MATIMYÁS MATEMATIKA
ISSN 0115-6926

Journal of the Mathematical Society of the Philippines
Vol. 45 No. 1 (2022) pp. 63-76

A characterization of irreducible modules of Terwilliger algebras of Doob graphs via quasi-isomorphism

John Vincent S. Morales
Department of Mathematics and Statistics
De La Salle University
2401 Taft Avenue, Manila, Philippines
john.vincent.morales@dlsu.edu.ph

Tessie M. Palma
Mathematics Learning Area
De La Salle Lipa
1962 J.P. Laurel National Highway Lipa City, Batangas, Philippines
tessie.palma@dlsl.edu.ph

Abstract

Let Γ denote an arbitrary distance-regular graph with vertex set X and adjacency matrix A. Fix $x \in X$ and let $T = T(x)$ denote the Terwilliger algebra Γ with respect to x. Then A decomposes into

$$A = L + F + R$$

where $L, F, R \in T$. The quantum adjacency algebra $Q = Q(x)$ of Γ with respect to x is the subalgebra of T generated by L, F, R. Recently, Terwilliger and Žitnik (J. Comb. Th. Ser. A 166: 297–314, 2019) introduced the notion of quasi-isomorphism between T-modules and gave equivalent conditions for $Q \neq T$. To provide examples, they showed $Q = T$ in Hamming graphs and $Q \neq T$ in bipartite dual-polar graphs. It is interesting to know which distance-regular graphs show $Q \neq T$. In this paper, we consider the Doob graph $D = D(n, m)$ formed by the Cartesian product of n copies of Shrikhande graph S and m copies of complete graph K_4. Using Tanabe's result (JAC 6: 173–195, 1997) on characterization of irreducible T-modules of D, we show a necessary and sufficient condition for two irreducible T-modules to be quasi-isomorphic. Moreover, we show $Q \neq T$ in Doob graphs. This paper provides an alternative proof of Corollary 5.7 in (JAC 54: 979–998, 2021) via quasi-isomorphism. This paper aims to explicitly carry out the computation mentioned in Remark 5.8 in (JAC 54: 979–998, 2021

Key words: Terwilliger algebra, quantum adjacency algebra, Doob graphs, distance-regular graphs, quasi-isomorphic modules

2020 MSC: 05E30, 05C50

1 Introduction

The subconstituent algebra was first introduced by Terwilliger in [30] with subsequent papers [31] and [32]. This is a finite-dimensional and semi-simple matrix $\mathbb{C}$-algebra attached to a fixed vertex of a graph or an association scheme. Since its introduction, the subconstituent algebra became a useful tool in the study of combinatorial and algebraic structures of graphs (see e.g., [7], [8]) as well as association schemes (see e.g., [29], [23], [13], [3]). Later on, the subconstituent algebra became known as Terwilliger algebra (see [3, 4, 6, 7, 8, 10, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 26, 27, 28, 34] for some recent works on the area). Independent of the notion of Terwilliger algebra, Hora and Obata introduced the quantum adjacency algebra in [11]. This algebra became useful in describing limiting spectral distributions of infinite sequences of graphs growing in terms of order and size.

Since the quantum adjacency algebra Q of a graph is generated by matrices that are contained in the Terwilliger algebra T, it is interesting to determine which graph shows $Q \neq T$. If the graph is distance-regular (see Section 2 for definition), then the above problem translates to existence of a pair of quasi-isomorphic modules that are non-isomorphic as T-modules. Recently, Terwilliger and Žitnik [33] introduced the notion of quasi-isomorphism between T-modules and gave equivalent conditions for $Q \neq T$. To provide examples, they showed $Q = T$ in Hamming graphs and $Q \neq T$ in bipartite dual-polar graphs.

Doob graphs belong to the family of distance-regular graphs and are formed by taking the Cartesian product of n copies of Shrikhande graph S and m copies of complete graph K_4 on four vertices. In [29], Tanabe studied the Terwilliger algebras of Doob graphs and gave a detailed characterization of irreducible modules based on four parameters – the integers v, d, p, t where v is the endpoint and $d + p$ is the diameter. He proved that two irreducible T-modules are isomorphic if and only if these modules coincide on the four parameters [29, Proposition 3]. In this paper, we show that two irreducible T-modules are quasi-isomorphic if and only if these modules coincide on these parameters except possibly the endpoint. This provides an alternative proof of [24, Corollary 5.7] via quasi-isomorphism. Consequently, there exists a pair of quasi-isomorphic modules that are not isomorphic as T-modules and thus, $Q \neq T$ in Doob graphs. This paper aims to carry out the computation mentioned in [24, Remark 5.8].

The paper is arranged as follows: In Section 2, we review some concepts on distance-regular graphs and related algebras. In Section 3, we mention results relating the Terwilliger algebras and the quantum adjacency algebras as well as their irreducible modules. We also review results about quasi-isomorphism in T-modules. In Section 4, we recall some properties of the Doob graphs and the characterization of irreducible modules. In Section 5, we prove the main result.

