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On the well-posedness of a quasilinear elliptic problem with semilinear terms in a two-component domain

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Abstract

This paper deals with the existence, uniqueness and boundedness of a weak solution of a quasilinear elliptic problem with semilinear terms posed in a two-component domain in $\mathbb{R}^N$ where $N \geq 2$. We show firstly, some well-posedness property for a related problem. To this aim, we study several properties of an associated operator and then invoke a fixed point theorem. We proceed in achieving the main goal by applying these preliminary results together with another fixed point theorem, lastly.

Key words: fixed point theorem, quasilinear problem, semilinear term, two-component domain, well-posed problem

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1 Introduction

The goal of this work is to show that a certain type of a quasilinear problem with semilinear terms in an open bounded set with two components, admits a unique solution and that this solution depends continuously on its data. Particularly, we consider the boundary value problem given by

$$\begin{cases} -\mathrm{div}\,(A\,(x,u_1)\,\nabla u_1) = f(u_1) & \text{in } O_1, \\ -\mathrm{div}\,(A\,(x,u_2)\,\nabla u_2) = f(u_2) & \text{in } O_2, \\ A\,(x,u_1)\,\nabla u_1 \cdot n_1 = -h(x)(u_1 - u_2) & \text{on } \Gamma, \\ A\,(x,u_2)\,\nabla u_2 \cdot n_2 = h(x)(u_1 - u_2) + g(x) & \text{on } \Gamma, \\ u_1 = 0 & \text{on } \partial O, \end{cases} \tag{1}$$

where n_i is the unit exterior normal to O_i for $i = 1, 2$. The domain O is an open bounded set in $\mathbb{R}^N$ ($N \geq 2$) that has two components O_1 and O_2 with $\overline{O}_2 \subseteq O_1$ and that $O = O_1 \cup \overline{O}_2$. The subdomains are separated by an imperfect interface Γ. One can refer to [1] for the first study in such type of domain, and to [6], [13]-[17] for more works, among others.

We suppose that the quasilinear factor $A(x,t)$ is a Caratheodory function and at the same time, a bounded and uniformly elliptic matrix field satisfying some weak Lipschitz-continuous-type condition as in [10] for the Dirichlet case (see also [2], [8] for nonlinear Robin condition and [3] for linear terms). We also assume that the semilinear term $f(t)$ is monotonuously decreasing and also a Lipschitz continuous function, h is a bounded nonnegative function and lastly, g is a square-integrable function on the interface Γ.

Physically, the problem models a thermal diffusion in a medium formed by two constituents (see [9] for an application). The functions u_i describe the temperature at a point in O_i for $i = 1, 2$. The matrix field A represents the thermal conductivity, g a heat flux and h can be taken as some function affecting the total heat flux on the interface. The first two equations describe the heat flow in components O_i for $i = 1, 2$. The semilinear and quasilinear terms in these equations imply that the temperature affects the heat source and conductivity, respectively and in essence, the system has a cyclic-like behavior. The third and fourth equations imply that the flux is not continuous on the boundary while the last equation implies that the temperature vanishes on the outer boundary.

There are a number of results in the literature related to this work. To name some, for one-component domain, one can refer to [10] for an elliptic problem with Dirichlet boundary condition (see also [2]), to [4] and [5] for a quasilinear elliptic problem with linear Robin condition in perforated domains, and to [8] for a quasilinear problem with nonlinear Robin conditions in a domain with a hole, all with the help of Schauder's Fixed Point Theorem in showing the existence part. The last three works all adapted the arguments presented in [2] and [10] in proving the uniqueness part. For an elliptic problem with nonlinear Robin condition, [11] provides an existence and uniqueness results by applying the Minty-Browder Theorem.

For two-component domains separated by an imperfect interface, the existence and uniqueness results for several types of elliptic problems hold via Lax-Milgram Theorem. Some of these can be found in [6]-[13], and [18]. The reader can also check [3] for an application of the Schauder's Fixed Point Theorem. For the parabolic case, we have the works [16] and [17] (for one with semilinear terms) where the abstract Galerkin Method and theory on semigroups together with Gronwall's Lemma were used, respectively. The reader is also referred to [14] and [19] for the hyperbolic case with the help of the abstract Galerkin Method. The homogenization of the problems were also studied by the authors in some of these references.

