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Modularity of generalized Weber functions $\nu_{l,k,N/l}$

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Abstract

For a complex variable z, the Dedekind η-function is defined by the infinite product $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$, where $q = \exp(2\pi i z)$. Let $s = 24/\gcd(N-1, 24)$ be the integer measuring how far $N-1$ is from being divisible by 24. For a positive divisor l of N and an integer k with $0 \leq k < \frac{N}{l}$ and $\gcd(l, k, N/l) = 1$, we study the *generalized Weber function* $\nu_{l,k,N/l}^s$, where

$$\nu_{l,k,N/l}(z) = \sqrt{l}\cdot\frac{\eta(\frac{lz+k}{N/l})}{\eta(z)}.$$

We show that the functions $\nu_{l,k,N/l}(z)$ are modular functions of level $24N$. We also present a technique for computing the modular polynomial associated to the function $\nu_{1,0,6}^{24}$.

Key words: modular functions, generalized Weber functions, modular polynomial
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1 Introduction

One of the central features of class field theory focuses on describing abelian extensions of number fields explicitly. The celebrated Kronecker-Weber theorem asserts that abelian extensions over the rational field $\mathbb{Q}$ are subfields of the cyclotomic fields $\mathbb{Q}(\zeta)$ obtained by adjoining a root of unity $\zeta \in \mathbb{C}$ to $\mathbb{Q}$. Such abelian extensions can be constructed using the roots of a cyclotomic polynomial Φ_N for some positive integer N, which is irreducible over $\mathbb{Q}$. By Galois theory, we can find explicit generators for the subfields of $\mathbb{Q}(\zeta)$.

For imaginary quadratic fields K, there is an analogue of the Kronecker-Weber theorem known as the *theory of complex multiplication*. This theory describes abelian extensions of K as *ray class fields* generated by the values of suitable *modular functions*, which are meromorphic functions defined on the upper half plane $\mathbb{H}$. One can use modular functions of higher level, which form the *modular function field* $\mathcal{F}$, to generate such class fields. One of the important results in the theory of complex multiplication is the following.

Theorem 1.1. *Let K be imaginary quadratic and let $\theta \in K \cap \mathbb{H}$. Then the maximal abelian extension K^{ab} over K is generated by the finite values of $h(\theta)$ for all modular functions $h \in \mathcal{F}$.*

An immediate consequence of the above theorem is the existence of *ring class fields* over K. For an element $\theta \in \mathbb{H}$ in an order $\mathcal{O}$ of K with discriminant $\Delta(\mathcal{O})$, the value $j(\theta)$, where

$$j(z) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots$$

is the *j-invariant*, and $q = e^{2\pi i z}$, is an algebraic integer which generates the ring class field $H_{\mathcal{O}}$ over K. The minimal polynomial of $j(\theta)$ over K is called the *ring class polynomial* $P^j_{\Delta(\mathcal{O})}$ which has degree $h(\Delta(\mathcal{O}))$, the class number of $\Delta(\mathcal{O})$, and surprisingly integer coefficients. We remark that if $\mathcal{O}$ is the ring of integers of K, then $j(\theta)$ generates the *Hilbert class field* H_K over K.

For small absolute values of discriminants $\Delta(\mathcal{O})$ of $\mathcal{O}$, the class polynomial $P^j_{\Delta(\mathcal{O})}$ becomes unwieldy. For an order $\mathcal{O} = \mathbb{Z}[\sqrt{-38}]$ of $K = \mathbb{Q}(\sqrt{-38})$ with class number 6, we have

$$\begin{aligned} P^j_{-152}(X) = X^6 &- 66246265919280000X^5 \\ &+ 17024071380555203520000000X^4 \\ &+ 6854544294799483688960000000000X^3 \\ &+ 2783058624787093614292992000000000000X^2 \\ &- 1380504171426125758791680000000000000000X \\ &+ 472390748138731280269312000000000000000000 \end{aligned}$$

and as the discriminant "grows", the coefficients of $P^j_{\Delta(\mathcal{O})}$ grows rapidly. Weber [13] discovered that in some cases, the values of modular functions of higher levels at algebraic integers can generate ring class fields whose minimial polynomials have smaller coefficients. In the example above, using the Weber function $\mathfrak{f}_2(z) = \eta(\frac{z}{2})/\eta(z)$ of level 48, we find that $\mathfrak{f}_2^2(\sqrt{-38})$ generates $H_{\mathcal{O}}$ over K with minimal polynomial

$$P^{\mathfrak{f}_2}_{-152} = X^6 - 16X^4 - 40X^3 - 32X^2 + 8$$

over K. His theory of these *class invariants* relies on a potpourri of clever tricks, numerical observations and open questions that have no clear distinctions to one another and thus can be hardly regarded as a "systematic" procedure. The regained interest in class invariants was due partly to Stark's comments [12] that Heegner's Weber-inspired proof of the *class one problem* for imaginary quadratic fields was correct and mainly to Shimura's contributions to modular forms through his book [10]. The interested reader may read the works of Schertz [9], Gee [6] and Sotakova [11] about several results concerning these class invariants.

