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On Certain Expressions for the Identity Operator on Hilbert Space

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Abstract

In this paper, we will discuss the resolution of identity formula arising from unitary representations of various groups, those consisting of orthonormal bases and those consisting of overcomplete systems in Hilbert space. We will construct coherent states on the discrete cylinder and apply these to the quantization of time of arrival functions. We will also construct localization operators on the L^p-spaces on the infinite cylinder viewed as parameter space of translations and rotations, for $p = 1, 2$, and ∞. The case for general $p \in [1, \infty]$ requires some interpolation formula on the cylinder which the authors do not possess.

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1 Introduction

Coherent states were first introduced by E. Schrödinger [37] in his investigations of quantum states whose behavior are nearest to the classical one, for instance, the averages satisfy Newton's Second Law [24]. What are now called canonical coherent states or harmonic oscillator coherent states are characterized by several properties that make them mathematically interesting and useful [3, 29]. In this paper, the most important property of coherent states is their use in the decomposition of the identity operator in Hilbert space, from which most of the important applications arise. In [5], canonical coherent states were used in the study of Hilbert spaces of analytic functions. R. Glauber used the canonical coherent states [10] in quantum optics to model coherent phenomena such as the then recently discovered laser.

In [10, 11], coherent states were used extensively to decompose, akin to Fourier decomposition, quite general quantum states and observables. In these works and also in [1, 2] and [41], coherent states were used in the study of phase space representation of states and observables via what are now called *quasiprobability distributions*, as well the fundamental problem of *operator ordering*. The ordering problem has been one of the important problems recognized quite early in the history of quantum mechanics, and is crucial in quantum measurement. For instance, the Wigner quasiprobability distribution appear as symbols for the Weyl ordering, while the $P-$distribution, also called the Glauber-Sudarshan distribution, and the $Q-$distribution (or the Husimi distribution) are symbols of the normal and anti-normal orderings, respectively [3, 7, 10, 11, 29, 12]. Symbols are phase space functions corresponding to quantum operators under a quantization/dequantization scheme. The use of coherent states in quantization is advocated by J.-P. Gazeau and coworkers [3, 7, 24], due to the success of previous investigations on the $P-$ and $Q-$distributions and their utility in quantum optics. It was Klauder and Berezin who introduced the general theory of coherent state quantization and dequantization, with Berezin giving emphasis on the mathematical developments and connections [28, 29]. Applications to signal analysis was taken up by Daubechies and coworkers [13, 14, 15] via the wavelet transform, introduced in [26], and whose development is intimately connected with unitary group representations. Most modern mathematical treatments of generalizations of the wavelet transform at present rely on unitary group representations as this provide the most fruitful line of investigation.

This work presents myriad expressions for the identity operator on Hilbert space $\mathcal{H}$. The main object of focus is that of coherent states arising from, but not restricted to, unitary irreducible representations of groups. We provide several new examples from our own work in phase *space representation of quantum mechanics* [31, 36]. Coherent states will mostly be constructed from the harmonic analysis of groups via unitary group representations. In particular, we know of no example of coherent state quantization of a confined time of arrival function [23]. On the other hand, our coherent state quantization of the discrete cylinder provides provides a simpler and viable quantization of angle, in contrast to the complicated Weyl quantization in [32, 33]. Furthermore, the theory of coherent states is very rich that we strongly feel that they should be introduced to mathematics students (especially in the Philippines). For advanced mathematics students, coherent states is an excellent path to learn quantum mechanics, which is historically the source of much important development in mathematics, for example, linear operators, spectral theory, unitary group representations and C^*-algebras, to name a few [5, 6, 8, 15, 18, 35, 42].

The plan of the paper is as follows. In Section 2, a quick introduction to quantum mechanics is given, emphasizing the state-observable formalism, in analogy with classical mechanics. Then the bra-ket notation of Dirac is explained [34]. Next, the basic properties of the harmonic oscillator coherent states are listed and discussed, one of which is the resolution of the identity operator in the Hilbert space $L^2(\mathbb{R})$. Other basic examples of the resolution of identity are given coming from the Fourier theory. Section 3 generalizes the idea using unitary group representations mainly through the Peter-Weyl Theorem. In Section 4, genuine expressions for the resolution of the identity will be given using overcomplete families of spanning vectors or coherent states. Section 5 continues the discussion in Section 4, but through the development of basic ideas in phase space quantum mechanics, the time-frequency, and wavelet representation of signals. In the final section, Section 6, recent results of the authors on the applications of coherent states to quantization and localization operators will be presented.

2 Preliminaries and First Examples

We now give a brief introduction to the mathematical formalism of standard quantum mechanics and some of the notations used which may not be familiar to a mathematical audience. In classical mechanics, for a system with one degree of freedom and without constraints, the phase space is $\mathbb{R}^2 = \{(q,p)\}$, the set of all possible position-momentum pairs (q,p). Elements of the phase space are called states. Classical observables are functions $f(q,p) = f(q(t),p(t))$ on the set of states. There is a distinguished observable $H(q,p)$ called the Hamiltonian or energy of the system which determines the evolution of the system, and is governed by Hamilton's equation $df/dt = \{H,f\}$, in terms of the Poisson bracket between a pair of functions, defined by $\{f,g\} = \partial_q f \partial_p g - \partial_p f \partial_q g$. For the simple harmonic oscillator the energy observable is $H(q,p) = p^2/2m + (1/2)m\omega^2 q^2$, where m is mass and ω is frequency. A simple computation tells us that Hamilton's equation implies Newton's Second Law $d^2q/dt = -\omega^2 q$ for this system. Similar to classical mechanics, quantum mechanics is a state-observable system. Quantum states are normalized vectors ψ in a Hilbert space $\mathcal{H}$, called the system Hilbert space, with the proviso that two vectors ψ and ψ' represent the same state if $\psi' = e^{ik}\psi$, for some real number k. Quantum observables are self-adjoint operators $\widehat{A} : \mathcal{H} \longrightarrow \mathcal{H}$, and there is a distinguished observable $\widehat{H}$ which determines the evolution of the system, given by $d\widehat{A}/dt = (i/\hbar)[\widehat{H},\widehat{A}]$, where $[\widehat{H},\widehat{A}]$ is the commutator $\widehat{H} \circ \widehat{A} - \widehat{A} \circ \widehat{H}$. Canonical quantization gives $\widehat{H} = \widehat{P}/2m + (1/2)m\omega^2\widehat{Q}^2$ for the quantum Hamiltonian of the harmonic oscillator. The operators $\widehat{Q}$, $\widehat{P} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ are defined by $\widehat{Q}\psi(q) = q\psi(q)$, $\widehat{P}\psi(q) = -i\hbar\psi'(q)$, respectively [20, 42].

For our purposes, throughout this paper, we will use the bra-ket notation of Dirac. States will be written $|\psi\rangle$ instead of ψ. In a Hilbert space, there is a one-to-one correspondence between vectors and linear functionals. Linear functionals will be denoted by bra vectors $\langle\psi| : \mathcal{H} \longrightarrow \mathbb{C}$ so that the action on vectors is consistent with the inner product notation: $\langle\psi|(|\varphi\rangle) = \langle\psi|\varphi\rangle$. An important linear operator that will be considered all throughout is the projection $|\psi\rangle\langle\varphi| : \mathcal{H} \longrightarrow \mathcal{H}$ given by $|\psi\rangle\langle\varphi|(|\alpha\rangle) = \langle\varphi|\alpha\rangle|\psi\rangle$.

The family of harmonic oscillator coherent states, which are the most important example of coherent states, consists of the states $|q,p\rangle = |z\rangle \in L^2(\mathbb{R})$ parametrized by $z = q + ip \in \mathbb{C}$. The characteristic properties of these states are the following:

1. $\langle\triangle\widehat{Q}\rangle_z \langle\triangle\widehat{P}\rangle_z = \frac{\hbar}{2}$
2. $|z\rangle = e^{\frac{i}{\hbar}(z\widehat{a}^\dagger - \overline{z}\widehat{a})}|0\rangle$
3. $\widehat{a}|z\rangle = z|z\rangle$
4. $I = \frac{1}{\pi}|z\rangle\langle z|dzd\overline{z}$

Item (1) states that coherent states $|z\rangle$ minimize the Heisenberg uncertainty inequality, while the second property says that such states belong to the orbit of a certain group action. The last two say that the coherent states form a complete system of eigenstates of the annihilation operator $\widehat{a}$, familiar from textbook analysis of the quantum harmonic oscillator. The so-called resolution of the identity, in item (4), is interpreted in the weak sense. That is

$$\langle\psi|\varphi\rangle = \frac{1}{\pi}\int_{\mathbb{C}} \langle\psi|z\rangle\langle z|\varphi\rangle dzd\overline{z}.$$

In many important cases, the much stronger relation of state or signal recovery is possible $|\varphi\rangle = \frac{1}{\pi}\int_{\mathbb{C}} |z\rangle\langle z|\varphi\rangle dzd\overline{z}$, in complete analogy with Fourier decomposition of square-integrable functions. If such a relation can be discretized, then many important applications are possible [12, 25]. Further details on the canonical coherent states may be found in [19, 24].