The differences of this work to those above are, the problem considered, (1), is situated in a two-component domain with an imperfect interface, we have the simultaneous presence of the quasilinear and semilinear terms which give the difficulty in showing that the problem has a solution, and the way the Schauder's Fixed Point Theorem is applied. Before applying the Schauder's Fixed Point Theorem which provides the existence of the solution, we consider first another problem related to (1), and show that this has a unique solution satisfying some a priori estimate. We proceed with studying some properties of an associated operator and then apply the Minty-Browder Theorem. For the uniqueness part, we adapt the arguments presented in [10] for a Dirichlet case.

We have the following organization of this paper. Section 2 gives the geometric setting and the statement of the problem as well as the assumptions on the data. Section 3 provides the necessary properties in achieving the goal and lastly, Section 4 contains the main result together with its proof.

All throughout the paper, we will use the term “generic constant” to mean that the constants appearing in a sentence are not necessarily the same.

2 Setting and Assumptions

Let O be an open subset of $\mathbb{R}^N$ with Lipschitz continuous boundary, ∂O and where $N \geq 2$. Let O_1 and O_2 be two nonempty disjoint connected open subsets of O such that $\overline{O}_2 \subseteq O$ and $O = O_1 \cup \overline{O}_2$. We further suppose that $\Gamma \doteq \partial O_2$ is also Lipschitz continuous. By construction, one has

$$\partial O_1 = \partial O \cup \Gamma \quad \text{with} \quad \partial O \cap \Gamma = \varnothing.$$

We introduce the Sobolev spaces

$$V = \left\{ u_1 \in H^1(O_1) \mid u_1 = 0 \text{ on } \partial O \right\},$$

equipped with norm

$$\|u_1\|_V = \|\nabla u_1\|_{L^2(O_1)}, \tag{2}$$

and

$$H = V \times H^1(O_2),$$

together with the norm

$$\|u\|_H^2 = \|\nabla u_1\|_{L^2(O_1)}^2 + \|\nabla u_2\|_{L^2(O_2)}^2 + \|u_1 - u_2\|_{L^2(\Gamma)}^2, \tag{3}$$

where $u = (u_1, u_2) \in H$.

Remark 1. We take note of the following properties.

(a) A Poincaré inequality holds in the space V. That is, there exists $C > 0$ dependent on the diameter of O_1 such that for all $v \in V$,

$$\|v\|_{L^2(O_1)} \leq C \|\nabla v\|_{L^2(O_1)}.$$

(b) For all $u = (u_1, u_2) \in H$, the sum

$$\|\nabla u_1\|_{L^2(O_1)} + \|\nabla u_2\|_{L^2(O_2)} + \|u_1 - u_2\|_{L^2(\Gamma)},$$

is equivalent to the norm $\|u\|_H$. Indeed, we have

$$\|u\|_H^2 \leq \left(\|\nabla u_1\|_{L^2(O_1)} + \|\nabla u_2\|_{L^2(O_2)} + \|u_1 - u_2\|_{L^2(\Gamma)}\right)^2 \leq 3\|u\|_H^2.$$

(c) From [18] (see also [12]), one has

$$\|v_2\|_{L^2(O_2)} \leq C \left(\|\nabla v_2\|_{L^2(O_2)} + \|v_2\|_{L^2(\Gamma)}\right),$$

for any $v_2 \in H^1(O_2)$ and

$$\|v_1\|_{L^2(\Gamma)} \leq C \left(\|v_1\|_{L^2(O_1)} + \|\nabla v_1\|_{L^2(O_1)}\right),$$

for any $v_1 \in H^1(O_1)$ for some generic constant $C > 0$.

(d) For every $u \in L^2(O_2)$ one has,

$$\|u\|_{L^2(O_2)}^2 \leq C(\|\nabla u\|_{L^2(O_2)}^2 + \|u\|_{L^2(\Gamma)}^2),$$

for some constant $C > 0$ (see [18]).

We will also need the following set in this paper.