2 Preliminaries

In this section, we briefly review some basic concepts concerning distance-regular graphs and related algebras. For more information, see [1, 2, 5, 9, 22, 30].

Let X denote a nonempty finite set and let $V = \mathbb{C}^X$ denote the $\mathbb{C}$-vector space of column vectors with complex entries whose coordinates are indexed by X. For $x \in X$, define the vector $\hat{x} \in V$ such that

$$y\text{-coordinate of } \hat{x} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y, \end{cases} \qquad (y \in X).$$

Note that $\{\hat{x} \mid x \in X\}$ is an orthonormal basis for V with respect to the inner product

$\langle u, v\rangle = u^t\bar{v}$ where ^{t} and $^-$ denote transpose and complex conjugate, respectively. For convenience, let $\dagger$ denote conjugate-transpose. Let $\mathrm{End}(V)$ denote the set of all linear transformations from V to V. We identify $\mathrm{End}(V)$ with the $\mathbb{C}$-algebra of complex matrices with rows and columns indexed by X. Note that $\langle Bu, v\rangle = \langle u, B^\dagger v\rangle$ for $u, v \in V$ and for $B \in \mathrm{End}(V)$.

Let $\Gamma = (X, R)$ denote a finite, undirected, simple connected graph with vertex set X and edge set R. The *distance* $\partial(a, b)$ from a to b is the length of a shortest path from a to b. By the *diameter* of Γ, we mean the scalar

$$s := \max\{\partial(a, b) \mid a, b \in X\}.$$

We say Γ is *distance-regular* if for $x, y \in X$ such that $\partial(x, y) = h$ the scalar

$$p^h_{ij} := |\{z \in X \mid \partial(x, z) = i \text{ and } \partial(z, y) = j\}| \tag{1}$$

is independent of the choice of x and y and depends only on h, i, j $(0 \le h, i, j \le s)$. The scalars (1) are called the *intersection numbers* of Γ. From here on, we assume Γ is distance regular with diameter $s \ge 1$.

We recall the Bose–Mesner algebra of Γ. For $i \in \{0, 1, \ldots, s\}$, define $A_i \in \mathrm{End}(V)$ such that

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \ne i, \end{cases} \quad (x, y \in X). \tag{2}$$

The matrices (2) are the *distance matrices* of Γ. We abbreviate $A := A_1$ and call A the *adjacency matrix* of Γ. Observe that

(i) $A_0 = I$, the identity matrix;

(ii) $\sum_{h=0}^{s} A_h = J$, the all ones matrix;

(iii) $A_i^t = A_i \quad (0 \le i \le s)$;

(iv) $\bar{A}_i = A_i \quad (0 \le i \le s)$;

(v) $A_iA_j = \sum_{h=0}^{s} p^h_{ij} A_h \quad (0 \le i, j \le s)$.

Since $p^h_{ij} = p^h_{ji}$, we have $A_iA_j = A_jA_i$ for all i, j $(0 \le i, j \le s)$. Note that $\{A_0, \ldots, A_s\}$ is linearly independent and forms a basis for the commutative subalgebra M of $\mathrm{End}(V)$. We call M the *Bose–Mesner algebra* of Γ. By [1, p. 190], M is generated by A. By [1, pp. 59, 64], M has a second basis $E_0, E_1, \ldots, E_s$ such that

$$\begin{aligned} E_0 + E_1 + \cdots + E_s &= I, \\ E_0 &= |X|^{-1}J, \\ E_i^t &= E_i \quad (0 \le i \le s), \\ \overline{E_i} &= E_i \quad (0 \le i \le s), \\ E_iE_j &= \delta_{ij}E_i \quad (0 \le i, j \le s). \end{aligned}$$

$E_0, E_1, \ldots, E_s$ are called the *primitive idempotents* of Γ. Moreover, there exist scalars $\theta_0, \theta_1, \ldots, \theta_s$ such that

$$A = \theta_0E_0 + \theta_1E_1 + \cdots + \theta_sE_s.$$

Observe that $AE_i = \theta_i E_i$ $(0 \le i \le s)$. The scalars $\{\theta_0, \theta_1, \ldots, \theta_s\}$ are real and mutually distinct. We call θ_i the *eigenvalue of A associated with E_i*. The *standard module* V decomposes into an orthogonal direct sum $V = E_0V + E_1V + \cdots + E_sV$ where E_iV is the *eigenspace of A associated with θ_i*. For convenience, $E_i := 0$ whenever $i < 0$ or $i > s$.

We recall the Terwilliger algebra of Γ. Fix a *base vertex* $x \in X$. For $i \in \{0, 1, \ldots, s\}$, we define the diagonal matrix $E_i^* = E_i^*(x)$ in $\mathrm{End}(V)$ such that

$$(E_i^*)_{yy} = (A_i)_{xy} \quad (y \in X).$$

$E_0^*, E_1^*, \ldots, E_D^*$ are called the *dual primitive idempotents* of Γ. For convenience, $E_i^* := 0$ whenever $i < 0$ or $i > s$. Observe that

(i) $\sum_{h=0}^{s} E_h^* = I$;

(ii) $E_i^{*t} = E_i^* \quad (0 \le i \le s)$;

(iii) $\bar{E}_i^* = E_i^* \quad (0 \le i \le s)$;

(iv) $E_i^* E_j^* = \delta_{ij} E_j^* \quad (0 \le i, j \le s)$.