We now look at the simplest example of a complex finite-dimensional vector space $V \cong \mathbb{C}^n$. Let $\{|k\rangle \in V : k = 1, ..., n\}$ be an orthonormal basis, so that we have the usual Fourier decomposition $|v\rangle = \sum_{k=1}^{n} \langle k|v\rangle |k\rangle$. This is equivalent to Parseval's identity $\langle u|v\rangle = \sum_{k=1}^{n} \langle u|k\rangle\langle k|v\rangle$. We, therefore, see here the simplest instance of the resolution of the identity

$$I_V = \sum_{k=1}^{n} |k\rangle\langle k|. \tag{1}$$

Consider the sequence space $\ell^2(\mathbb{N})$ consisting of vectors $|a\rangle = (a_1, a_2, a_3, ...)$, where $a_k \in \mathbb{C}$. We have the exact same Fourier decomposition, using say the standard basis elements $|k\rangle$ consisting of 1 at the kth site and zero elsewhere: $|a\rangle = \sum_{k=1}^{\infty} \langle k|a\rangle |k\rangle$. This is equivalent to Parseval's identity $\langle b|a\rangle = \sum_{k=1}^{\infty} \langle b|k\rangle\langle k|a\rangle$, so that we again obtain the resolution of identity $I_{\ell^2(\mathbb{N})} = \sum_{k=1}^{\infty} |k\rangle\langle k|$. Any separable Hilbert space $\mathcal{H}$ is isomorphically isometric to $\ell^2(\mathbb{N})$ so that a resolution of the identity on $\mathcal{H}$ is always available [20, 40].

We look particularly at the important case of the Hilbert space $\mathcal{H} = L^2(\mathbb{T})$, where $\mathbb{T}$ is the one-dimensional torus $[0, 2\pi)$. An orthonormal basis is given by the functions $\chi_n : \mathbb{T} \longrightarrow \mathbb{C}$, $\chi_n(x) = e^{inx}$, $n \in \mathbb{Z}$. If $f, g \in L^2(\mathbb{T})$, then we have Parseval's identity

$$\langle g|f\rangle = \sum_{n\in\mathbb{Z}} \langle g|\chi_n\rangle\langle \chi_n|f\rangle = \sum_{n\in\mathbb{Z}} \overline{\widehat{g}(n)}\widehat{f}(n),$$

where $\widehat{f}(n) = \langle \chi_n|f\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$ is the nth Fourier coefficient of the function f. We thus obtain the resolution of the identity

$$I_{L^2(\mathbb{T})} = \sum_{n\in\mathbb{Z}} |\chi_n\rangle\langle\chi_n|.$$

This identity implies the stronger Fourier series decomposition (equivalent to Parseval's identity, of course) $|f\rangle = \sum_{n\in\mathbb{Z}} \langle \chi_n|f\rangle |\chi_n\rangle = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{in(\cdot)}$[20, 40].

A non-discrete example. Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. Similar to the Fourier series decomposition of square-integrable functions on the circle, square-integrable complex-valued functions on the real line $\mathbb{R}$ may be expanded in terms of pure plane waves $\chi_\xi(x) = e^{i\xi x}$. Indeed, we have the Fourier inversion formula $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{i\xi x}d\xi$, where $\widehat{f}(\xi) = \langle\chi_\xi|f\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$ defines the Fourier transform $f \mapsto \widehat{f}$. The resolution of the identity in this case takes the integral form

$$I_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |\chi_\xi\rangle\langle\chi_\xi|\, d\xi.$$

In these examples, except for the last one, orthonormal basis elements were used to decompose the identity operator. Notice that in the last example, the functions χ_ξ do not belong to $L^2(\mathbb{R})$[20, 40]!

3 Harmonic Analysis on Groups

To unify the examples above, we look at the unitary representations of abelian groups. We introduce some general definitions first. If G is a group, a unitary representation of G

in the inner product space $\mathcal{H}$ is a group homomorphism $T : G \longrightarrow \mathcal{U}(V)$ into the group of unitary operators on $\mathcal{H}$. Thus, $T(e) = I_{\mathcal{H}}$ is the identity operator, $T(gh) = T(g)T(h)$ is the homomorphism property, $T(g)^{-1} = T(g^{-1})$ is the inverse and $\langle T(g)u, T(g)v\rangle = \langle u, v\rangle$ for any $g \in G$ and $u, v \in \mathcal{H}$. This last property is unitarity of the operators $T(g) : \mathcal{H} \longrightarrow \mathcal{H}$. For many applications, the groups are also smooth manifolds, so some continuity property is required of T. For instance, if G is a Lie group, then the mapping $G \longrightarrow V$ given by $g \mapsto T(g)v$ must be continuous for each $v \in V$. If U is a subspace of $\mathcal{H}$, it is called an invariant subspace of T if $T(g)u \in U$ for all $g \in G, u \in U$. The representation T is said to be irreducible if the only invariant subspaces are $\{0\}$ and $\mathcal{H}$. For finite and compact groups, and most groups of importance (classical Lie groups), the unitary irreducible representations, abbreviated to UIR, are sufficient to decompose L^2-functions on the groups and implement harmonic analysis on the groups and homogeneous spaces of the groups. For finite and compact groups G, there are countably many UIRs, up to equivalence, and each is finite-dimensional, so one may choose orthonormal bases in the representation spaces and express matrix elements in the bases. A well-known and important result is that the matrix elements of the UIRs form an orthonormal basis for $L^2(G)$ [38, 40]. This allows for harmonic analysis on G, which is completely parallel to the standard ones on $G = S^1$ (Fourier series) and on $G = \mathbb{R}$ (Fourier transform). For abelian groups, the UIRs are all one-dimensional.

1. In case of the one-dimensional torus $\mathbb{T}$, the UIRs are the homomorphisms $\chi_n : \mathbb{T} \longrightarrow \mathcal{U}(\mathbb{C}) \cong \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$, parametrized by the integers, and given by $\chi_n(x) : z \mapsto e^{inx}z$, a multiplication operator.

2. In case $G = \mathbb{R}$, the UIRs are parametrized by the real numbers: $\chi_\xi : \mathbb{R} \longrightarrow \mathcal{U}(\mathbb{C})$, again given by multiplication $\chi_\xi(x) : z \mapsto e^{i\xi x}z$.

3. For the finite-dimensional case in the very first example, $G = \mathbb{Z}_n$ and $V = \mathcal{F}(G)$, which is the space of all complex-valued functions on a finite set of n elements. This is nothing but $\mathbb{C}^n$, as indicated above in the paragraph of equation (1). For the group $\mathbb{Z}_n$, the UIRs form a group $\{|k\rangle : k = 0, ..., n-1\} = \{\chi_k : \mathbb{Z}_n \longrightarrow \mathbb{C}^\times : \chi_k(j) = e^{2\pi ikj/n}\}$ isomorphic to $\mathbb{Z}_n$ and the Fourier decomposition of functions in $\mathcal{F}(\mathbb{Z}_n)$ is precisely what is given above, entirely analogous to the case $G = \mathbb{Z}$ or $G = \mathbb{R}$.

Further details may be found in [38, 40].