Definition 2. *Let $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < \beta$. Denote by $M(\alpha, \beta, O)$ the set of $N \times N$ matrices $A = (a_{ij})_{1 \leq i,j \leq N} \in (L^\infty(O))^{N \times N}$ such that*

$$(A(x)\lambda, \lambda) \geq \alpha |\lambda|^2 \quad \textit{and} \quad |A(x)\lambda| \leq \beta |\lambda|,$$

for any $\lambda \in \mathbb{R}^N$ and almost everywhere in O.

Consider the problem in (1). We suppose that:

$(A1)$ $A : (x,t) \in O \times \mathbb{R} \ \mapsto \ A(x,t) \in \mathbb{R}^{N \times N}$, a matrix field that satisfies the following:
- *(a)* A is Caratheodory function,
- *(b)* for every $t \in \mathbb{R}$, $A(\cdot, t) \in M(\alpha, \beta, O)$,
- *(c)* there exists a function ω from $\mathbb{R}$ to $\mathbb{R}$ such that
  - $(c.1)$ ω is continuous and non-decreasing, with $\omega(t) > 0$ for all $t > 0$,
  - $(c.2)$ $|A(x,t) - A(x,t_1)| \leq \omega(|t - t_1|)$ a.e. in O for $t \neq t_1$, and
  - $(c.3)$ for every $y > 0$
$$\lim_{x \to 0^+} \int_x^y \frac{dt}{\omega(t)} = \infty;$$

$(A2)$ $f : \mathbb{R} \to \mathbb{R}$ is a function that satisfies the following:
- *(a)* f is globally Lipschitz continuous,
- *(b)* $f(0) = 0$, and
- *(c)* f is monotonously decreasing;

$(A3)$ $g \in L^2(\Gamma)$; and

$(A4)$ $h : \mathbb{R} \to \mathbb{R}$ such that h is a non-negative function in $L^\infty(\Gamma)$ such that there exists $h_0 \in \mathbb{R}$ with $0 < h_0 < h(y)$ on Γ.

Remark 3. We have the next observations.

(a) Assumption $(A1)(c)$ is very essential in proving the uniqueness of the solution of (1). In this premise, $(c.2)$ provides the "modulus of continuity" of the matrix field A while $(c.3)$ quantifies this condition. Moreover, as observed in [10] (see also [7]),
  - (i) condition $(c.3)$ implies that $\omega(t) \to 0$ as $t \to 0$;
  - (ii) since ω is continuous, in order to have $(c.3)$ it is sufficient that it is true for a given $y_0 > 0$;
  - (iii) if A is uniformly Lipschitz continuous in t, that is there exists a constant $L > 0$ such that
$$|A(x,t) - A(x,t_1)| \leq L|t - t_1|,$$
for almost every $x \in \mathcal{O}$ and for every $t, t_1 \in \mathbb{R}$, then the function
$$\omega(t) = Lt$$
satisfies condition $(A1)(c)$.

(b) From property $(A2)(a) - (b)$, if $u_i \in L^2(O_i)$, one has
$$|f(u_i)| \leq C|u_i|, \tag{4}$$
for some positive generic constant C and where $i = 1, 2$.

In this work, we will show that problem (1) is well-posed. We note here that since the boundary terms are not regular enough, one cannot guarantee a solution $u = (u_1, u_2)$ in a classical sense, that is, $(u_1, u_2) \in C^2(O_1 \times O_2) \cap C(\bar{O}_1 \times \bar{O}_2)$. Hence, we need to consider its weak or variational formulation and then prove that this formulation has a unique solution, in particular, in the space H, and that this solution behaves continuously with the initial conditions. This will then bring us to our goal.

By Green's formula and the boundary conditions imposed on Γ, the weak formulation of (1) is then given by

$$\begin{cases} \text{Find } \; u = (u_1, u_2) \in H \text{ such that } \forall\ v = (v_1, v_2) \in H, \\ \displaystyle\int_{O_1} A(x, u_1)\nabla u_1 \nabla v_1 \ dx + \int_{O_2} A(x, u_2)\nabla u_2 \nabla v_2 \ dx \\ \qquad \displaystyle + \int_{\Gamma} h(x)(u_1 - u_2)(v_1 - v_2) \ ds \\ \displaystyle = \int_{O_1} f(u_1)v_1 \ dx + \int_{O_2} f(u_2)v_2 \ dx + \int_{\Gamma} g(x)v_2 \ ds. \end{cases} \tag{5}$$

3 Some Preliminary Results

In this section, we consider a problem related to (5) and prove that this admits a solution via Minty-Browder Theorem. We will also show that this solution is unique and bounded. These will then help us achieve the main goal of the paper.