Note that $\{E_0^*, \ldots, E_s^*\}$ forms a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{End}(V)$. We call M^* the *dual Bose–Mesner algebra* of Γ with respect to x. Let $T = T(x)$ denote the subalgebra of $\mathrm{End}(V)$ generated by M and M^*. We call T the *Terwilliger algebra* of Γ with respect to x. Since M is generated by A and M^* is spanned by $\{E_0^*, E_1^*, \ldots, E_s^*\}$, T is generated by $\{A, E_0^*, E_1^*, \ldots, E_s^*\}$.

We now recall T-modules. Let W denote a subspace of V. For $B \in \mathrm{End}(V)$, define $BW = \{Bw : w \in W\}$. We say W is *B-invariant* whenever $BW \subseteq W$. We say W is a *T-module* if W is B-invariant for all $B \in T$. Since T is generated by $\{A, E_0^*, \ldots, E_s^*\}$, W is a T-module if W is B-invariant for each $B \in \{A, E_0^*, \ldots, E_s^*\}$. We call W an *irreducible T-module* if $W \neq 0$ and W contains no other T-modules other than 0 and itself. If W is a T-module, then so is its *orthogonal complement* $W^\perp$. In particular, if W is a T-module containing another T-module W', then $W'^\perp \cap W$ is also a T-module and $W = W' \oplus \left(W'^\perp \cap W\right)$. Hence, any nonzero T-module such as V can be decomposed into an orthogonal direct sum of irreducible T-modules. By a *T-module isomorphism*, we mean a vector space isomorphism σ from W onto another T-module W' such that $\sigma(Bw) - B\sigma(w) = 0$ for all $B \in T$ and $w \in W$. If σ exists, then we call W and W' *isomorphic T-modules*. Suppose W is an irreducible T-module. By the *support* of W, we mean the set

$$\mathrm{supp}(W) = \{i \in \mathbb{Z} : E_i^* W \neq 0\}.$$

We call $|\mathrm{supp}(W)| - 1$ and $\min(\mathrm{supp}(W))$ the *diameter* and *endpoint* of W, respectively. By the *dual support* of W, we mean the set

$$\mathrm{dsupp}(W) = \{i \in \mathbb{Z} : E_i W \neq 0\}.$$

We define the *dual diameter* and *dual endpoint* of W analogously. If v (resp. μ) is the endpoint (resp. dual endpoint) of W, then

$$W = \textstyle\sum_{k=0}^{\infty} E_{v+k}^* W$$
$$W = \textstyle\sum_{k=0}^{\infty} E_{\mu+k} W$$

are orthogonal direct sum decompositions of W. We say W is *thin* (resp. *dual thin*) whenever $\dim E^*_i W \leq 1$ (resp. $\dim E_i W \leq 1$) for all $i \in \mathbb{Z}$.

We recall the quantum adjacency algebra of Γ. Define the matrices $L = L(x)$, $F = F(x)$, and $R = R(x)$ by

$$L = \textstyle\sum_{i=0}^{s} E^*_{i-1} A E^*_i, \quad F = \sum_{i=0}^{s} E^*_i A E^*_i, \quad R = \sum_{i=0}^{s} E^*_{i+1} A E^*_i. \tag{3}$$

We call L, F, and R the *lowering*, *flat*, and *raising* matrices, respectively. Since T is generated by $\{A, E^*_0, \ldots, E^*_s\}$, we have $L, F, R \in T$. Let $Q = Q(x)$ denote the subalgebra of T generated by L, F, R. We call Q the *quantum adjacency algebra* of Γ with respect to x. Since $E^*_j A E^*_k = 0$ whenever $|j - k| > 1$, we have

$$\begin{aligned} A &= (\textstyle\sum_{i=0}^{s} E^*_i)\, A\, (\sum_{h=0}^{s} E^*_h) \\ &= \textstyle\sum_{i=0}^{s} E^*_{i-1} A E^*_i + \sum_{i=0}^{s} E^*_i A E^*_i + \sum_{i=0}^{s} E^*_{i+1} A E^*_i \\ &= L + F + R. \end{aligned}$$

Observe that

$$\bar{L} = L, \quad \bar{F} = F, \quad \bar{R} = R, \quad F^t = F, \quad R^t = L. \tag{4}$$

By (4), Q is closed under $\dagger$ and is semi-simple. Moreover,

$$L E^*_i V \subseteq E^*_{i-1} V, \quad F E^*_i V \subseteq E^*_i V, \text{ and } R E^*_i V \subseteq E^*_{i+1} V. \tag{5}$$

We define *Q-modules*, *irreducible Q-modules*, and *Q-module isomorphism* analogous to that of T-modules, irreducible T-modules, and T-module isomorphism, respectively. Note that each T-module turns into a Q-module by restricting the action of T to Q. In particular, each irreducible T-module is irreducible as Q-module [33, Proposition 6.3].