Nonabelian examples. It is precisely through unitary group representations that harmonic analysis on groups other than the abelian ones is possible. If G is a compact group, including the finite ones, there are countably many inequivalent unitary irreducible representations $\rho^n : G \longrightarrow \mathcal{U}(V_n)$ and each V_n is finite-dimensional, say $\dim V_n = d_n$. Choosing orthonormal basis $\{e_k\}$ in V_n, matrix elements $u^n_{ij} : G \longrightarrow \mathbb{C}$ are given by $u^n_{ij}(g) = \langle e_i|\rho^n(g)e_j\rangle$. The Peter-Weyl Theorem states that the set $\{\sqrt{d_n}u^n_{ij}\}_{n,i,j}$ forms an orthonormal basis for $\mathcal{H} = L^2(G)$ [40]. This means that completely analogous to classical Fourier series, for $f \in L^2(G)$,

$$f = \sum_{n,i,j} d_n\langle u^n_{ij}|f\rangle u^n_{ij}.$$

Using the bra-ket notation, $|f\rangle = \sum_{n,i,j} d_n\langle u^n_{ij}|f\rangle|u^n_{ij}\rangle$. We therefore have the resolution of identity

$$I_{L^2(G)} = \sum_{n,i,j} d_n|u^n_{ij}\rangle\langle u^n_{ij}|.$$

The permutation group $G = S_3$.

$G = S_3$ has 3 unitary irreducible representations:

- identity representation$-$1-dimensional
- permutation representation$-$1-dimensional
- standard action by $S_3 \cong D_3$$-$2-dimensional

The matrix elements are the six functions: $f_1 \equiv 1, f_2(\sigma) = (-1)^{\operatorname{sgn}\sigma}$, and the functions f_k below

$$\begin{pmatrix} f_3 & f_4 \\ f_5 & f_6 \end{pmatrix}$$

where the values are the corresponding entries of the following matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix},$$

$$\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}.$$

Here, $\operatorname{sgn}\sigma$ is 1 if σ is an even permutation, and -1 if σ is an odd permutation and the values of f_3 are the entries a_{11}, the values of f_4 are the entries f_{12}, and so on. The resolution of identity on $\mathcal{H} = L^2(S_3) \cong \mathbb{C}^6$ is given by the sum

$$\widehat{I} = \sum_{k=1}^{6} d_k |f_k\rangle\langle f_k|.$$

Here, $d_1 = d_2 = 1$ while $d_i = 2, i = 3, 4, 5, 6$.

The spin group $G = \mathrm{SU}(2)$.

SU(2) is the group of two by two unitary complex matrices $\begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix}$, $|a|^2 + |b|^2 = 1$. Equivalently, in terms of Euclidean norm, $||g \cdot X|| = ||X||$, and $\det g = 1$. Here, $g \in \mathrm{SU}(2)$ and $X \in \mathbb{C}^2$.

It is well known that the UIRs of SU(2) are parametrized by the nonnegative integers $U^n : G \longrightarrow \mathcal{H}_n$, where $\mathcal{H}_n \cong \mathbb{C}^{n+1}$ and U^n is standard action of the matrices g on homogeneous polynomials in 2 variables. This means that $[U^n(g)P](z) = P(g^{-1} \cdot z)$. In more details, the action is entirely determined by $U^n(g)(z_1^k z_2^{n-k}) = (az_1 + bz_2)^k (cz_1 + dz_2)^{n-k}$, where $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By the Peter-Weyl theorem, the matrix elements given by the inner product $U^n_{ij} := \langle e_i | U^n(g) e_j \rangle$ form an orthonormal basis for $L^2(\mathrm{SU}(2))$ [40] and this gives the resolution of the identity $I = \sum_{n,i,j} (n+1) |U^n_{ij}\rangle\langle U^n_{ij}|$. SU(2) is a double cover of the three-dimensional rotation group SO(3) so that the representation U^n factors through a UIR of SO(3). Each n gives a UIR on $\mathcal{H} \cong \mathbb{C}^{2n+1}$ and once again, this gives a resolution of the identity $I_{L^2(\mathrm{SO}(3))}$.

4 Group Coherent States

It turns out that the ideas in the previous section can be generalized considerably. The idea is provided by the case $\mathcal{H} = L^2(\mathbb{R})$ and the resolution of identity arising from the Fourier inversion formula. However, we want the system of vectors $|e_\alpha\rangle$ used in the decomposition of the identity operator to lie in $\mathcal{H}$, which is not the case in Fourier inversion. On the other hand, as in Fourier inversion, we do not insist on orthonormal bases to decompose the identity, which is the case for all the other Hilbert spaces considered above. We allow what are called overcomplete systems in Hilbert space [9]. Proposition 3 below is a more general statement than what is found in the literature on tight group frames, and the example on the rotation group at the end of this section, although a direct application of a general construction, is our own observation, inspired by the planar example before it.

Definition 4.1. Let $\mathcal{H}$ be a Hilbert space and X be a measure space with measure $d\mu$. The family of vectors $\mathcal{A} = \{|e_\alpha\rangle : \alpha \in X\} \subset \mathcal{H}$ is called an overcomplete system in $\mathcal{H}$ or a system of coherent states if

$$\langle f|g\rangle = \int_X \langle f|e_\alpha\rangle\langle e_\alpha|g\rangle d\mu(\alpha)$$

for any pair $|f\rangle, |g\rangle \in \mathcal{H}$.

Therefore, a vector $|f\rangle$ in $\mathcal{H}$ uniquely determines a function $\widetilde{f} \in L^2(X)$ given by $\widetilde{f}(\alpha) = \langle f|e_\alpha\rangle$ so that $\mathcal{H}$ embeds in $L^2(X)$ and since $\langle f|f\rangle = \int_X \langle f|e_\alpha\rangle\overline{\langle f|e_\alpha\rangle} d\mu(\alpha)$, the embedding is an isometry, that is $||f||_{\mathcal{H}} = ||\widetilde{f}||_{L^2(X)}$. Usually, there is a further requirement that the mapping $\alpha \mapsto |e_\alpha\rangle$ be continuous [29]. We do not impose this requirement as we will consider finite and discrete measure spaces X. The discrete topology restores this continuity condition but is immaterial in this case. What is more important in the discrete case is the concept of frames [15].

Definition 4.2. Let $\mathcal{H}$ be a Hilbert space and J be a countable set. A family of vectors $\mathcal{A} = \{|e_j\rangle : j \in J\}$ is called a frame in $\mathcal{H}$ if there exist positive constants $\alpha \leq \beta$ such that

$$\alpha||f||^2 \leq \sum_{j\in J} |\langle f|e_j\rangle|^2 \leq \beta||f||^2, \text{ for all } f \in \mathcal{H}.$$

If $\alpha = \beta$ then the $|e_j\rangle$ are coherent states or that $\mathcal{A}$ is a tight frame. The inequality on the left implies that the embedding $|f\rangle \mapsto \widetilde{f}(i) = \langle f|e_j\rangle$ is one-to-one, while the inequality on the right implies that this mapping is bounded. In this paper, we will be only interested in the case where $\alpha = \beta$.

A basic example. Let G be a finite group. Fix a unitary irreducible representation $\rho : G \longrightarrow \mathcal{U}(\mathcal{H})$ in the Hilbert space $\mathcal{H}$ and fix a vector $|0\rangle \in \mathcal{H}$. Write $|x\rangle = \rho(x)|0\rangle$.

In the following proposition, we obtain whole families of resolution of the identity on finite-dimensional Hilbert spaces and on the space of linear operators on them. Note that the standard formulation is $\sum_{x\in G} |x\rangle\langle x| = c\widehat{I}$, while our statement is $\sum_{x\in G} |x\rangle\langle y| = c\widehat{I}$. We do not find this exact statement in the literature, but it is probably folklore. We do not find it used in an explicit way either. A work in preparation of the first author on discrete quantum mechanics uses this expression in a formulation of sequential position-momentum and sequential momentum-position measurments.

Proposition 4.3. *Let G be a finite group and ρ be a UIR of G in $\mathcal{H}$.*

1. *Write $|x\rangle = \rho(x)|0\rangle$. Then $\{|x\rangle : x \in G\}$ is a tight frame in $\mathcal{H}$.*
   In fact, more generally
$$\frac{\dim \mathcal{H}}{\alpha} \sum_{x,y \in G} |x\rangle\langle y| = \widehat{I}, \tag{2}$$
   where $\alpha = \mathrm{Tr}\big(\sum |x\rangle\langle y|\big)$.