Consider the problem

$$\begin{cases} \text{Find } u = (u_1, u_2) \in H \text{ such that } \forall\ v = (v_1, v_2) \in H \\ \displaystyle\int_{O_1} B(x)\nabla u_1 \nabla v_1 \ dx + \int_{O_2} B(x)\nabla u_2 \nabla v_2 \ dx \\ \qquad \displaystyle + \int_{\Gamma} h(x)(u_1 - u_2)(v_1 - v_2) \ ds \\ \displaystyle = \int_{O_1} f(u_1)v_1 \ dx + \int_{O_2} f(u_2)v_2 \ dx + \int_{\Gamma} g(x)v_2 \ ds, \end{cases} \tag{6}$$

where B is a matrix field in $M(\alpha, \beta, O)$ and the functions f, g and h satisfy assumptions $(A2) - (A4)$.

Theorem 4. *Under the given assumption above, problem (6) admits a unique solution $u \in H$. Moreover, there exists some positive constant C independent of $B(x)$, such that*

$$\|u\|_H \le C.$$

We recall the following theorem before we provide the proof to this result.

Theorem 5 (Minty-Browder Theorem)**.** *Let X be a real, separable reflexive Banach space. Suppose F is monotone, coercive and continuous function that maps X into its dual space X'. Then F is automatically surjective. That is, for all $x' \in X'$, there exists $x \in X$ such that $F(x) = x'$.*

For us to be able to use this property, we pass through the next result first.

Theorem 6. *Under the assumptions for* (6)*, define the operator*

$$F : u = (u_1, u_2) \in H \mapsto F(u) \in H',$$

by

$$\begin{aligned}\langle F(u),v\rangle_{H',H} = &\int_{O_1} B(x)\nabla u_1\nabla v_1 \ dx + \int_{O_2} B(x)\nabla u_2\nabla v_2 \ dx \\ &+ \int_{\Gamma} h(x)(u_1-u_2)(v_1-v_2) \ ds - \int_{O_1} f(u_1)v_1 \ dx \\ &- \int_{O_2} f(u_2)v_2 \ dx - \int_{\Gamma} g(x)v_2 \ ds,\end{aligned} \tag{7}$$

where H' is the dual space of H and $v=(v_1,v_2)\in H$. Then, F is monotone, coercive and continuous.

Proof:

Let $u,w\in H$ with $u=(u_1,u_2)$ and $w=(w_1,w_2)$. By (7), the ellipticity of B, assumptions $(A2)(c)$ and $(A4)$, it is the case that

$$\begin{aligned}&\langle F(u)-F(w),u-w\rangle_{H',H} \\ &= \int_{O_1} B(x)\nabla(u_1-w_1)\nabla(u_1-w_1) \ dx + \int_{O_2} B(x)\nabla(u_2-w_2)\nabla(u_2-w_2) \ dx \\ &\quad + \int_{\Gamma} h(x)((u_1-u_2)-(w_1-w_2))((u_1-w_1)-(u_2-w_2)) \ ds \\ &\quad + \int_{O_1} (f(w_1)-f(u_1))(u_1-w_1) \ dx + \int_{O_2} (f(w_2)-f(u_2))(u_2-w_2) \ dx \\ &\geq 0.\end{aligned}$$

Hence, F is monotone.