3 Irreducible T-modules and Q-modules

Since every T-module turns into a Q-module by restricting the action of T to Q, every pair of isomorphic T-modules must be a pair of isomorphic Q-modules. In some distance-regular graphs, it is possible to have a pair of isomorphic Q-modules that are non-isomorphic as T-modules. In this section, we look into conditions equivalent to existence of non-isomorphic T-modules that are isomorphic as Q-modules. Basically, all results in this section are taken from Terwilliger and Žitnik [33]. Throughout this section, we assume the following: Let Γ denote a distance-regular with diameter s and adjacency matrix A. Let V denote the standard module of Γ. For a base vertex $x \in X$, write $T = T(x)$, $Q = Q(x)$, and $E^*_i = E^*_i(x)$ $(0 \leq i \leq s)$.

Lemma 3.1. *[33, Proposition 6.3] With reference to above assumption, every irreducible T-module is an irreducible Q-module.*

Lemma 3.2. *[33, Proposition 7.5] With reference to above assumption, let W and W' denote irreducible T-modules. If W and W' are isomorphic as Q-modules, then they have the same diameter.*

Definition 3.3. [33, Definition 8.1] Let W and W' denote irreducible T-modules with endpoints μ and μ', respectively. Let $\gamma = \mu' - \mu$. By a *quasi-isomorphism of T-modules from W to W'* we mean a $\mathbb{C}$-linear bijection $\sigma : W \to W'$ such that on W,

$$\sigma L = L\sigma, \quad \sigma F = F\sigma, \quad \sigma R = R\sigma, \tag{6}$$

and

$$\sigma E_i^* = E_{i+\gamma}^* \sigma \quad \forall i \in \mathbb{Z}. \tag{7}$$

If a quasi-isomorphism exists between W and W', then we call them *quasi-isomorphic T-modules.*

Lemma 3.4. *[33, Corollary 8.5] With reference to above assumption, let W and W' denote irreducible T-modules. Then the following are equivalent:*

(i) the T-modules W and W' are quasi-isomorphic and have the same endpoint;

(ii) the T-modules W and W' are isomorphic.

Lemma 3.5. *[33, Corollary 8.7] With reference to above assumption, let W and W' denote irreducible T-modules. Then the following are equivalent:*

(i) the Q-modules W and W' are isomorphic;

(ii) the T-modules W and W' are quasi-isomorphic.

It turns out that a pair of quasi-isomorphic irreducible T-modules constitutes a pair of isomorphic irreducible Q-modules. However, some quasi-isomorphic irreducible T-modules are isomorphic T-modules. To prove that Q is properly contained in T, it suffices to show the existence of a pair of non-isomorphic irreducible T-modules that are quasi-isomorphic.

Proposition 3.6. *[33, Theorem 9.1] The following are equivalent:*

(i) $Q \neq T$;

(ii) Q is properly contained in T;

(iii) there exists a pair of non-isomorphic irreducible T-modules that are isomorphic as Q-modules;

(iv) there exists a pair of quasi-isomorphic irreducible T-modules that have different endpoints.

4 Doob graphs and their Terwilliger algebras

In this section, we recall Doob graphs and their properties. To begin, consider the graph $S = (X', R')$ such that X' consists of all cyclic permutations of the binary codewords 000000, 110000, 010111, and 011011 and R' is given by

$$R' = \{(a, b) \in X' \times X' : a \text{ and } b \text{ differ in exactly two coordinates}\}.$$

We call S the *Shrikhande graph*. On the other hand, consider the graph $K_4 = (X'', R'')$ such that $|X''| = 4$ and $R'' = \{(a, b) \mid a, b \in X'' \text{ and } a \neq b\}$. We call K_4 the *complete graph on four vertices*. Observe that S has diameter 2 and K_4 has diameter 1. Let A'_i (resp. A''_i) denote the ith distance matrix of S (resp. K_4). For fixed integers $n \geq 1$ and $m \geq 0$, let $D = D(n, m)$ denote the graph formed by the Cartesian product of n copies of S and m copies of K_4. We call D the *Doob graph*. Observe that D has diameter $2n+m$. The distance matrices of D are given by

$$A_i = \sum A'_{i_1} \otimes \cdots \otimes A'_{i_n} \otimes A''_{j_1} \otimes \cdots \otimes A''_{j_m} \quad (i \in \{0, 1, \ldots, 2n+m\}) \tag{8}$$

where the sum ranges to all $i_1, i_2, \ldots, i_n \in \{0, 1, 2\}$ and $j_1, j_2, \ldots, j_m \in \{0, 1\}$ such that $i_1 + \cdots + i_n + j_1 + \cdots + j_m = i$ and that $\otimes$ denotes Kronecker product of matrices. Similar equations hold for the primitive idempotents.