In this paper, we will not dwell on finding class invariants and instead focus on examining the properties of the generalized Weber functions

$$\nu_{l,k,N/l}(z) = \sqrt{l} \cdot \frac{\eta(\frac{lz+k}{N/l})}{\eta(z)},$$

where l is a positive divisor of N and k is an integer with $0 \leq k < \frac{N}{l}$ and $\gcd(l, k, N/l) = 1$. Enge and Morain [4] studied the modularity of these functions in the case $l = 1$ and $k = 0$

and proved that for some divisor e of $s = 24/\gcd(N-1, 24)$, $\nu_{1,0,N}^e$ is a modular function of level $24N$. Gee [6] also examined these functions and showed that e can be taken as 1, that is, $\nu_{1,0,N}$ is a modular function of level $24N$ using Meyer's formula. She then remarked that the functions $\nu_{1,k,N}$ for $1 \le k < N$ and $\nu_{N,0,1}$ are modular functions of level $24N$ as a consequence of her result. Our main theorem is a generalization of Gee's result.

Theorem 1.2. *For all positive divisors l of N and integers k with $0 \le k < \frac{N}{l}$ and $\gcd(l,k,N/l) = 1$, $\nu_{l,k,N/l}$ is a modular function of level $24N$.*

The flow of this paper is as follows. In Section 2, we present the actions of the generators of the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ on the generalized Weber functions $\nu_{l,k,N/l}$ and prove the above theorem using Meyer's formula. In Section 3, we show that $\nu_{l,k,N/l}^s$ form a set of conjugates of $\nu_{1,0,N}^s$ using coset representatives of the congruence subgroup $\Gamma^0(N)$ over Γ. We also provide a method of constructing the modular polynomial of $\nu_{1,0,N}^s$ using these representatives and illustrate the process for $N = 6$ using resultants.

2 Properties of the generalized Weber functions $\nu_{l,k,N/l}$

We consider functions that are invariant under the action of the *congruence subgroups* of Γ, namely

$$\Gamma^0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma : N \mid b \right\}, \quad \Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma : N \mid c \right\}$$

and

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma : a \equiv d \equiv 1 \pmod{N}, b \equiv c \equiv 0 \pmod{N} \right\}$$

for some positive integer N. For a congruence subgroup Γ' of Γ, we denote the field of all modular functions that are invariant under Γ' by $\mathbb{C}_{\Gamma'}$. Thus, $\mathbb{C}_{\Gamma} = \mathbb{C}(j)$ and the extension $\mathbb{C}_{\Gamma^0(N)}/\mathbb{C}_{\Gamma}$ is not normal. Equivalently, since

$$\Gamma_0(N) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} \Gamma^0(N) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

any modular function $f(z)$ for $\Gamma^0(N)$ gives rise to a modular function $f([\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}]z)$ for $\Gamma_0(N)$. We define a modular function of level N by a function f that is invariant under $\Gamma(N)$ and whose q-expansion has coefficients in the cyclotomic field $\mathbb{Q}(\zeta_N)$, that is, $f \in \mathbb{Q}(\zeta_N)((q^{1/N}))$, where $q = e^{2\pi i z}$.

For a complex variable z, we recall Dedekind η-function defined by

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

Let $s = 24/\gcd(N-1, 24)$ be the integer measuring how far $N-1$ is from being divisible by 24. For a positive divisor l of N and an integer k with $0 \le k < \frac{N}{l}$ and $\gcd(l,k,N/l) = 1$, we study the *generalized Weber function* $\nu_{l,k,N/l}^s$, where

$$\nu_{l,k,N/l}(z) = \sqrt{l} \cdot \frac{\eta(\frac{lz+k}{N/l})}{\eta(z)}.$$

In order to establish that generalized Weber functions are invariant under the action of $\Gamma(24N)$, we need to examine the transformation behavior of these functions under the action of Γ. Meyer's formula provides an explicit action of Γ on η.

Lemma 2.1 (Meyer's formula)**.** *Let* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$ *be normalized so that* $c \geq 0$ *and* $d > 0$ *if* $c = 0$. *Set* $c = 2^r c_0$ *with* c_0 *odd if* $c \neq 0$, *for* $c = 0$ *set* $c_0 = 1$ *and* $r = 0$. *Then*

$$\eta \circ M(z) = \varepsilon(M)\sqrt{cz+d}\,\eta(z)$$

where the real part of $\sqrt{cz+d}$ *is positive and*

$$\varepsilon(M) = \left(\frac{a}{c_0}\right)\zeta_{24}^{ab+cd(1-a^2)-ca+3c_0(a-1)+3r(a^2-1)/2}.$$

Proof. See [9, Prop. 2]. □

Lemma 2.2. *Let* l, k, N *be integers with* l *and* N *positive such that* $l \mid N, 0 \leq k < \frac{N}{l}$ *and* $\gcd(l, k, \frac{N}{l}) = 1$. *Then there exist matrices* $U, V \in \Gamma$ *such that*

$$U\begin{bmatrix} l & k \\ 0 & \frac{N}{l} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & N \end{bmatrix} V.$$