We will show next that F is coercive. By the ellipticity of B, $(A4)$ and (7), one has

$$\begin{aligned}\langle F(u),u\rangle_{H',H} \geq &\ \alpha\left\|\nabla u_1\right\|^2_{L^2(O_1)} + \alpha\left\|\nabla u_2\right\|^2_{L^2(O_2)} + C\left\|u_1-u_2\right\|^2_{L^2(\Gamma)} \\ &- \int_{O_1} f(u_1)u_1 \ dx - \int_{O_2} f(u_2)u_2 \ dx - \int_{\Gamma} g(x)u_2 \ ds,\end{aligned} \tag{8}$$

where $C>0$. For the last term on the right-hand side of this inequality, we apply Cauchy-Schwarz and triangle inequalities to obtain

$$-\left\|g\right\|_{L^2(\Gamma)}\left\|u_2\right\|_{L^2(\Gamma)} \geq -\left\|g\right\|_{L^2(\Gamma)}\left\|u_1\right\|_{L^2(\Gamma)} - \left\|g\right\|_{L^2(\Gamma)}\left\|u_1-u_2\right\|_{L^2(\Gamma)}. \tag{9}$$

By trace theorem, Poincaré inequality, (3), and Remark 1 (b),

$$-\left\|g\right\|_{L^2(\Gamma)}\left\|u_1\right\|_{L^2(\Gamma)} \geq -C_1\left\|g\right\|_{L^2(\Gamma)}\left\|u\right\|_H, \tag{10}$$

for some positive constant C_1.

By (3), we also have

$$-\left\|g\right\|_{L^2(\Gamma)}\left\|u_1-u_2\right\|_{L^2(\Gamma)} \geq -\left\|g\right\|_{L^2(\Gamma)}\left\|u\right\|_H.$$

This together with (10) in (9), we obtain

$$\begin{aligned}-\left\|g\right\|_{L^2(\Gamma)}\left\|u_2\right\|_{L^2(\Gamma)} &\geq -C_1\left\|g\right\|_{L^2(\Gamma)}\left\|u\right\|_H - \left\|g\right\|_{L^2(\Gamma)}\left\|u\right\|_H \\ &\geq S\left\|u\right\|_H,\end{aligned} \tag{11}$$

where

$$S = \min\{-C_1 \|g\|_{L^2(\Gamma)}, -\|g\|_{L^2(\Gamma)}\}. \tag{12}$$

Moreover, assumption $(A2)(c)$ guarantees that

$$-\int_{O_i} f(u_i)u_i \, dx \geq 0, \quad \text{for } i = 1, 2. \tag{13}$$

This, with (8) and (11) gives

$$\begin{aligned}
\langle F(u), u\rangle_{H',H} &\geq \alpha \|\nabla u_1\|^2_{L^2(O_1)} + \alpha \|\nabla u_2\|^2_{L^2(O_2)} + C\|u_1 - u_2\|^2_{L^2(\Gamma)} + S\|u\|_H \\
&\geq C^* \left(\|\nabla u_1\|^2_{L^2(O_1)} + \|\nabla u_2\|^2_{L^2(O_2)} + \|u_1 - u_2\|^2_{L^2(\Gamma)}\right) + S\|u\|_H \\
&= C^* \|u\|^2_H + S\|u\|_H,
\end{aligned}$$

where $C^* = \min\{\alpha, C\}$. Hence,

$$\frac{\langle F(u), u\rangle_{H',H}}{\|u\|_H} \geq C^* \|u\|_H + S,$$

which implies

$$\frac{\langle F(u), u\rangle_{H',H}}{\|u\|_H} \to \infty \text{ as } \|u\|_H \to \infty,$$

and therefore F is coercive.

Lastly, we manifest that F is continuous from H to H'. That is, if $\{u_n\} = \{(u_{1_n}, u_{2_n})\}$ is a sequence in H such that there exists $u = (u_1, u_2) \in H$ with

$$u_n \to u \quad \text{strongly in } H, \tag{14}$$

or equivalently, if

$$\begin{cases} u_{1_n} \to u_1, & \text{in } V, \\ u_{2_n} \to u_2, & \text{in } H^1(O_2), \end{cases} \tag{15}$$

then

$$\|F(u_n) - F(u)\|_{H'} \to 0. \tag{16}$$

In view of (7), invoking triangle inequality provides

$$\begin{aligned}
&\left|\langle F(u_n) - F(u), v\rangle_{H',H}\right| \\
&\leq \int_{O_1} |B(x)\nabla(u_{1_n} - u_1)\nabla v_1| \, dx + \int_{O_2} |B(x)\nabla(u_{2_n} - u_2)\nabla v_2| \, dx \\
&\quad + \int_\Gamma |h(x)((u_{1_n} - u_1) - (u_{2_n} - u_2))(v_1 - v_2)| \, ds \\
&\qquad + \int_{O_1} |f(u_{1_n}) - f(u_1)|\,|v_1| \, dx + \int_{O_2} |f(u_{2_n}) - f(u_2)|\,|v_2| \, dx
\end{aligned} \tag{17}$$