2. *The set $\{\rho(x) = |\rho(x)\rangle : x \in G\}$ is a tight frame on $\mathcal{B}(\mathcal{H})$, where $\langle \widehat{A}|\widehat{B}\rangle = Tr(\widehat{A}^* \widehat{B})$ and $\mathcal{B}(\mathcal{H})$ is the space of all linear operators on $\mathcal{H}$. Indeed,*
$$\left(\frac{\dim \mathcal{H}}{|G|}\right)^2 \sum_{x,y} |\rho(x)\rangle\langle \rho(y)| = \widehat{I}.$$

Proof. The proof is a standard Schur Lemma argument [38, 40]. If $\widehat{S} = \sum_{x,y\in G} |x\rangle\langle y|$ we have that $\rho(z)\widehat{S}|u\rangle = \sum_{x,y} |zx\rangle\langle z^{-1}zy|u\rangle = \sum_{x,y} |zx\rangle\langle zy|\rho(z)u\rangle = \widehat{S}\rho(z)|u\rangle$. Thus, $\rho(z)\widehat{S} = \widehat{S}\rho(z)$ for any $z \in G$. By Schur Lemma, this implies the the operator $\widehat{S}$ is a multiple of the identity operator on $\mathcal{H}$: $\widehat{S} = c\widehat{I}$. Taking the trace of both sides gives the constant $c = \frac{\mathrm{Tr}\widehat{S}}{\dim \mathcal{H}}$. The homomorphism property was used in $\rho(z)|x\rangle = |zx\rangle$.

For the second assertion, the Schur Lemma again applies, so that $\sum_{x,y\in G} |\rho(x)\rangle\langle\rho(y)| = cI$. Taking the trace of both sides give
$$\sum_{1\le k,l\le \dim\mathcal{H}} \sum_{x,y\in G} \langle E_{kl}|\rho(x)\rangle\langle\rho(y)|E_{kl}\rangle,$$
where $\langle E_{kl}|\rho(x)\rangle = \mathrm{Tr}(E_k^* \rho(x)) = a_{kl}(x)$, E_{kl} being the matrix consisting of 1 at the (k,l) entry and zero elsewhere. We then obtain, using properties of matrix elements of UIRs, $\sum_{x,y}\sum_{k,l} a_{kl}(x)a_{kl}(y^{-1}) = c\dim\mathcal{H}$, which gives $c = \sum_{x,y} \frac{|G|}{(\dim\mathcal{H})^2}\delta_{x,y^{-1}} = (\frac{|G|}{\dim\mathcal{H}})^2$.

□

Due to the equation $|G| = d_1^2 + \cdots d_r^2$, where $|G|$ denotes the cardinality of G, d_i is the dimension of the ith UIR and r is the total number of inequivalent UIRs (equal to the number of conjugacy classes of G), the families $\{|x\rangle : x \in G\}$ and $\{|\rho(x)\rangle = \rho(x)I : x \in G\}$ of coherent states far exceeds the dimension of $\mathcal{H}$, that is, we obtain a resolution of the identity without the use of orthonormal bases.

A limiting case [24]. For the group $\mathbb{Z}_n$, irreducible representations $\chi_k : G \longrightarrow \mathbb{C}^\times$, $\chi_k(l) = e^{2\pi ikl/n}$ are considered as rotation matrices
$$a^k = \begin{pmatrix} \cos(2\pi i/n) & -\sin(2\pi i/n) \\ \sin(2\pi i/n) & \cos(2\pi i/n) \end{pmatrix}^k.$$
A frame consisting of l elements $\{|k,l\rangle = a^{kl}|u\rangle : l = 0,\dots,n-1\}$ is formed on $\mathbb{R}^2$, where $|u\rangle = \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \end{pmatrix}$. Then
$$\widehat{I} = \frac{2}{n}\sum_{l=0}^{n-1} |k,l\rangle\langle k,l|$$

goes to

$$\widehat{I} = \frac{1}{\pi}\int_0^{2\pi} |\theta\rangle\langle\theta|\, d\theta = \frac{1}{\pi}\int_0^{2\pi} d\theta \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \cos^2\theta \end{pmatrix}$$

for the choice $\theta_0 = 0$ in the limit $n \to \infty$ [24].

The rotation group. Consider again the compact group $G = \mathrm{SO}(3)$. Recall that for each odd dimension $2n+1$, there is a UIR on $\mathcal{H}_n \cong \mathbb{C}^{2n+1}$. Choose a spanning set of $2n+1$ by $2n+1$ elements, say, an orthonormal basis of $2n+1$ elements repeated $2n+1$ times, $\{|e^n_{st}\rangle\}$. Construct $|x\rangle \in \mathcal{H}_n$ by $G = SO(3) \ni x \mapsto |x\rangle = \sum_{1\le s,t\le 2n+1} a^n_{st}(x)|e^n_{st}\rangle$, where a^n_{st} are matrix elements of the nth UIR. Using the normalized Haar integral on the group and orthogonality properties of matrix elements [38],

$$\begin{aligned}\int |x\rangle\langle x| d\mu(x) &= \sum_{st,ij}\left[\int a^n_{st}(x)\overline{a^n_{ji}(x)}\, d\mu(x)\right] |e^n_{st}\rangle\langle e^n_{ji}| \\ &= \frac{2n+1}{d_n}\sum_{st} |e^n_{st}\rangle\langle e^n_{st}| \\ &= \widehat{I},\end{aligned}$$

since the dimension $d_n = 2n+1$. This gives a family of coherent states on $\mathbb{C}^{2n+1}$ parametrized by $G = \mathrm{SO}(3)$. This example is a direct application of a more general construction [7] than what is considered in this paper. It is similar to the limiting case above, where a continuous family of coherent states is available for finite-dimensional space, but here we obtain them for all the representation spaces.

In the next section, we present further examples of coherent states arising from UIRs of groups, and point out their role in quantum mechanics and signal analysis.

5 Phase Space Quantum Mechanics

The phase space representation of a quantum system is a formulation relying on phase space functions instead of on Hilbert space and linear operators on the Hilbert space. In the correspondence $\widehat{A} \leftrightarrow f_{\widehat{A}}$, the product of quantum operators $\widehat{A}\circ\widehat{B}$ gives rise to the relation $f_{\widehat{A}\circ\widehat{B}} = f_{\widehat{A}} \star f_{\widehat{B}}$. As long as the correspondence is one-to-one, which is the case for square integrable functions and Hilbert-Schmidt operators, this relation is well-defined, and all C^*-algebra properties of operators are transferred to the space of phase space functions under the operation of star-product $\star : (f,g) \mapsto f \star g$. This gives an autonomous and equivalent formulation of quantum mechanics in terms of classical phase space functions. In general, the correspondence is of the form $\widehat{A} \leftrightarrow f_{\widehat{A}} = f_0 + \hbar f_1 + \hbar^2 f_2 + \hbar^3 f_3 + \cdots$ [4, 6, 18] so that one is lead to the conclusion that quantum mechanics is a deformation of classical mechanics. The classical limit is obtained by letting $\hbar \to 0$ [6, 9, 18]. Furthermore, the commutator bracket $[\widehat{A},\widehat{B}]$ goes to $[f_{\widehat{A}}, f_{\widehat{B}}]_\star = \frac{1}{2i\hbar}(f_{\widehat{A}}\star f_{\widehat{B}} - f_{\widehat{B}}\star f_{\widehat{A}})$ so that as $\hbar \to 0$ the classical limit is the classical Poisson bracket $\{f_{\widehat{A}}, f_{\widehat{B}}\}$. It is the authors' belief that the classical-to-quantum correspondence is best analyzed through phase space quantum mechanics, also called deformation quantization [4, 6, 18, 39, 46]. This one-to-one correspondence may be implemented using the Weyl-Wigner correspondence to be discussed next.

For purposes of illustrating the main ideas, consider a one-particle system where the system Hilbert space is $\mathcal{H} = L^2(\mathbb{R})$. For the simplest operator given by the projection $|\psi\rangle\langle\varphi|$:

$\mathcal{H} \longrightarrow \mathcal{H}$, the phase space function corresponding to it in the Weyl-Wigner formalism is given by the Wigner transform $(\psi, \varphi) \mapsto W_{\psi,\varphi}$, where

$$\begin{aligned} W_{\psi,\varphi}(q,p) &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}py}\psi(q+y/2)\overline{\varphi(q-y/2)}dy \\ &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}py}\langle q+y/2|\psi\rangle\langle\varphi|q-y/2\rangle dy. \end{aligned}$$

In general, for Hilbert-Schmidt operators $\widehat{A}$, which are limits of sequences of finite linear combinations of projections $|\psi\rangle\langle\varphi|$, the Wigner distribution function is

$$W_{\widehat{A}}(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}py}\langle q-y/2|\widehat{A}|q+y/2\rangle dy.$$

The original proposal of E. Wigner in 1932 [43] is

$$W_{\psi}(q,p) = W_{\psi,\psi}(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}py}\psi(q+y/2)\overline{\psi(q-y/2)}dy,$$

in his investigation of thermodynamic equilibrium at low temperature, where quantum effects are important. The Wigner function $W_{\psi}(q,p)$ has many important and useful properties which makes it a more than viable representation for the quantum state ψ [12, 21, 44]. If $||\psi|| = 1$, which is the condition for being a quantum state, then $\iint W_{\psi}(q,p)dqdp = 1$. The marginality properties hold $\int_{-\infty}^{\infty} W_{\psi}(q,p)dp = |\psi(q)|^2$, $\int_{-\infty}^{\infty} W_{\psi}(q,p)dq = |\widehat{\psi}(p)|^2$ as well as some covariance property, which says the "probabilities" are preserved under translation. The Wigner function is not a true probability distribution since it may take negative values [20, 27]. The Weyl transform $\sigma(q,p) \mapsto \mathcal{W}_{\sigma}$ which maps phase space functions to Hilbert-Schmidt operators may be obtained as the unique linear operator for which the following fundamental formula is true: $\langle\psi|\mathcal{W}_{\sigma}\varphi\rangle = \iint_{\mathbb{R}^2} \sigma(q,p)W_{\psi,\varphi}(q,p)dqdp$ [44].