Proof. Since $\gcd(l, k, \frac{N}{l}) = 1$, there exists $t \in \mathbb{Z}$ such that $\gcd(lt - k, \frac{N}{l}) = 1$. By extended Euclidean algorithm, we can find integers x and y such that $x(lt-k) - \frac{yN}{l} = 1$. Choosing the matrices

$$U = \begin{bmatrix} x & y \\ \frac{N}{l} & lt-k \end{bmatrix}, \qquad V = \begin{bmatrix} xl & xlt-1 \\ 1 & t \end{bmatrix}$$

completes the proof. □

The action of Γ on the generalized Weber functions can be obtained from the action of the generators

$$S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad T := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

on η via the formulas

$$\eta \circ S(z) = \zeta_8^{-1}\sqrt{z}\eta(z) \qquad \text{and} \qquad \eta \circ T(z) = \zeta_{24}\eta(z).$$

Lemma 2.3. *Let* l, k, N *be integers with* l *and* N *positive such that* $l \mid N, 0 \leq k < \frac{N}{l}$ *and* $\gcd(l, k, \frac{N}{l}) = 1$. *For* $i \in \{1, 2\}$, *there exist integers* l_i, k_i *and matrices* $M_i \in \Gamma$ *such that*

1. $l_i > 0, l_i \mid N, 0 \leq k_i < \frac{N}{l_i}, \gcd(l_i, k_i, \frac{N}{l_i}) = 1$, *and*
2. $\begin{bmatrix} l & k \\ 0 & \frac{N}{l} \end{bmatrix} S = M_1 \begin{bmatrix} l_1 & k_1 \\ 0 & \frac{N}{l_1} \end{bmatrix}$ *and* $\begin{bmatrix} l & k \\ 0 & \frac{N}{l} \end{bmatrix} T = M_2 \begin{bmatrix} l_2 & k_2 \\ 0 & \frac{N}{l_2} \end{bmatrix}$.

Proof. As $\gcd(l,k,\frac{N}{l})=1$, there exists $t\in\mathbb{Z}$ such that $\gcd(lt-k,\frac{N}{l})=1$. By extended Euclidean algorithm, we can find integers x and y such that $x(lt-k)-\frac{yN}{l}=1$. By Lemma 2.2, we have,

$$\begin{bmatrix} l & k \\ 0 & \frac{N}{l}\end{bmatrix} S = U^{-1}\begin{bmatrix} 1 & 0 \\ 0 & N\end{bmatrix} VS = U^{-1}\begin{bmatrix} 1 & 0 \\ 0 & N\end{bmatrix} VS \begin{bmatrix} l_1 & k_1 \\ 0 & \frac{N}{l_1}\end{bmatrix}^{-1}\begin{bmatrix} l_1 & k_1 \\ 0 & \frac{N}{l_1}\end{bmatrix}$$

where

$$U=\begin{bmatrix} x & y \\ \frac{N}{l} & lt-k\end{bmatrix}, \qquad V=\begin{bmatrix} xl & xlt-1 \\ 1 & t\end{bmatrix}.$$

We compute

$$\begin{aligned} U_1 &:= \begin{bmatrix} 1 & 0 \\ 0 & N\end{bmatrix} VS \begin{bmatrix} l_1 & k_1 \\ 0 & \frac{N}{l_1}\end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & N\end{bmatrix}\begin{bmatrix} xlt-1 & -xl \\ t & -1\end{bmatrix}\begin{bmatrix} \frac{1}{l_1} & -\frac{k_1}{N} \\ 0 & \frac{l_1}{N}\end{bmatrix} \\ &= \begin{bmatrix} \frac{xlt-1}{l_1} & \frac{-k_1(xlt-1)-xll_1}{N} \\ \frac{Nt}{l_1} & -l_1-k_1t\end{bmatrix}. \end{aligned}$$

Choosing $M_1=U^{-1}U_1\in\Gamma$, we require that $U_1\in\Gamma$. Thus, we take $l_1=\gcd(xlt-1,N)$ and since the upper right entry of U_1 is an integer, we have

$$N \mid -k_1(xlt-1)-xll_1 \iff \frac{N}{l_1} \mid -k_1\left(\frac{xlt-1}{l_1}\right)-xl.$$

As $\frac{N}{l_1}$ and $\frac{xlt-1}{l_1}$ are relatively prime, $\frac{xlt-1}{l_1}$ is invertible modulo $\frac{N}{l_1}$. We then take $k_1 = -xl(\frac{xlt-1}{l_1})^{-1} \pmod{\frac{N}{l_1}}$. Let $d=\gcd(l_1,k_1,\frac{N}{l_1})$. Then $d\mid k_1$ and $d\mid l_1\mid xlt-1$. But $d\mid\frac{N}{l_1}$ so $d\mid -k_1(\frac{xlt-1}{l_1})-xl$. This implies $d\mid xl$ so that $d\mid xlt-(xlt-1)=1$ and $d=1$. On the other hand, we have

$$\begin{bmatrix} l & k \\ 0 & \frac{N}{l}\end{bmatrix} T = U^{-1}\begin{bmatrix} 1 & 0 \\ 0 & N\end{bmatrix} VT = U^{-1}\begin{bmatrix} 1 & 0 \\ 0 & N\end{bmatrix} VT \begin{bmatrix} l_2 & k_2 \\ 0 & \frac{N}{l_2}\end{bmatrix}^{-1}\begin{bmatrix} l_2 & k_2 \\ 0 & \frac{N}{l_2}\end{bmatrix}$$