For any two vertices x, y of D, there exists an automorphism $\varphi \in \mathrm{Aut}(D)$ such that $\varphi(x) = y$. Hence, the Terwilliger algebras $T(x)$ and $T(y)$ are isomorphic [29, p. 176]. Let x' and x'' denote respective base vertices of S and K_4. Then we can pick

$$x = (\underbrace{x', x', \ldots, x'}_{n \text{ copies}}, \underbrace{x'', x'', \ldots, x''}_{m \text{ copies}}) \tag{9}$$

as base vertex so the dual primitive idempotents of D with respect to x are given by

$$E^*_i = \sum E^{*\prime}_{i_1} \otimes \cdots \otimes E^{*\prime}_{i_n} \otimes E^{*\prime\prime}_{j_1} \otimes \cdots \otimes E^{*\prime\prime}_{j_m} \quad (i \in \{0, 1, \ldots, 2n+m\}) \tag{10}$$

where the sum ranges to all $i_1, i_2, \ldots, i_n \in \{0, 1, 2\}$ and $j_1, j_2, \ldots, j_m \in \{0, 1\}$ such that $i_1 + \cdots + i_n + j_1 + \cdots + j_m = i$. To prove our main results, we shall use the assumption below.

Assumption 4.1. For fixed integers $n \geq 1$ and $m \geq 0$, we consider the Doob graph $D = D(n, m)$ with adjacency matrix A and standard module V. Let x be as in (9) and write $E^*_i = E^*_i(x)$ for the dual primitive idempotents of D. Let $L = L(x)$, $F = F(x)$, and $R = R(x)$ denote respectively the lowering, flat, and raising matrices. Finally, let $T = T(x)$ (resp. $Q = Q(x)$) denote the Terwilliger (resp. quantum adjacency) algebra of D.

Note that V decomposes into a direct sum of irreducible T-modules. Tanabe [29] gave a detailed characterization of these irreducible modules.

Proposition 4.2. *[29, Proposition 3] With reference to Assumption 4.1, consider integers v, d, p, t such that $v, d, p \geq 0$. Let $W = W(n, m; v, d, p, t)$ denote a subspace of V with basis $\{w_{ij} \in E^*_{v+i+j}W : 0 \leq i \leq d,\ 0 \leq j \leq p\}$ satisfying*

$$Aw_{ij} = 3(d - i + 1)w_{i-1,j} + (p - j + 1)w_{i,j-1} + (t + 2i - 2j)w_{ij}$$

$$+3(j+1)w_{i,j+1}+(i+1)w_{i+1,j} \tag{11}$$

such that $w_{kl} := 0$ *if* $k \notin \{0,1,\ldots,d\}$ *or* $l \notin \{0,1,\ldots,p\}$*. If* $W \neq 0$*, then each of statements (i)–(v) below holds:*

(i) W *is an irreducible* T*-module with endpoint* v *and diameter* $d+p$*.*

(ii) W *is thin if and only if* $dp = 0$*.*

(iii)

$$dim\ E^*_{v+k}W = \begin{cases} k+1 & \text{if } 0 \leq k \leq min\{d,p\}, \\ min\{d,p\}+1 & \text{if } min\{d,p\} < k < max\{d,p\}, \\ d+p+1-k & \text{if } max\{d,p\} < k \leq d+p. \end{cases}$$

(iv) If μ *is the dual-endpoint of* W*, then*

$$\begin{aligned} \mu &= \tfrac{1}{4}[3(2n+m)-t-3d-p], \\ dim\ E_{\mu+k}W &= dim\ E^*_{v+k}W. \end{aligned}$$

(v) For $W' := W(n,m;v',d',p',t')$*,* W *and* W' *are isomorphic* T*-modules if and only if* $(v,d,p,t) = (v',d',p',t')$*.*

Proposition 4.3. *[29, Proposition 4] If* W *is an irreducible* T*-module, then it is isomorphic to* $W(n,m;v,d,p,t)$ *for some integers* v,d,p,t*.*

By Proposition 4.2, two irreducible T-modules are isomorphic if and only if they coincide on the four parameters v,d,p,t. We end this section with explicit actions of the dual primitive idempotents and the lowering, raising, and flat matrices on the irreducible modules.

Lemma 4.4. *Let* $W = W(n,m;v,d,p,t)$ *denote an irreducible* T*-module with endpoint* v*, diameter* $d+p$*, and basis* $\{w_{ij} \in E^*_{v+i+j}W :\ 0 \leq i \leq d,\ 0 \leq j \leq p\}$ *satisfying the conditions of Proposition 4.2. Then each of statements (i)–(ii) below holds:*

(i) For all $\alpha \in \mathbb{Z}$*,* $0 \leq i \leq d$*, and* $0 \leq j \leq p$*, we have*

$$E^*_\alpha w_{ij} = \begin{cases} 0 & \text{if } \alpha \neq v+i+j, \\ w_{ij} & \text{if } \alpha = v+i+j. \end{cases}$$