There is therefore a one-to-one correspondence $\sigma \mapsto \mathcal{W}_{\sigma}$ so that one may define the so-called star-product of phase space functions by

$$\sigma_1 \star \sigma_2 = \mathcal{W}^{-1}(\mathcal{W}_{\sigma_1} \circ \mathcal{W}_{\sigma_2}).$$

Most important for us is the following relation involving the Wigner function, called Moyal's identity

$$\iint W_{\psi}(q,p)W_{\varphi}(q,p)dqdp = |\langle\psi|\varphi\rangle|^2.$$

To see why the Moyal formula is important, we look again at unitary irreducible group representations in the following subsections.

5.1 Coherent States and the Heisenberg group

The Heisenberg group $G = \mathbb{H}_3$ has the underlying space $\mathbb{R}^3 = \{(q,p,t)\}$. The group operation is given by

$$(q_1,p_1,t_1)\cdot(q_2,p_2,t_2) = (q_1+q_2, p_1+p_2, t_1+t_2+\frac{1}{2}(q_1p_2-q_2p_1)).$$

Associate to (q,p,t) the matrix $m(q,p,t) = \begin{pmatrix} 0 & q & t \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix}$ so that

$$\exp m(q_1,p_1,t_1) \exp m(q_2,p_2,t_2) = \exp m(q_1+q_2, p_1+p_2, t_1+t_2+\frac{1}{2}(q_1p_2-q_2p_1)).$$

This multiplication of matrices gives us the group operation. Here the exponential of a matrix is the usual one: $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$.

The UIRs of $\mathbb{H}_3$ are all equivalent to the Schrödinger representation

$$\rho : \mathbb{H}_3 \longrightarrow \mathcal{U}(L^2(\mathbb{R})), \;\; [\rho(q,p,t)f](x) = e^{it+\frac{i}{2}qp+iqx} f(x+p).$$

In terms of infinitesimal operators of $\mathbb{H}_3$ (that is, the generators of the Lie algebra of $\mathbb{H}_3$), $\widehat{I}, \widehat{Q}, \widehat{P}$, where $[\widehat{Q}, \widehat{P}] = i\widehat{I}$, the Schrödinger representation may be written $\rho(q,p,t) = e^{i(t\widehat{I}+q\widehat{Q}+p\widehat{P})}$. Restricting to $\mathbb{H}_{\text{red}} = \mathbb{R}^2 \times (\mathbb{R}/2\pi\mathbb{Z})$ and then extending to an algebra representation to $L^1(\mathbb{H}_{\text{red}})$ we obtain $\widetilde{\rho}(q,p,0)F = \int_{\mathbb{R}^2} F(q,p)\rho(q,p)\, dqdp : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$. This is an irreducible unitary representation of the algebra $L^1(\mathbb{R}^2)$ [20]. This is all standard procedure, and we mention it for the sake of completeness. The construction is a step away from the Weyl quantization, already encountered above, and given by $F \mapsto \mathcal{W}_F = \widetilde{\rho}(q,p,0) \circ (\mathcal{F}F) = \int_{\mathbb{R}^2} (\mathcal{F}F)(x,\xi)\rho(x,\xi) dx d\xi : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$. Here, $\mathcal{F}F$ is the Fourier transform of the phase space function F [42].

On the other hand, the Wigner function corresponding to the state $\psi \in L^2(\mathbb{R})$ may be given, like Weyl quantization, in terms of the Schrödinger representation of $\mathbb{H}_3$. It is a Fourier transform $W_{\psi,\varphi}(q,p) = \mathcal{F}\langle\psi|\rho(q,p)\varphi\rangle$. The Moyal identity is then given by

$$\langle\psi|\phi\rangle\langle\varphi|\varphi\rangle = \int_{\mathbb{R}^2} \langle\psi|\rho(q,p)\varphi\rangle\langle\rho(q,p)\varphi|\phi\rangle dqdp,$$

by using the Plancherel identity $||f|| = ||\widehat{f}||$. If φ is a fixed unit vector, we obtain $\langle\psi|\phi\rangle = \int_{\mathbb{R}^2} \langle\psi|\rho(q,p)\varphi\rangle\langle\rho(q,p)\varphi|\phi\rangle dqdp$, which leads to the most well known resolution of identity [24, 44]

$$I_{L^2(\mathbb{R})} = \int_{\mathbb{R}^2} |\rho(q,p)\varphi\rangle\langle\rho(q,p)\varphi| dqdp.$$

5.2 Time-Frequency Analysis and the Heisenberg Group

Suppressing Planck's constant h one obtains the main tool of time-frequency analysis. Let $f, g \in L^2(\mathbb{R})$, where $g \neq 0$ is a fixed window function. The short-time Fourier transform of f with respect to g is defined by $(V_g f)(x,\omega) = \int_{-\infty}^{\infty} f(t)\overline{g(t-x)} e^{-2\pi i t\omega} dt$. In terms of the UIR of the Heisenberg group,

$$\langle f|\rho(x,\omega)g\rangle = e^{\pi i x\omega} V_g f(x,\omega).$$

Due to the simple exponential factor $e^{\pi i x\omega}$, the Moyal identity holds also for V_g. Thus, the resolution of identity implies the Moyal identity for the Fourier-Wigner transform, and in fact, the apparently stronger signal recovery formula is true:

$$f = \frac{1}{||g||^2} \int_{\mathbb{R}^2} V_g f(x,\omega) e^{-\pi i x\omega} \rho(x,\omega) g dx d\omega.$$

See [25] for more details on time-frequency analysis.

5.3 Ladder Operators and the Harmonic Oscillator Coherent States

In (1), the harmonic oscillator coherent states $|z\rangle$ were introduced, and their important characteristic properties were listed. This family of coherent states is obtained as an orbit of the Schrödinger representation. It is best studied in terms of the ladder operators: the *annihilation* and *creation operators*, introduced next. Let $\widehat{a} = \frac{1}{\sqrt{2}}(\widehat{Q}+i\widehat{P})$, $\widehat{a}^\dagger = \frac{1}{\sqrt{2}}(\widehat{Q}-i\widehat{P})$ and $z = q + ip \in \mathbb{C}$. Then

$$|z\rangle = e^{i(z\widehat{a}^\dagger - \overline{z}\widehat{a})}|0\rangle,\ \widehat{a}|0\rangle = 0.$$

That is, the harmonic oscillator coherent states are elements in the orbit $\{|z\rangle = \rho(q,-p)|0\rangle : (q,-p) \in \mathbb{R}^2\}$. The unique normalized solution $|0\rangle$ to the differential equation $\widehat{a}|0\rangle = 0$ is the harmonic oscillator ground state $\langle x|0\rangle = \frac{1}{\sqrt[4]{2\pi}}e^{-x^2/2}$. In [1, 2, 5, 10], the study of the harmonic oscillator coherent states was fully developed mathematically. In [5], the development is in terms of what is at present called reproducing kernel Hilbert spaces in general, and Bargmann space in particular. In the papers [1, 2, 10, 41] coherent states were used in the quantization of the electromagnetic field, with a view towards applications to quantum optics. In the latter papers, the fact that the problem of quantum mechanical measurement was recognized to be intimately related to phase space distributions led to deep mathematical and physical insights. For instance, the $P-$distribution or Glauber-Sudarshan distribution is what corresponds to coherent state quantization, which in turn requires the resolution of identity.