We compute

$$\begin{aligned} V_1 &:= \begin{bmatrix} 1 & 0 \\ 0 & N\end{bmatrix} VT \begin{bmatrix} l_2 & k_2 \\ 0 & \frac{N}{l_2}\end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & N\end{bmatrix}\begin{bmatrix} xl & xlt-1+xl \\ 1 & 1+t\end{bmatrix}\begin{bmatrix} \frac{1}{l_2} & -\frac{k_2}{N} \\ 0 & \frac{l_2}{N}\end{bmatrix} \\ &= \begin{bmatrix} \frac{xl}{l_2} & \frac{-k_2lx+l_2(xlt-1+xl)}{N} \\ \frac{N}{l_2} & l_2-k_2+l_2t\end{bmatrix}. \end{aligned}$$

Again, choosing $M_2=U^{-1}V_1$, we require that $V_1\in\Gamma$. We can take $l_2=l$ and as the upper right entry of V_1 is an integer, we have

$$N \mid -k_2lx+l(xlt-1+xl) \iff \frac{N}{l} \mid -k_2x+xlt+xl-1.$$

Note that x is invertible modulo $\frac{N}{l}$, so we take $k_2 = (xlt + xl - 1)x^{-1} \pmod{\frac{N}{l}}$. Let $e = \gcd(l, k_2, \frac{N}{l})$ so that $e \mid xl$ and $e \mid k_2$. But $e \mid \frac{N}{l}$ so $e \mid -k_2x + xlt + xl - 1$. This implies $e = 1$. In both cases, condition (1) and (2) hold for the values of l_i, k_i and $\frac{N}{l_i}, i \in \{1, 2\}$. $\square$

Remark 2.4. In the proof of Lemma 2.3, we explicitly compute the matrices

$$M_1 = U^{-1}U_1 = \begin{bmatrix} * & * \\ \dfrac{N/l}{l_1} & -\dfrac{k_1}{l} \end{bmatrix} \quad \text{and} \quad M_2 = U^{-1}V_1 = \begin{bmatrix} * & * \\ 0 & 1 \end{bmatrix},$$

where the asterisks denote integer entries. Note that the entries on the second row of M_1 are integers. Indeed, since $l_1 = \gcd(xlt - 1, N)$, we have $\gcd(l_1, l) = 1$ and $l_1 \mid N$. As $l \mid N$ and $N \mid -k_1(xlt - 1) - xll_1$, we see that $l \mid k_1$ and $l_1 \mid \frac{N}{l}$. Thus, the assertion follows and we can apply Meyer's formula to M_1 and M_2.

Using Lemmata 2.1 − 2.3, we derive the transformation formulas

$$\begin{aligned}
\nu_{l,k,N/l} \circ S(z) &= \sqrt{l} \cdot \frac{\eta \circ [\begin{smallmatrix} l & k \\ 0 & N/l \end{smallmatrix}] S(z)}{\eta \circ S(z)} = \sqrt{l} \cdot \frac{\eta \circ M_1 [\begin{smallmatrix} l_1 & k_1 \\ 0 & N/l_1 \end{smallmatrix}](z)}{\eta \circ S(z)} \\
&= \sqrt{l} \cdot \frac{\varepsilon(M_1) \left(\sqrt{\frac{N/l}{l_1} [\begin{smallmatrix} l_1 & k_1 \\ 0 & N/l_1 \end{smallmatrix}] z - \frac{k_1}{l}} \right) \eta \circ [\begin{smallmatrix} l_1 & k_1 \\ 0 & N/l_1 \end{smallmatrix}](z)}{\eta \circ S(z)} \\
&= \sqrt{l} \cdot \frac{\varepsilon(M_1) \sqrt{\frac{N/l}{l_1} (\frac{l_1 z + k_1}{N/l_1}) - \frac{k_1}{l}} \, \eta(\frac{l_1 z + k_1}{N/l_1})}{\zeta_8^{-1} \sqrt{z} \, \eta(z)} \\
&= \sqrt{l} \cdot \frac{\varepsilon(M_1) \sqrt{\frac{l_1 z}{l}} \, \eta(\frac{l_1 z + k_1}{N/l_1})}{\zeta_8^{-1} \sqrt{z} \, \eta(z)} = \zeta_8 \varepsilon(M_1) \nu_{l_1, k_1, N/l_1}(z) \qquad (1)
\end{aligned}$$

