *and let* $w \in \mathbb{R}^{p+q}$ *satisfy* $w^T J_{p,q} w \neq 0$.

(a) $RAR^{-1} \in O(q,p)$, *and* $RXR^{-1} \in \mathfrak{so}(q,p)$

(b) *The unit vectors* $e_k \in \mathbb{R}^{p+q}$ *satisfy* $Re_k = e_{p+q-k+1}$, $1 \leq k \leq p+q$.

(c) $w^T J_{p,q} w = -(Rw)^T J_{q,p} Rw$.

(d) $RS_w R^{-1} = S'_{Rw}$

(e) If $S_w X S_w^{-1} = -X$*, then* $S'_{Rw}(RXR^{-1})(S'_{Rw})^{-1} = -RXR^{-1}$.

(f) If $S_w A S_w^{-1} = A$*, then* $S'_{Rw}(RAR^{-1})(S'_{Rw})^{-1} = RAR^{-1}$.

Theorem 3.2. *Let* $w = e_p$*, and let* $A \in O(p,q)$*. Then there exists* $(X, K) \in \mathfrak{p}_w \times K_w$ *such that* $A = e^X K$ *is a* S_w*-polar decomposition.*

Proof From Lemma 3.1, we find $Re_p = e_{q+1}$ and $RAR^{-1} \in O(q,p)$. Applying Theorem 2.16 to $O(q,p)$, there exist $X \in \mathfrak{so}(q,p)$ and $K \in O(q,p)$ such that $S'_{e_{q+1}} X S'^{-1}_{e_{q+1}} = -X$, $S'_{e_{q+1}} K S'^{-1}_{e_{q+1}} = K$, and $RAR^{-1} = e^X K$. Thus, $A = e^{RXR^{-1}}(RKR^{-1})$ is a S_{e_p}-polar decomposition.

□

Finally, we combine Theorems 2.16 and 3.2, and the discussion in the introduction regarding S_w-polar decompositions and the sign of $w^T J_{p,q} w \neq 0$. Recall, $e_{p+1}^T J_{p,q} e_{p+1} = -1$ is negative and $e_p^T J_{p,q} e_p = 1$ is positive.

Theorem 3.3. *Let* $w \in \mathbb{R}^{p+q}$ *satisfy* $w^T J_{p,q} w \neq 0$*. If* $A \in O(p,q)$*, then there exist* $X \in \mathfrak{p}_w$ *and* $K \in K_w$ *such that* $A = e^X K$ *is a* S_w*-polar decomposition.*

An algorithm for finding a real square root of real matrix is discussed in [4]. It would be interesting to know whether the algorithm can find a square root of $AA^{[w]}$ in $\exp(\mathfrak{p}_w)$, if $w = e_{p+1}$.

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