5.4 Coherent states, the Affine group, and the Wavelet transform

In this subsection, we present some general definitions regarding square integrable representations of groups and give the most important example of wavelet analysis. The idea is due to Duflo-Moore [20, 21] and the connection with coherent states was due to Berezin, Perelomov, and Gilmore. It is a natural development from the considerations in (5.1,5.2,5.3) that lead to the idea of using the Real Affine group $G = \mathrm{Aff}(\mathbb{R})$ in the same vein and using it in quantum mechanics, due to Klauder and Aslaksen [29], and for wavelets, due to Daubechies, Grossmann, Morlet and Paul [15, 26]. It is where we take off towards the applications in the last section. It is in [26] that square integrable representations were used as the basis of generalized wavelet analysis, while in [15], the analogous consideration was done for a discrete set of vectors in Hilbert space, where the setting is in terms of the more general concept of frames [3, 35].

Definition 5.1. Let $T : G \longrightarrow \mathcal{U}(\mathcal{H})$ be a unitary irreducible representation of a group G in the Hilbert $\mathcal{H}$. T is called square-integrable if there exists a vector $\varphi \in \mathcal{H}, \varphi \neq 0$ such that $c_\varphi = \int_G |\langle\varphi|T(x)\varphi\rangle|^2 d\mu(x) < \infty$. The vector φ is called an admissible vector for T.

The orthogonality relation for square-integrable representations is the Moyal identity: $\int_G \langle\psi_1|T(x)\varphi_1\rangle\langle\psi_2|T(x)\varphi_2\rangle d\mu(x) = \lambda\langle\psi_1|\psi_2\rangle\langle\varphi_2|\varphi_1\rangle$ for some complex number λ depending only on T, in the case of the unimodular groups such as the Heisenberg group. For nonunimodular groups, such as the affine group, the Moyal identity has the form

$$\int_G \langle\psi_1|T(x)\varphi_1\rangle\langle\psi_2|T(x)\varphi_2\rangle d\mu(x) = \lambda\langle\psi_1|\psi_2\rangle\langle C\varphi_2|C\varphi_1\rangle,$$

for some unique positive self-adjoint operator C on $\mathcal{H}$, called the Duflo-Moore operator.

Thus, if we fix $\varphi_1 = \varphi_2$, and choosing it so that $\langle C\varphi_1 | C\varphi_1\rangle = \frac{1}{\lambda}$, we get

$$I_{\mathcal{H}} = \int_G |T(x)\varphi\rangle\langle T(x)\varphi| d\mu(x) = \int_G |x\rangle\langle x| d\mu(x),$$

the resolution of identity in terms of coherent states $|x\rangle = |T(x)\varphi\rangle$.

For a fixed "ground state" φ, the embedding $\mathcal{H} \ni \psi \mapsto \widetilde{\psi} = W\psi \in L^2(G)$, $\widetilde{\psi}(x) = \langle\psi|T(x)\varphi\rangle$ is called the *wavelet transform.* The Moyal identity states that the wavelet transform is unitary $\langle W\psi_1|W\psi_2\rangle = \langle\psi_1|\psi_2\rangle$ as a mapping $W : \mathcal{H} \longrightarrow \operatorname{Ran} W$ onto the range of W in $L^2(G)$.

Let us now consider the real affine group $G = \mathrm{Aff}(\mathbb{R})$, which underpins the original wavelet transform. This group consists of elements $(a, b) \in \mathbb{R}_+ \times \mathbb{R}$ with group operation $(a_1, b_1) \cdot (a_2, b_2) = (a_1a_2, a_1b_2 + a_2)$. Under the association $(a, b) \mapsto \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ the group operation is simply the usual matrix multiplication

$$\begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1b_2 + b_1 \\ 0 & 1 \end{pmatrix}.$$

The standard action on $\mathbb{R} = \left\{ x \leftrightarrow \begin{pmatrix} x \\ 1 \end{pmatrix} \right\}$ is by affine transformations: $(a, b) \cdot x = ax + b$.

The UIRs of $\mathrm{Aff}(\mathbb{R})$ are given by $\rho^{\pm} : \mathrm{Aff}(\mathbb{R}) \longrightarrow \mathcal{U}(\mathcal{H}_+)$ on the Hardy spaces $\mathcal{H}_+ = \{f \in L^2(\mathbb{R}) : \operatorname{supp} \widehat{f} \subset [0, \infty)\}, \mathcal{H}_- = \{\psi \in L^2(\mathbb{R}) : \operatorname{supp} \widehat{\psi} \subset (-\infty, 0]\}$, respectively, and given by the same formula

$$[\rho^{\pm}(a, b)\psi](x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x - b}{a}\right).$$

The wavelet transform is given by $W_\psi(f) = \langle f|\rho(a, b)\psi\rangle = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(x)\overline{\psi\left(\frac{x-b}{a}\right)} dx$ for an admissible vector ψ, that is, $\int_{-\infty}^{\infty} \frac{|\widehat{\psi}(x)|^2}{|x|} dx < \infty$ and the Moyal formula gives the resolution of identity

$$I = \int_{\mathrm{Aff}(\mathbb{R})} |\rho(a, b)\psi\rangle\langle\rho(a, b)\psi| \frac{dadb}{a^2}.$$

This implies the signal recovery formula $|f\rangle = \int_0^{+\infty} \int_{-\infty}^{+\infty} |\rho(a, b)\psi\rangle\langle\rho(a, b)\psi|f\rangle \frac{dadb}{a^2}$. The wavelet analysis of signals based on the affine group is much preferred by scientists and engineers than the short-time Fourier transform due to the fact that wavelet decomposition models real-life signals better. One way, among many, that wavelet decomposition is better is due to the fact that signals are cut up into different frequency components whose resolutions are matched to their scale. That is, high-frequency components correspond to short-time localization while low-frequency portions of the signal are broader in their localization. Thus, the wavelet transform is able to zoom in better on high-frequency parts of signals [13]. Discretization of the signal recovery formula leads to multiresolution analysis, where it is standard to employ the Haar wavelet, as it provides the simplest orthonormal wavelet basis [30].

6 Applications of the Resolution of Identity

In this section, we give several applications of coherent states to quantization. The quantization of angle is via coherent states arising from unitary operators representing the discrete cylinder $G = \mathbb{Z} \times S^1$. This quantization gives a viable scheme for functions of angle, and is much simpler than those in [32, 33]. Although G is a group, the operators do not form a representation [45]. They are however a direct discrete analogue of the Schrödinger representation. The quantization of time again arises from coherent states on the discrete cylinder, which are direct discrete analogues of the canonical coherent states, arising from unitary irreducible representations of G. We quantize confined time of arrival functions [22, 23], and coherent states on the discrete cylinder, we feel, form an appropriate phase space quantum mechanical setting for this quantization problem. Finally, localization operators on the Euclidean motion group on the plane $E(2)$ are constructed via coherent states arising from the left-regular representation and an appropriate admissible vector. The UIRs of $E(2)$ do not admit any admissible vectors but it is possible to bypass this difficulty via the regular representation [16, 28]. Thus, localization is now possible on the phase space of translations and rotations, and boundedness properties for functions in L^p, $p = 1, 2, +\infty$ are proved (see (6.3) below).

6.1 Coherent State Quantization of Angle

Definition 6.1. [3] Suppose $\{|x\rangle : x \in X\} \subset \mathcal{H}$ is a family of coherent states in the Hilbert space $\mathcal{H}$ parametrized by the measure space X, that is, $I_{\mathcal{H}} = \int_X |x\rangle\langle x| d\mu(x)$. If X is a discrete measure space, the integral is replaced by a sum. The coherent state quantization of the square-integrable function $f : X \longrightarrow \mathbb{C}$ is

$$\widehat{A}_f = \int f(x)|x\rangle\langle x|d\mu(x) : \mathcal{H} \longrightarrow \mathcal{H}.$$

Quantization is any procedure that assigns quantum observables to classical observables. This is the intentionally vague definition in physics and mathematics. In signal analysis, quantization is usually meant mapping a large set of inputs to a smaller set of outputs. We follow the idea of Gazeau-Bergeron [7, 34] in viewing quantization as assigning labeled observables to the functions on the label set X. This is an attractive general definition because it can accommodate more general classical spaces such as measure spaces, in contrast to the standard one of symplectic manifolds. Moreover, one is allowed to quantize discrete and finite spaces, where coherent state quantization is the simplest procedure. It is precisely the point of view of phase space representation of quantum mechanics that makes coherent state quantization quite attractive, e.g., the $P-$distribution of Glauber-Sudarshan [10, 41] and the quantization method due to Berezin [9, 8]. Coherent state quantization is also known by the names Toeplitz quantization and anti-Wick quantization [35].