and

$$\begin{aligned}
\nu_{l,k,N/l} \circ T(z) &= \sqrt{l} \cdot \frac{\eta \circ [\begin{smallmatrix} l & k \\ 0 & N/l \end{smallmatrix}] T(z)}{\eta \circ T(z)} = \sqrt{l} \cdot \frac{\eta \circ M_2 [\begin{smallmatrix} l & k_2 \\ 0 & N/l \end{smallmatrix}](z)}{\eta \circ T(z)} \\
&= \sqrt{l} \cdot \frac{\varepsilon(M_2) \eta(\frac{lz + k_2}{N/l})}{\eta \circ T(z)} = \sqrt{l} \cdot \frac{\varepsilon(M_2) \eta(\frac{lz + k_2}{N/l})}{\zeta_{24} \eta(z)} \\
&= \zeta_{24}^{-1} \varepsilon(M_2) \nu_{l, k_2, N/l}(z) \qquad (2)
\end{aligned}$$

where l_1, k_1, k_2 are integers and $M_1, M_2 \in \Gamma$ are matrices satisfying the conditions of Lemma 2.3.

Enge and Morain used Meyer's formula to find the action of S and T on $\nu_{1,0,N}$, which proves the modularity of a power of $\nu_{1,0,N}$ as shown by the following theorem.

Theorem 2.5. *Let e and t be positive integers such that $s \mid t \mid 24$ and $e \mid t$. Write $N = 2^\lambda N_1$ with N_1 odd. If N_1 is a square or e is even, then $\nu_{1,0,N}^e$ is invariant under $\Gamma(\frac{t}{e}) \cap \Gamma^0(\frac{t}{e}N)$. Otherwise, $\nu_{1,0,N}^e$ is invariant under $\Gamma(\frac{t}{e}N_1) \cap \Gamma^0(\frac{t}{e}N)$. In both cases, $\nu_{1,0,N}^e$ is a modular function of level $24N$.*

Proof. See [4, Thm. 3.2]. □

Gee utilized the same formula and established that the exponent can be taken as 1, as shown by the following theorem.

Theorem 2.6. *The function $\nu_{1,0,N}$ is a modular function of level $24N$. Consequently, the functions $\nu_{1,k,N}, 1 \leq k < N$ and $\nu_{N,0,1}$ are modular functions of level $24N$.*

Proof. See [6, p. 76, Thm. 5] and [6, p. 77, Lem. 8]. The second statement follows from the fact that $\Gamma(24N)$ is a normal subgroup of Γ (cf. [6, p. 77]). □

Proof. (of Theorem 1.2). Let $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma(24N)$. By replacing M with $-M$ if necessary, we may assume that $c \geq 0$ and $d > 0$ if $c = 0$. Consider the matrix

$$M_{l,k} = \begin{bmatrix} l & k \\ 0 & \frac{N}{l} \end{bmatrix} M \begin{bmatrix} l & k \\ 0 & \frac{N}{l} \end{bmatrix}^{-1} = \begin{bmatrix} a + \dfrac{ck}{l} & \dfrac{bl + dk - k(a + \frac{ck}{l})}{\frac{N}{l}} \\ \dfrac{cN}{l^2} & d - \dfrac{ck}{l} \end{bmatrix}.$$

Since $c \equiv 0 \pmod{24N}$, $l \mid c$ and $\frac{N}{l} \mid \frac{c}{l}$. Also, since M is equivalent to the 2-by-2 identity matrix modulo $24N$, we have $bl+dk-k(a+\frac{ck}{l}) \equiv 0 \pmod{\frac{N}{l}}$. Thus $M_{l,k} \in \Gamma$, and applying Meyer's formula to both $M \in \Gamma$ and $M_{l,k}$ yields

$$\begin{aligned} \nu_{l,k,N/l} \circ M &= \sqrt{l} \cdot \frac{\eta \circ \left[\begin{smallmatrix} l & k \\ 0 & N/l \end{smallmatrix}\right] M(z)}{\eta \circ M(z)} = \sqrt{l} \cdot \frac{\eta \circ M_{l,k} \left[\begin{smallmatrix} l & k \\ 0 & N/l \end{smallmatrix}\right](z)}{\eta \circ M(z)} \\ &= \sqrt{l} \cdot \frac{\varepsilon(M_{l,k}) \left(\sqrt{\frac{cN}{l^2} \left[\begin{smallmatrix} l & k \\ 0 & N/l \end{smallmatrix}\right] z + d - \frac{ck}{l}} \right) \eta \circ \left[\begin{smallmatrix} l & k \\ 0 & N/l \end{smallmatrix}\right](z)}{\eta \circ M(z)} \\ &= \sqrt{l} \cdot \frac{\varepsilon(M_{l,k}) \sqrt{\frac{cN}{l^2} (\frac{lz+k}{N/l}) + d - \frac{ck}{l}}\, \eta(\frac{lz+k}{N/l})}{\varepsilon(M) \sqrt{cz+d}\, \eta(z)} \\ &= \sqrt{l} \cdot \frac{\varepsilon(M_{l,k}) \sqrt{cz+d}\, \eta(\frac{lz+k}{N/l})}{\varepsilon(M) \sqrt{cz+d}\, \eta(z)}. \end{aligned}$$