For the first two terms of the right-hand side of this inequality, applying the boundedness of B and (15) yield,

$$\int_{O_i} |B(x)\nabla(u_{i_n} - u_i)\nabla v_i| \, dx \leq \beta \|\nabla(u_{i_n} - u_i)\|_{L^2(O_i)} \|\nabla v_i\|_{L^2(O_i)} \to 0. \tag{18}$$

Next, for the third term of right-hand side of (17), by $(A4)$, Cauchy-Schwarz inequality and (15),

$$\begin{aligned}\int_\Gamma &\left|h(x)((u_{1_n}-u_1)-(u_{2_n}-u_2))(v_1-v_2)\right|\ ds \\ &\leq C\left\|(u_{1_n}-u_1)-(u_{2_n}-u_2)\right\|_{L^2(\Gamma)}\left\|v_1-v_2\right\|_{L^2(\Gamma)}\to 0.\end{aligned} \tag{19}$$

For the last two terms, assumption (A2) (a), Cauchy-Schwarz inequality and (15) produce

$$\int_{O_1}\left|f(u_{1_n})-f(u_1)\right|\left|v_1\right|\ dx+\int_{O_2}\left|f(u_{2_n})-f(u_2)\right|\left|v_2\right|\ dx\to 0.$$

This, together with inequalities (18) and (19) in (17) gives us

$$\left\|F(u_n)-F(u)\right\|_{H'}=\sup_{v\neq 0}\frac{\langle F(u_n)-F(u),v\rangle_{H',H}}{\left\|v\right\|_H}\to 0$$

and this proves the continuity of F on H. □

Let us now have the proof of Theorem 4, stating the well-posedness of the variational formulation (6).

Proof of Theorem 4:
The existence of the solution of (6) follows immediately from Theorem 6 and the Minty-Browder Theorem.

Next, we suppose that (6) has two solutions $u=(u_1,u_2)$ and $w=(w_1,w_2)$ in H. For every $v=(v_1,v_2)\in H$,

$$\begin{aligned}\int_{O_1}B(x)\nabla u_1\nabla v_1\ dx+\int_{O_2}B(x)\nabla u_2\nabla v_2\ dx+\int_\Gamma h(x)(u_1-u_2)(v_1-v_2)\ ds \\ -\int_{O_1}f(u_1)v_1\ dx-\int_{O_2}f(u_2)v_2\ dx-\int_\Gamma g(x)v_2\ ds=0,\end{aligned} \tag{20}$$

and

$$\begin{aligned}\int_{O_1}B(x)\nabla w_1\nabla v_1\ dx+\int_{O_2}B(x)\nabla w_2\nabla v_2\ dx+\int_\Gamma h(x)(w_1-w_2)(v_1-v_2)\ ds \\ -\int_{O_1}f(w_1)v_1\ dx-\int_{O_2}f(w_2)v_2\ dx-\int_\Gamma g(x)v_2\ ds=0.\end{aligned} \tag{21}$$

Taking $v_1=u_1-w_1\in V$ in (20) and $v_2=u_2-w_2\in H^1(O_2)$ in (21) and then subtracting (21) from (20), we get

$$\begin{aligned}\int_{O_1}B(x)(\nabla u_1-\nabla w_1)(\nabla u_1-\nabla w_1)\ dx+\int_{O_2}B(x)(\nabla u_2-\nabla w_2)(\nabla u_2-\nabla w_2)\ dx \\ +\int_\Gamma h(x)((u_1-w_1)-(u_2-w_2))((u_1-w_1)-(u_2-w_2))\ ds \\ +\int_{O_1}(f(w_1)-f(u_1))(u_1-w_1)\ dx+\int_{O_2}(f(w_2)-f(u_2))(u_2-w_2)\ dx=0.\end{aligned} \tag{22}$$