Consider the following family of unitary operators [45] $\rho(n, \theta) : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ given by

$$(\rho(n,\theta)F)(k) = \begin{cases} e^{i\left(k+\frac{n}{2}\right))\theta} F(k+n), n \in \mathbb{Z}_{even} \\ e^{i\left(k+\frac{n-1}{2}\right))\theta} F(k+n), n \in \mathbb{Z}_{odd} \end{cases}$$

where $(n, \theta) \in G = \mathbb{Z} \times S^1$, $F \in \ell^2(\mathbb{Z})$. The matrix elements are, for $G \in \ell^2(\mathbb{Z})$

$$V(F,G) = \langle \rho(n,\theta)F|G\rangle = \sum_{m\in\mathbb{Z}_e} e^{im\theta} F\left(m+\frac{n}{2}\right) \overline{G\left(m-\frac{n}{2}\right)}$$

$$+\sum_{m\in\mathbb{Z}_o} e^{im\theta} F\left(m+\frac{n+1}{2}\right)\overline{G\left(m-\frac{n-1}{2}\right)}$$

and we have the Moyal identity (see [45] for the proof)

$$\langle V(F_1,G_1)|V(F_2,G_2)\rangle_{L^2(\mathbb{Z}\times S^1)} = \langle F_1|F_2\rangle_{\ell^2(\mathbb{Z})}\langle G_1|G_2\rangle_{\ell^2(\mathbb{Z})}$$

which gives the resolution of identity

$$\widehat{I}_{\ell^2(\mathbb{Z})} = \frac{1}{||F||^2}\sum_n \frac{1}{2\pi}\int_0^{2\pi} \big|\rho(n,\theta)F\rangle\langle\rho(n,\theta)F\big|d\theta.$$

The coherent quantization of the angle is then given by

$$\widehat{A}_\theta G = \sum_n \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta|\rho(n,\theta)F\rangle\langle\rho(n,\theta)F|G\rangle d\theta,$$

whose matrix elements are

$$\langle l|\widehat{A}_\theta|k\rangle = \frac{\pm 4i}{(k-l)}\sum_{n\in\mathbb{Z}} F(l+n)\overline{F(k+n)}.$$

This gives $\langle l|\widehat{A}_\theta|k\rangle = \frac{\pm 4i}{k-l}\sum \exp[\frac{-(l+n)^2-(k+n)^2}{2\sigma^2}]$ for a Gaussian F and is a superposition of Gaussians with peaks at $\frac{k+l}{2}$. Localization at particular lattice points may be achieved by localization operators

$$\widehat{L}_F = \sum_n \frac{1}{2\pi}\int_0^{2\pi} \theta w(n)|\rho(n,\theta)F\rangle\langle\rho(n,\theta)F|\, d\theta$$

using a weight function $w(n)$.

6.2 Coherent State Quantization of Time

The quantization of time of arrival functions, has received focused theoretical considerations in the last two decades, mainly by E. Galapon and collaborators [22, 23]. However, among the most important conjugate pairs such as the angle-angular momentum pair, the time-energy pair received the least attention. This is mainly due to a well-known folklore attributed to W. Pauli, stating that there exists no self-adjoint time operator conjugate to a semibounded Hamiltonian [22, 23]. Time has been relegated mostly as an external parameter, a situation that cannot be long sustained due to the operator theoretic nature of the various Heisenberg uncertainty principles involving conjugate pairs of observables.

We consider the following discrete coherent states parametrized by $(\ell,\sigma)\in\mathbb{Z}\times S^1$: $|(\ell,\sigma)\rangle = \hat{U}(\ell,\sigma)|0\rangle = \frac{\sqrt{2}}{\sqrt[4]{\pi}}e^{-i\sigma\ell/2}e^{i\ell\theta}e^{-\frac{(\theta-\sigma)^2}{2}}$ for a confined particle on a line segment and the time of arrival function $T(\theta,n\hbar) = -\mu\frac{\theta}{n\hbar}$, where T is a function on the phase space (which is also a group) $G = S^1\times\mathbb{Z}$ [32, 33, 36]. The coherent state quantization arises from the resolution of the identity $\widehat{I} = \sum_{\ell\in\mathbb{Z}}\int_{-\pi}^{\pi}|(\ell,\sigma)\rangle\langle(\ell,\sigma)|d\sigma$ and is given by

$$\widehat{T} = \sum_{\ell\in\mathbb{Z}}\int_{-\pi}^{\pi}\check{T}(\sigma,\ell)|(\ell,\sigma)\rangle\langle(\ell,\sigma)|d\sigma,$$

$$(\widehat{T}\phi)(\theta) = -\frac{i\mu}{4\sqrt{2}\hbar} \sum_{\ell,n\neq 0\in\mathbb{Z}} \frac{(-1)^\ell e^{-n^2/4}}{n\ell} e^{-in\theta}\phi(\theta).$$

The proof for the resolution of identity is a simple computation: it uses the Fourier decomposition of the function $e^{-\frac{(\varphi-\sigma)^2}{2}}\eta(\varphi)$, with the assumption that $\eta \in L^2(S^1)$,

$$\begin{aligned}\sum_{\ell\in\mathbb{Z}} \int_{-\pi}^{\pi} |\sigma,\ell\rangle\langle\sigma,\ell|\eta\rangle d\sigma &= \mathcal{C}^2 \sum_{\ell\in\mathbb{Z}} \int_{-\pi}^{\pi} e^{i\ell\omega} e^{-\frac{(\omega-\sigma)^2}{2}} \int_{-\pi}^{\pi} e^{-i\ell\varphi} e^{-\frac{(\varphi-\sigma)^2}{2}} \eta(\varphi) d\varphi d\sigma \\ &= 2\pi\mathcal{C}^2 \int_{-\pi}^{\pi} e^{-\frac{(\omega-\sigma)^2}{2}} \eta(\omega) d\sigma \\ &= 2\pi\eta(\omega).\end{aligned}$$

Thus it gives us

$$\sum_{\ell\in\mathbb{Z}} \int_{-\pi}^{\pi} |\sigma,\ell\rangle\langle\sigma,\ell| d\sigma = 2\pi \hat{I}_{\mathcal{H}}.$$

We have then the coherent state quantization of T :

$$\begin{aligned}\hat{T} &= \int_{-\pi}^{\pi} \check{T}(\sigma,\ell)|\sigma,\ell\rangle\langle\sigma,\ell| d\sigma \\ &= -\frac{i\mu}{2\pi\hbar}\frac{1}{\sqrt{2\pi}} \sum_{\substack{\ell,n\in\mathbb{Z}\\ n,\ell\neq 0}} \int_{-\pi}^{\pi} \frac{(-1)^\ell e^{-in\sigma}}{n\ell} e^{-(\theta-\sigma)^2} d\sigma \\ &= -\frac{i\mu}{4\sqrt{2}\hbar} \sum_{\substack{\ell,n\in\mathbb{Z}\\ n,\ell\neq 0}} \frac{(-1)^\ell e^{-n^2/4}}{n\ell} e^{-in\theta}.\end{aligned}$$

This means that for $\phi(\theta) \in \mathcal{H} = L^2(S^1)$ then

$$(\hat{T}\phi)(\theta) = -\frac{i\mu}{4\sqrt{2}\hbar} \sum_{\substack{\ell,n\in\mathbb{Z}\\ n,\ell\neq 0}} \frac{(-1)^\ell e^{-n^2/4}}{n\ell} e^{-in\theta}\phi(\theta).$$

6.3 Localization on the Euclidean Motion Group

We now look at the construction of coherent states and localization operators on the Euclidean motion group on the plane. The Euclidean motion group of rank two, $E(2)$, is defined to be the semi-direct product of the translation group, $\mathbb{R}^2$, and the circle group, S, denoted by $\mathbb{R}^2 \rtimes S$. We also consider the group action of S on $\mathbb{R}^2$, given by the standard action by rotation

$$\sigma : S \to Aut(\mathbb{R}^2), \sigma(z) = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix},$$

where $\theta = arg z$, $\theta \in [0, 2\pi)$. The group multiplication on $E(2)$ is given by

$$(x_1, z_1)(x_2, z_2) = (x_1 + R_{z_1}x_2, z_1z_2)$$

for all $(x_1, z_1), (x_2, z_2) \in \mathbb{R}^2 \rtimes S, x_i = (b_{i_1}, b_{i_2})$.