As $M \in \Gamma(24N)$, we see that $M_{l,k}$ is equivalent to the 2-by-2 identity matrix modulo 24. Write $c = 2^r c_0, l = 2^s l_0$ and $N = 2^t N_0$ with c_0, l_0 and N_0 odd. Then

$$\begin{aligned} \varepsilon(M_{l,k}) &= \left(\frac{a + \frac{ck}{l}}{\frac{c_0 N_0}{l_0^2}} \right) = \left(\frac{a + \frac{ck}{l}}{\frac{c_0}{l_0}} \right) \left(\frac{a + \frac{ck}{l}}{\frac{N_0}{l_0}} \right) = \left(\frac{a}{\frac{c_0}{l_0}} \right) \left(\frac{a}{\frac{N_0}{l_0}} \right) \\ &= \left(\frac{a}{c_0} \right) \left(\frac{a}{l_0} \right)^{-1} = \left(\frac{a}{c_0} \right) = \varepsilon(M), \end{aligned}$$

since $a \equiv 1 \pmod{24N}$ implies $a \equiv 1 \pmod{l_0}$ and $a \equiv 1 \pmod{\frac{N_0}{l_0}}$. Hence, $\nu_{l,k,N/l} \circ M = \nu_{l,k,N/l}$ for all such values of l and k. As $\sqrt{l} \in \mathbb{Q}(\zeta_{24N})$, the coefficients of the q-expansion of $\nu_{l,k,N/l}$ lie on $\mathbb{Q}(\zeta_{24N})$. We conclude that $\nu_{l,k,N/l}$ is a modular function of level $24N$. □

3 Modular polynomial for $\nu_{1,0,6}^{24}$

We introduce the modular polynomial associated to the modular function $\nu_{1,0,N}^s$ which is its characteristic polynomial with respect to the field extension $\mathbb{C}_{\Gamma^0(N)}/\mathbb{C}_\Gamma$. This extension is not Galois as $\Gamma^0(N)$ is not a normal subgroup of Γ, so we consider the embeddings of $\mathbb{C}_{\Gamma^0(N)}$ into the algebraic closure of $\mathbb{C}_\Gamma$ corresponding to the right coset representatives of $\Gamma^0(N)\backslash\Gamma$. We now define

$$\Phi[\nu](X) := \prod_{g\in\Gamma^0(N)\backslash\Gamma}(X - \nu_{1,0,N}^s \circ g).$$

We see clearly that $\Phi[\nu](X) \in \mathbb{C}(j)[X]$. Because $\nu_{1,0,N}^s$ and $\nu_{1,0,N}^s \circ S$ are holomorphic in $\mathbb{H}\cup\mathbb{P}^1(\mathbb{Q})$ and have rational q-expansions, by Hasse q-expansion principle [3], we have $\Phi[\nu](X) \in \mathbb{Z}[j, X]$. In order to compute $\Phi[\nu](X)$ explicitly, we need an exact description of the right cosets $\Gamma^0(N)\backslash\Gamma$, which can be obtained by using right cosets of $\Gamma_0(N)\backslash\Gamma$.

Theorem 3.1 (Orive)**.** *Let d be a proper divisor of N and let*

$$M_d = \left\{dt : 0 < t < \frac{N}{d}, \gcd(t, d, \frac{N}{d}) = 1\right\}.$$

Denote M as the union of all M_s for all proper divisors s of N and for each $m \in M$, let $n := n(m)$ be the smallest positive integer such that $N \mid nm^2$. Then a complete set of representatives for right cosets of $\Gamma_0(N)\backslash\Gamma$ is given by

$$\{I\} \cup \{ST^k : 0 \le k \le N-1\} \cup \{ST^mST^j : m \in M, 0 \le j \le n-1\}.$$

Proof. See [8, Prop. 2]. □

Corollary 3.2. *Under the conditions of Theorem 3.1, a complete set of representatives for right cosets of $\Gamma^0(N)\backslash\Gamma$ is given by*

$$\{I\} \cup \{SR^k : 0 \le k \le N-1\} \cup \{SR^mSR^j : m \in M, 0 \le j \le n-1\}$$

where $R = (TST)^{-1}$. We remark that $[\Gamma : \Gamma^0(N)] = \psi(N) = \prod_{p\ prime, p\mid N}(1 + \frac{1}{p})$.