(ii) For $0 \leq i \leq d$ *and* $0 \leq j \leq p$*, we have*

$$Lw_{ij} = 3(d-i+1)w_{i-1,j} + (p-j+1)w_{i,j-1} \tag{12}$$

$$Fw_{ij} = (t+2i-2j)w_{ij} \tag{13}$$

$$Rw_{ij} = (i+1)w_{i+1,j} + 3(j+1)w_{i,j+1} \tag{14}$$

where $w_{kl} := 0$ *whenever* $k \notin \{0,1,\ldots,d\}$ *or* $l \notin \{0,1,\ldots,p\}$*.*

Proof. (i) holds by properties of dual primitive idempotents. (ii) follows immediately from (i), (3), and (11). □

5 Quasi-isomorphic modules of Doob graphs

In this section, we prove a necessary and sufficient condition for irreducible T-modules to be isomorphic as Q-modules but non-isomorphic as T-modules. We establish this by considering all possible pairs of irreducible T-modules and by proving whether or not a quasi-isomorphism exists in each case. In this section, let $W := W(n, m; v, d, p, t)$ and $W' := W(n, m; v', d', p', t')$.

Lemma 5.1. *With reference to Assumption 4.1, suppose $(v, d, p, t) = (v', d', p', t')$. Then W and W' are isomorphic irreducible Q-modules.*

Proof. Follows from Lemma 3.1, Proposition 4.2(v), and the fact that $Q \subseteq T$. □

Lemma 5.2. *With reference to Assumption 4.1, assume $v = v'$ and $(d, p, t) \neq (d', p', t')$. Then W and W' are non-isomorphic irreducible Q-modules.*

Proof. By Lemma 3.1, W and W' must be irreducible Q-modules. Suppose they are isomorphic as Q-modules. By Lemma 3.5 and since $v = v'$, the two are quasi-isomorphic T-modules with the same endpoint. By Lemma 3.4, the two are isomorphic T-modules. Hence, $(v, d, p, t) = (v', d', p', t')$ which contradicts assumption. □

Lemma 5.3. *With reference to Assumption 4.1, suppose $v \neq v'$ and $t \neq t'$. Then W and W' are non-isomorphic irreducible Q-modules.*

Proof. Let $\{w_{ij}\}$ (resp. $\{w'_{ij}\}$) denote a basis for W (resp. W') satisfying the conditions of Proposition 4.2. Suppose W and W' are isomorphic as Q-modules. By Lemma 3.5, there exists a quasi-isomorphism σ from W to W'. Since σ is a bijection and w'_{00} is a nonzero vector in W', there exist scalars $\{c_{kl} \mid 0 \leq k \leq d, 0 \leq l \leq p\}$ such that

$$\sigma(c_{00}w_{00} + c_{10}w_{10} + c_{01}w_{01} + \cdots + c_{dp}w_{dp}) = w'_{00}. \tag{15}$$

Pre-multiplying (15) by $E^*_{v'}$ and using Lemma 4.4(i) with $\alpha = v'$, we obtain

$$E^*_{v'}\sigma(c_{00}w_{00} + c_{10}w_{10} + c_{01}w_{01} + \cdots + c_{dp}w_{dp}) = w'_{00}. \tag{16}$$

Using (7) with $i = v$ and $\gamma = v' - v$, (16) becomes

$$\sigma(E^*_v(c_{00}w_{00} + c_{10}w_{10} + c_{01}w_{01} + \cdots + c_{dp}w_{dp})) = w'_{00}. \tag{17}$$

Using Lemma 4.4(i) with $\alpha = v$, (17) becomes $\sigma(c_{00}w_{00}) = w'_{00}$. Observe that $c_{00} \neq 0$ since σ is a bijection and w_{00} and w'_{00} are nonzero vectors. By (6), we have $\sigma F = F\sigma$ on W. By this and (13), we have

$$tw'_{00} = \sigma F\,(c_{00}w_{00}) = F\sigma\,(c_{00}w_{00}) = Fw'_{00} = t'w'_{00}$$

and so $t = t'$ which is a contradiction. So, W and W' are non-isomorphic as Q-modules. Irreducibility follows from Lemma 3.1. □

Lemma 5.4. *With reference to Assumption 4.1, assume $v \neq v'$ and $(d, p, t) = (d', p', t')$. Then W and W' are isomorphic irreducible Q-modules.*

Proof. Let $\{w_{ij}\}$ (resp. $\{w'_{ij}\}$) denote a basis for W (resp. W') satisfying the conditions of Proposition 4.2. To prove W and W' are isomorphic Q-modules, it suffices to show that the two are quasi-isomorphic. We define the $\mathbb{C}$-linear map $\sigma : W \to W'$ such that $\sigma(w_{ij}) = w'_{ij}$ for all $i, j \in \mathbb{N} \cup \{0\}$. We claim σ is a quasi-isomorphism. Clearly, σ is a bijection. By (12), we obtain

$$\sigma L(w_{ij}) = 3(d - i + 1)w'_{i-1,j} + (p - j + 1)w'_{i,j-1} = L\sigma(w_{ij}) \quad (0 \leq i \leq d, 0 \leq j \leq p).$$