The conventional approach in the construction of coherent states for $E(n)$ where $n \geq 2$ fails because none of the unitary irreducible representations of $E(n)$ on $L^2(S^{n-1})$ are square-integrable [28]. This means that

$$\int_{E(2)} \left|\langle \varphi, U_g^s \varphi \rangle\right|^2 dg = \infty.$$

We modify this conventional approach, so that, instead of using the unitary irreducible representations of the group, we use the regular representation of $E(2)$ on $L^2(\mathbb{R}^2)$, which is the composition of the representations T and $\bar{R}$ of the translation group, $\mathbb{R}^2$, and the circle group, S, respectively. The action of the unitary representation with the function f on $L^2(\mathbb{R}^2)$ is given by

$$(U_g f)(x) = (T_t \bar{R}_z f)(x) = f(R_z^{-1}(x - t))$$

where $g \in E(2)$ parametrized by $g = (t, z)$, $t \in \mathbb{R}^2$ and $z \in S$. Here, we have $(T_t f)(x) = f(x - b)$ and $(\bar{R}_z f)(x) = f(R_z^{-1} x)$.

Let $\psi_\alpha \in L^2(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ be a non-vanishing and admissible wavelet [16] defined by

$$\psi_\alpha : x \to \frac{\alpha}{4} e^{-|x_1| - \alpha |x_2|}.$$

We define a function c_ψ, which will be the wavelet constant for the regular representation,

$$\begin{aligned} c_\psi(\omega) &= \int_S \left|(\mathcal{F} \bar{R}_z \psi_\alpha(\omega)\right|^2 dz \\ &= \frac{\alpha^4}{4\pi^2} \int_0^{2\pi} \frac{d\theta}{(1 + \xi^2)^2 (\alpha^2 + \eta^2)^2} < \infty, \end{aligned} \tag{3}$$

where $\xi = \omega_1 \cos\theta + \omega_2 \sin\theta$, $\eta = \omega_2 \cos\theta - \omega_1 \sin\theta$ and $\theta = \arg z$. Here, $\mathcal{F} \bar{R}_z \psi_\alpha$ is the Fourier transform of $\bar{R}_z \psi_\alpha$ given by

$$(\mathcal{F} \bar{R}_\theta \psi_\alpha)(\omega) = \frac{\alpha^2}{2\pi [1 + (\omega_1 \cos\theta + \omega_2 \sin\theta)^2][\alpha^2 + (\omega_2 \cos\theta - \omega_1 \sin\theta)^2]},$$

for $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$.

The wavelet transform [16], $W_{\psi_\alpha} : L^2(\mathbb{R}^2) \to \mathbb{C}^{\mathbb{R}^2 \rtimes S}$, associated to ψ_α is given by

$$(W_{\psi_\alpha} f)(g) = \langle U_g \psi_\alpha, f \rangle_{L^2(\mathbb{R}^2)} = \langle \mathcal{F} U_g \psi_\alpha, \mathcal{F} f \rangle_{L^2(\hat{\mathbb{R}}^2)} \tag{4}$$

where $\hat{\mathbb{R}}^2$ is the dual of $\mathbb{R}^2$ and $\mathbb{C}^{\mathbb{R}^2 \rtimes S}$ is a functional Hilbert space which is a subspace of $L^2(E(2))$. It is simply the range of W_{ψ_α} in $L^2(E(2))$. If $\psi \in \mathcal{H}$ satisfies (3) the group representation is square-integrable and the functional Hilbert space is a closed subspace of $L^2(E(2))$.

Given the wavelet transform, W_{ψ_α}, in (4), we derive the resolution of identity in terms of W_{ψ_α} as follows (see Theorem 11.16 of [16])

$$\langle X, Y \rangle_{L^2(\mathbb{R}^2)} = \left\langle \frac{1}{\sqrt{c_{\psi_\alpha}}} W_{\psi_\alpha} X, \frac{1}{\sqrt{c_{\psi_\alpha}}} W_{\psi_\alpha} Y \right\rangle_{\mathbb{C}^{\mathbb{R}^2 \rtimes S}},$$

where c_{ψ_α} is a constant associated to ψ_α given in (3). This unitarity of the wavelet transform is nothing but the Moyal identity and this gives the resolution of the identity formula in $E(2)$ on $L^2(\mathbb{R}^2)$:

$$I_{L^2(\mathbb{R}^2)} = \frac{1}{c_\psi}\int |W_{\psi_\alpha}\rangle\langle W_{\psi_\alpha}|dg.$$

Define a bounded linear operator, $L_{F,\psi_\alpha} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ related to the wavelet transform, W_{ψ_α}, by

$$\begin{aligned}\langle L_{F,\psi_\alpha}X, Y\rangle_{L^2(\mathbb{R}^2)} &= \frac{1}{c_{\psi_\alpha}}\int_{E(2)} F(g)(W_{\psi_\alpha}X)(g)(\overline{W_{\psi_\alpha}Y})(g)dg\\ &= \frac{1}{c_{\psi_\alpha}}\int_{E(2)} F(g)\,\langle U_g\psi_\alpha, X\rangle\,\langle Y, U_g\psi_\alpha\rangle\,dg\end{aligned}$$

where $F \in L^1(E(2)) \cup L^\infty(E(2))$ and $X, Y \in L^2(\mathbb{R}^2)$. These bounded linear operators L_{F,ψ_α} are called localization operators in the group $E(2)$. Localization operators are important in signal analysis and have been known since the work of I. Daubechies [14] on signal analysis and was certainly foreshadowed by the works F. A. Berezin in his work on quantization [8, 9].
The following are some mathematical properties observed for the operators L_{F,ψ_α} [31]:

1. Let $F \in L^1(E(2))$. Then $L_{F,\psi_\alpha} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a bounded operator and

$$||L_{F,\psi_\alpha}||_* \leq \frac{1}{c_{\psi_\alpha}}||F||_{L^1(E(2))}||\psi_\alpha||^2.$$

   To prove this first property, let $F \in L^1(E(2))$. This means that $\int_{E(2)} |F(g)|\,dg < \infty$. Then, by applying the inequalities of Hölder's and Cauchy-Schwarz we obtain

$$\begin{aligned}|\langle L_{F,\psi_\alpha}X, Y\rangle| &= \left|\frac{1}{c_{\psi_\alpha}}\int_{E(2)} F(g)\,\langle U_g\psi_\alpha, X\rangle\,\langle Y, U_g\psi_\alpha\rangle\,dg\right|\\ &\leq \frac{1}{c_{\psi_\alpha}}||F||_{L^1(E(2))}.\end{aligned} \tag{5}$$

   Then $L_{F,\psi_\alpha} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a bounded operator with

$$||L_{F,\psi_\alpha}||_* \leq \frac{1}{c_{\psi_\alpha}}||F||_{L^1(E(2))},$$

   where $||\ ||_*$ denotes the operator norm.
   The proofs of the other two important inequalities are similar.

2. Let $F \in L^\infty(E(2))$. Then $L_{F,\psi_\alpha} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a bounded operator and

$$||L_{F,\psi_\alpha}||_* \leq \frac{1}{c_{\psi_\alpha}}||F||_{L^\infty(E(2))}||\psi_\alpha||^2$$

3. Let $F \in L^2(E(2))$. Then $L_{F,\psi_\alpha} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a bounded operator and

$$||L_{F,\psi_\alpha}||_* \leq \frac{1}{c_{\psi_\alpha}}||F||_{L^2(E(2))}||\psi_\alpha||^2.$$

7 Conclusion

In this paper, we presented several expressions for the identity operator in Hilbert space in terms of coherent states, called the resolution of identity formula. Coherent states are very important and useful objects in physics due to their classical-like behavior. Applications of coherent states to quantization and localization are presented, one arising from a unitary irreducible representation of a group, one from a family of unitary operators which are not group representations, although the parametrization comes from a group, and one from a unitary group representation which is not irreducible. The theory of coherent states and the mathematical developments arising from it, such as quantization, localization and generalized wavelet transforms, are mathematically very rich, as well as accessible to advanced students of mathematics. In this work, we broached to the local mathematical audience the mathematical theory of coherent states, possibly as an entry and motivation to study quantum mechanics, which is at the source of many important mathematical developments in the previous century. One such strong motivation should be the deep connection with unitary representations and harmonic analysis on groups.

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