Proof. Write Γ as a disjoint union $\Gamma = \sqcup_{k=1}^{\psi(N)}\Gamma_0(N)g_k$ for some right cosets representatives $g_1, \dots, g_{\psi(N)}$ of $\Gamma_0(N)$ over Γ and let $u \in \Gamma$. Then $u = vg_j$ for some $v \in \Gamma_0(N)$ and $j \in \{1, \dots, \psi(N)\}$. We have $u^t = (g_j)^tv^t$, where M^t denotes the transpose of a matrix M. As $u^t \in \Gamma$, we see that $\Gamma = \sqcup_{k=1}^{\psi(N)}(g_k)^t\Gamma^0(N)$ and applying the bijection $g\Gamma^0(N) \mapsto \Gamma^0(N)g^{-1}$ yields $\Gamma = \sqcup_{k=1}^{\psi(N)}\Gamma^0(N)((g_k)^t)^{-1}$. The desired conclusion follows from the fact that $(S^t)^{-1} = S$ and $(T^t)^{-1} = R$. □

Using the transformation formulas (1) and (2), and applying Lemma 2.3 and Corollary 3.2, we have $\nu_{1,0,N}^s \circ T^k = \zeta_{24}^{-ks}\nu_{1,k,N}^s, \nu_{1,0,N}^s \circ S = \nu_{N,0,1}^s$ and for a right coset representative M of $\Gamma^0(N)\backslash\Gamma$ that is not S nor a power of T, we have $\nu_{1,0,N}^s \circ M = \zeta_{l,k}^s\nu_{l,k,N/l}^s$ for some $\zeta_{l,k} \in \mathbb{Q}(\zeta_{24})$. Thus, we obtain the following.

Corollary 3.3. *The conjugates of* $\nu_{1,0,N}^s$ *are the modular functions* $\zeta_{l,k}^s \nu_{l,k,N/l}^s$ *for some positive divisor* l *of* N, *integer* k *with* $0 < k < \frac{N}{l}$ *and* $\gcd(k, l, \frac{N}{l}) = 1$, *and an element* $\zeta_{l,k} \in \mathbb{Q}(\zeta_{24})$. *Consequently, we have*

$$\Phi[\nu_{1,0,N}^s](X, j) := \prod_{l,k} (X - \zeta_{l,k}^s \nu_{l,k,N/l}^s)$$

and the degree of $\Phi[\nu_{1,0,N}^s](X, j)$ *in* X *and* j *are* $\psi(N)$ *and*

$$\frac{s}{24} \left[N - 1 + \sum_{1<k<N, 1<\gcd(k,N)<\sqrt{N}} \mu(k) \left(1 - \frac{\gcd(k,N)^2}{N} \right) \right],$$

respectively, where $\mu(k)$ *is the smallest integer such that* $\mu(k)k - 1$ *and* N *are coprime.*

Proof. See [4, Thm. 7.3]. □

Remark 3.4. We note that if $N = pq$, where $p < q$ are primes, then the degree of $\Phi[\nu_{1,0,N}^s](X, j)$ in j is $\frac{s}{24}(N - 1 + q - p)$ (see [4, Thm. 7.6]).

Remark 3.5. We can also find the modular polynomial $\Phi[\nu_{1,0,N}^s]$ in the case that $N = pq$ with p and q prime using resultants, which we describe as follows (cf. [7]). Write $\nu_{1,0,N}^s(z) = \nu_{1,0,p}^s(z)\nu_{1,0,q}^s(z/p)$. For some divisors s_1 and s_2 of s, the modular polynomials $\Phi[\nu_{1,0,p}^{s_1}]$ and $\Phi[\nu_{1,0,p}^{s_2}]$ satisfy

$$\Phi[\nu_{1,0,p}^{s_1}](\nu_{1,0,p}^{s_1}(z), j(z)) = 0, \qquad \Phi[\nu_{1,0,q}^{s_2}](\nu_{1,0,q}^{s_2}(z/p), j(z/p)) = 0.$$

On the other hand, there is a bivariate polynomial Φ_p such that $\Phi_p(j(z), j(z/p)) = 0$ (known as the *classical modular polynomial*). We then eliminate $\nu_{1,0,p}^{s_1}(z), \nu_{1,0,q}^{s_2}(z/p)$ and $j(z/p)$ using resultants to get a polynomial Φ with $\Phi(\nu_{1,0,N}^s, j) = 0$ and then factor Φ to obtain the correct modular polynomial with degree $\psi(N)$ in X. We illustrate this method for $N = 6$. We know that the conjugates of $\nu_{1,0,6}^{24}$ are the generalized Weber functions $\nu_{l,k,6/l}^{24}$. Write

$$\nu_{1,0,6}^{24}(z) = \left(\frac{\eta(\frac{z}{6})}{\eta(z)} \right)^{24} = \left(\frac{\eta(\frac{z/2}{3})}{\eta(z/2)} \right)^{12 \cdot 2} \left(\frac{\eta(\frac{z}{2})}{\eta(z)} \right)^{24} \tag{3}$$

and let $\mathfrak{g}(z) = \eta(z/3)/\eta(z)$ and $\mathfrak{f}(z) = \eta(z/2)/\eta(z)$. The modular polynomials for $\mathfrak{g}(z/2)^{12}$ and $\mathfrak{f}(z)^{24}$ are

$$\Phi[\mathfrak{g}(z/2)^{12}](X, j_1) = X^4 + 36X^3 + 270X^2 + (756 - j_1)X + 729 \tag{4}$$

where $j_1 := j_1(z) = j(z/2)$ (see [6, eq. 5 p. 73] or [13, p. 255]) and

$$\Phi[\mathfrak{f}(z)^{24}](X, j) = X^3 + 48X^2 + (768 - j)X + 4096, \tag{5}$$

(see [2, p. 99]) respectively. The functions j and j_1 are related by the classical modular polynomial (see [2, p. 75])

$$\begin{aligned}\Phi_2(j,j_1) := {} & j^3 + j_1^3 - j^2 j_1^2 + 1488(j^2 j_1 + j j_1^2) - 162000(j^2 + j_1^2) + 40773375 j j_1 \\ & + 8748000000(j + j_1) - 157464000000000. \end{aligned} \tag{6}$$