Since $\{w_{ij}\}$ is a basis for W, we get $\sigma L = L\sigma$ on W. Analogously, we prove $\sigma F = F\sigma$ and $\sigma R = R\sigma$ on W by (13)–(14). By Lemma 4.4(i), we have

$$\sigma E^*_\alpha(w_{ij}) = \begin{cases} 0 & \text{if } \alpha \neq v + i + j \\ w'_{ij} & \text{if } \alpha = v + i + j \end{cases} = E^*_{\alpha+(v'-v)}w'_{ij} = E^*_{\alpha+(v'-v)}\sigma(w_{ij})$$

for all $i, j \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{Z}$. Since $\{w_{ij}\}$ is a basis for W, we get $\sigma E^*_\alpha = E^*_{\alpha+v'-v}\sigma$ on W. Claim holds. Irreducibility immediately follows from Lemma 3.1. □

Lemma 5.5. *With reference to Assumption 4.1, let $v \neq v'$, $t = t'$, and $(d + 1)(p + 1) \neq (d' + 1)(p' + 1)$. Then W and W' are non-isomorphic irreducible Q-modules.*

Proof. By Proposition 4.2, we have $\dim W = (d + 1)(p + 1) \neq (d' + 1)(p' + 1) = \dim W'$. So, the two are non-isomorphic Q-modules. Irreducibility follows from Lemma 3.1. □

Lemma 5.6. *With reference to Assumption 4.1, let $v \neq v'$, $t = t'$, and $(d + 1)(p + 1) = (d' + 1)(p' + 1)$ but $(d, p) \neq (d', p')$. Then W and W' are non-isomorphic irreducible Q-modules.*

Proof. Let $A|_W$ (resp. $A|_{W'}$) denote the restriction of A on the irreducible module W (resp. W'). Suppose W and W' are isomorphic Q-modules. Since $A \in Q$, the trace of $A|_W$ must be equal to the trace of $A|_{W'}$. By (11), we have

$$\operatorname{trace}(A|_W) = (d + 1)(p + 1)(t + d - p)$$

$$\operatorname{trace}(A|_{W'}) = (d'+1)(p'+1)(t'+d'-p').$$

Since $(d+1)(p+1) = (d'+1)(p'+1)$ and $t = t'$, we get $d - p = d' - p'$. By Lemma 3.2 and Proposition 4.2(i), $d + p = d' + p'$. Consequently, we obtain $d = d'$ and $p = p'$ which contradict assumption. Thus, the two are non-isomorphic Q-modules. Irreducibility follows from Lemma 3.1. □

Theorem 5.7. *With reference to Assumption 4.1, W and W' are isomorphic irreducible Q-modules if and only if $(d,p,t) = (d',p',t')$. Equivalently, W and W' are quasi-isomorphic T-modules if and only if $(d,p,t) = (d',p',t')$.*

Proof. Immediate from Lemmas 5.1–5.6. □

Corollary 5.8. *With reference to Assumption 4.1, we have $Q \neq T$.*

Proof. Follows immediately from Proposition 3.6 and Theorem 5.7. □

Remark 5.9. It was shown in [24, Theorem 5.1] that there is a homomorphism from the universal enveloping algebra of the classical Lie algebra $\mathfrak{so}_4$ (also known as special orthogonal algebra) to the quantum adjacency algebra Q of D. In [24, Theorem 5.5], it was proven that every irreducible Q-module is an irreducible $\mathfrak{so}_4$-module from the perspective of highest weight theory. With this approach, Theorem 5.7 was proven in [24, Corollary 5.7] via Lie algebraic means. The current paper directly carries out the computation mentioned in [24, Remark 5.8].

Acknowledgment

The second author would like to acknowledge the following institutions for their support in this research project: Department of Science and Technology-Science Education Institute (DOST-SEI) and De La Salle Lipa (DLSL).

References

[1] Bannai, E., Ito, T.: Algebraic Combinatorics I: Association Schemes. The Benjamin/Cummings, Menlo Park, California (1984)

[2] Brouwer, A.E., Cohen, A.M., Neumaier, A.: Distance-Regular Graphs. Springer, Berlin (1989)

[3] Caughman, J.S., MacLean, M., Terwilliger, P.: The Terwilliger algebra of an almost-bipartite P-and Q-polynomial association scheme. Discrete Math. **292**, 17-44 (2005)

[4] Caughman, J.S., Wolff, N.: The Terwilliger algebra of a distance-regular graph that supports a spin model, J. Algebraic Combin. **21** 289–310 (2005)

[5] van Dam, E.R., Koolen, J.H., Tanaka, H.: Distance-regular graphs, Electron. J. Combin., #DS22, 156 pp., arXiv:1410.6294 (2016)

[6] Fernández, B., Miklavič, Š: On the Terwilliger algebra of distance-biregular graphs, Linear Algebra Appl. **597** 18–32 (2020)

[7] Gao, S., Zhang, L., Hou, B.: The Terwilliger algebras of Johnson graphs. Linear Algebra Appl. **443**, 164-183 (2014)

[8] Gao, X., Gao, S., Hou, B.: The Terwilliger algebras of Grassmann graphs. Linear Algebra Appl. **471**, 427-448 (2015)

[9] Godsil, C.D.: Algebraic Combinatorics, Chapman and Hall, New York, 1993.