Using MAGMA [1], we take the resultant of equations (3) to (6) and choose a factor whose degree in X is $\psi(6) = 12$, arriving at the modular polynomial

$$\begin{aligned}\Phi[\nu_{1,0,6}^{24}](X,j) = {} & X^{12} + 36288X^{11} + (3440226816 - 431460j)X^{10} \\
& + (109056j^2 - 50163935040j + 109207646945280)X^9 \\
& + (-6696j^3 + 3372539814j^2 - 3060482822737920j + 4284327340231557120)X^8 \\
& + (144j^4 + 35130240j^3 + 29781629564671296j^2 - 150945357921479884800j \\
& + 111106635441567808094208)X^7 \\
& + (-j^5 + 3864j^4 - 7502360202978084j^3 - 1266015680155533252096j^2 \\
& - 424265080525480124743680j + 2248666474050361425420877824)X^6 \\
& + (460319565496320j^4 - 232025985564586752960j^3 \\
& + 3266539722689396219412480j^2 - 16349318776164233382608240640j \\
& + 39752136338000494961380407902208)X^5 \\
& + (-9896812609536j^5 - 2407476972082120479j^4 - 9931694818294487485440j^3 \\
& + 2124144831431112866880159744j^2 + 359239342288242741416799177277440j \\
& + 404566267530357320253454333111173120)X^4 \\
& + (68719476736j^6 - 311711546474496j^5 + 507978495201116160j^4 \\
& - 130008450295278194863325184j^3 + 65427952163338431948417449066496j^2 \\
& - 870776926913051856042880101179719680j \\
& + 1565033484678756823775228573867342561280)X^3 \\
& + (2795153950927245451198464j^4 + 682156440320053120453815828480j^3 \\
& + 3345818930503173307774960259825664j^2 \\
& + 1302588988593527673994007935136563200j \\
& + 14084399260290413838252276776033058816)X^2 \\
& + (-19408409961765342806016j^5 + 72665086896849443465723904j^4 \\
& - 89717626423815861900574457856j^3 + 40562091222757453098173696311296j^2 \\
& - 5054530374576816035557692848209920j \\
& + 42250535333590949287706130842124288)X \\
& + 42247883974617233597120303333376.\end{aligned}$$

We see that the degree of $\Phi[\nu_{1,0,6}^{24}](X,j)$ in j is 6, as predicted by the previous remark.

The minimality of $\Phi[\nu_{1,0,6}^{24}](X, j)$ follows from the fact that $\nu_{1,0,6}^{24} \circ S$ is the only simple root having a positive order (see [5] for the definition of the order of a q-expansion). We deduce that $\mathbb{Q}(j) \subset \mathbb{Q}(\nu_{1,0,6})$ and $\mathbb{Q}(j) \subset \mathbb{Q}(\nu_{l,k,6/l})$ for all permissible values of l and k.

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References

[1] W. Bosma, J. Cannon and C. Playoust, *The MAGMA algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), 235–265.

[2] R. Bröker, *Constructing elliptic curves of prescribed order*, PhD thesis, Universiteit Leiden, 2006.

[3] M. Deuring, *Die Klassenkörper der komplexen Multiplikation*, Enzyklopädie der Math. Wiss., Band I, 2. Teil, Heft 10, Teil II, 1958.

[4] A. Enge and F. Morain, *Generalized Weber functions*, Acta Arith. **164** (2014), 309–341.

[5] A. Enge and R. Schertz, *Modular curves of composite level*, Acta Arith. **118** (2005), 129–141.

[6] A. Gee, *Class fields by Shimura reciprocity*, PhD thesis, University of Amsterdam, 2001.

[7] F. Morain, *Modular equations for some η-products*, Acta Arith. **161** (2013), 301–326.

[8] A. L. Orive, *Some presentations for $\overline{\Gamma}_0(N)$*, Conform. Geom. Dyn. **6** (2002), 33–60.

[9] R. Schertz, *Weber's class invariants revisited*, J. Théor. Nombres Bordeaux **14** (2002), 325–343.

[10] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton University Press, NJ, 1971.

[11] J. Sotakova, *Eta quotients and class fields of imaginary quadratic fields*, Master's thesis, ALGANT/Universiteit Leiden, 2017.

[12] H. Stark, *On a "gap" in a theorem of Heegner*, J. Number Theory **1** (1969), 16–27.

[13] H. Weber, *Lehrbuch der Algebra*, vol. 3, Friedrich Vieweg und Sohn, 1908.

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