[10] Hamid, N., Oura, M.: Terwilliger algebras of some group association schemes, Math. J. Okayama Univ. **61** 199–204 (2019)

[11] Hora, A., Obata, N.: Quantum Probability and Spectral Analysis of Graphs. Theoretical and Mathematical Physics, Springer, Berlin Heidelberg (2007)

[12] Ito, T., Tanabe, K., Terwilliger, P.: Some algebra related to P- and Q-polynomial association schemes, in: Codes and Association Schemes (Piscataway NJ, 1999), Amer. Math. Soc., Providence RI, 167–192 (2001)

[13] Levstein, F., Maldonado, C., Penazzi, D.: The Terwilliger algebra of a Hamming scheme $H(d, q)$. Eur. J. Combin. **27**, 1-10 (2006)

[14] Li, S.D., Fan, Y.Z., Ito, T., Karimi, M., Xu, J.: The isomorphism problem of trees from the viewpoint of Terwilliger algebras, J. Combin. Theory Ser. A **177** (2021)

[15] Liang, X., Ito, T., Watanabe, Y.: The Terwilliger algebra of the Grassmann scheme $J_q(N, D)$ revisited from the viewpoint of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$, Linear Algebra Appl. **596** 117–144 (2020)

[16] MacLean, M.S., Miklavič, Š.: On bipartite distance-regular graphs with exactly one non-thin T-module with endpoint two, European J. Combin. **64** 125–137 (2017)

[17] MacLean, M.S., Miklavič, Š.: On bipartite distance-regular graphs with exactly two irreducible T-modules with endpoint two, Linear Algebra Appl. **515** 275–297 (2017)

[18] MacLean, M.S. Miklavič, Š.: On a certain class of 1-thin distance-regular graphs, Ars Math. Contemp. **18** (2) 187–210 (2020)

[19] MacLean, M.S., Miklavič, Š., Penjić, S.: On the Terwilliger algebra of bipartite distance-regular graphs with $\Delta_2 = 0$ and $c_2 = 1$. Linear Algebra Appl. **496**, 307-330 (2016)

[20] MacLean, M.S., Miklavič, Š., Penjić, S.: An A-invariant subspace for bipartite distance-regular graphs with exactly two irreducible T-modules with endpoint 2, both thin, J. Algebraic Combin. **48** (2018) 511–548.

[21] Miklavič, Š.: On bipartite Q-polynomial distance-regular graphs with diameter 9, 10, or 11, Electron. J. Combin. **25** (1), #P1.52 (2018)

[22] Martin, W.J., Tanaka, H.: Commutative association schemes. Eur. J. Combin. **30** 1497-1525 (2009)

[23] Morales, J.V.S: On Lee association schemes over $\mathbb{Z}_4$ and their Terwilliger algebra. Linear Algebra Appl. **510** 311-328 (2016)

[24] Morales, J.V.S., Palma, T.M.: On quantum adjacency algebras of Doob graphs and their irreducible modules, J. Algebr. Comb. 54 979–998 (2021)

[25] Morales, J.V.S., Pascasio, A.: An action of the tetrahedron algebra on the standard module for the Hamming graphs and Doob graphs. Graphs Combin. **30**, 1513-1527 (2014)

[26] Muzychuk, M., Xu, B.: Terwilliger algebras of wreath products of association schemes, Linear Algebra Appl. **493** 146–163 (2016)

[27] Penjić, S.: On the Terwilliger algebra of bipartite distance-regular graphs with $\Delta_2 = 0$ and $c_2 = 2$, Discrete Math. **340** 452–466 (2017)

[28] Tan, Y.Y., Fan, Y.Z. Ito, T., Liang, X.: The Terwilliger algebra of the Johnson scheme $J(N, D)$ revisited from the viewpoint of group representations, European J. Combin. **80** 157–171 (2019)

[29] Tanabe, K.: The irreducible modules of the Terwilliger algebras of Doob schemes. J. Algebr. Combin. **6**, 173-195 (1997)

[30] Terwilliger, P.: The subconstituent algebra of an association scheme I. J. Algebr. Combin. **1**, 363-388 (1992)

[31] Terwilliger, P.: The subconstituent algebra of an association scheme II. J. Algebr. Combin. **2**, 73-103 (1993)

[32] Terwilliger, P.: The subconstituent algebra of an association scheme III. J. Algebr. Combin. **2**, 177-210 (1993)

[33] Terwilliger, P., Žitnik, A.: The quantum adjacency algebra and subconstituent algebra of a graph. J. Combin. Theory Ser. A **166**, 297-314 (2019)

[34] Tomiyama, M.: The Terwilliger algebra of the incidence graph of the Hamming graph, J. Algebraic Combin. **48** 77–118 (2018)

This page is intentionally left blank