Considering assumptions $(A4)$ and $(A2)(c)$, this equation yields the identity

$$\int_{O_1}B(x)(\nabla u_1-\nabla w_1)(\nabla u_1-\nabla w_1)\,dx+\int_{O_2}B(x)(\nabla u_2-\nabla w_2)(\nabla u_2-\nabla w_2)\,dx=0. \tag{23}$$

Applying Poincaré inequality and taking into account the ellipticity of B in this equation, we have

$$0 \geq \alpha \left(\|\nabla(u_1 - w_1)\|^2_{L^2(O_1)} + \|\nabla(u_2 - w_2)\|^2_{L^2(O_2)} \right) \geq C \|u_1 - w_1\|^2_{L^2(O_1)} \geq 0,$$

for $\alpha > 0$ and some constant $C > 0$ which means that

$$\|u_1 - w_1\|^2_{L^2(O_1)} = 0.$$

Thus, $u_1 = w_1$.

Now, since $u_1 = w_1$, (22) is equivalent to

$$\begin{aligned} \int_{O_2} B(x)(\nabla u_2 - \nabla w_2)(\nabla u_2 - \nabla w_2)\, dx + \int_\Gamma h(x)(u_2 - w_2)(u_2 - w_2)\, ds \\ + \int_{O_2} (f(w_2) - f(u_2))(u_2 - w_2)\, dx = 0. \end{aligned} \tag{24}$$

By assumptions $(A2)(c)$ and $(A4)$, the ellipticity of B and Remark 1 (d), we obtain

$$0 \geq C \|u_2 - w_2\|^2_{L^2(O_2)} \geq 0,$$

for some constant $C > 0$. Consequently,

$$\|u_2 - w_2\|^2_{L^2(O_2)} = 0.$$

giving $u_2 = w_2$. We therefore conclude that $u = (u_1, u_2) = (w_1, w_2) = w$, that is, the solution is unique.

Lastly, we now prove that the unique solution $u = (u_1, u_2) \in H$ of (6) is bounded, meaning, there is a constant $C > 0$ such that

$$\|u\|_H \leq C. \tag{25}$$

Take $u = (u_1, u_2) \in H$ as a test function on (6). We have

$$\begin{aligned} \int_{O_1} B(x)\nabla u_1 \nabla u_1\, dx + \int_{O_2} B(x)\nabla u_2 \nabla u_2\, dx + \int_\Gamma h(x)(u_1 - u_2)(u_1 - u_2)\, dx \\ = \int_{O_1} f(u_1)u_1\, dx + \int_{O_2} f(u_2)u_2\, dx + \int_\Gamma g(x)u_2\, ds. \end{aligned} \tag{26}$$

By the ellipticity of B, $(A4)$, (3), (13), Cauchy-Schwarz inequality, (11) with (26), one gets

$$\begin{aligned} \min\{\alpha, h_0\}\|u\|^2_H &= \min\{\alpha, h_0\} \left(\|\nabla u_1\|^2_{L^2(O_1)} + \|\nabla u_2\|^2_{L^2(O_2)} + \|u_1 - u_2\|^2_{L^2(\Gamma)} \right) \\ &\leq \int_{O_1} B(x)\nabla u_1 \nabla u_1\, dx + \int_{O_2} B(x)\nabla u_2 \nabla u_2\, dx + \int_\Gamma h(x)(u_1 - u_2)(u_1 - u_2)\, ds \\ &\leq \int_{O_1} B(x)\nabla u_1 \nabla u_1\, dx + \int_{O_2} B(x)\nabla u_2 \nabla u_2\, dx + \int_\Gamma h(x)(u_1 - u_2)(u_1 - u_2)\, ds \\ &\qquad - \int_{O_1} f(u_1)u_1\, dx - \int_{O_2} f(u_2)u_2\, dx \\ &= \int_\Gamma g(x)u_2\, ds \end{aligned}$$

$$\leq \|g\|_{L^2(\Gamma)}\|u_2\|_{L^2(\Gamma)}$$
$$\leq C_1\|u\|_H,$$

where $C_1 = -S$ with S given by (12). Hence, (25) holds by taking

$$C = \frac{C_1}{\min\{\alpha, h_0\}}, \tag{27}$$

and the proof is complete. □

4 Main Result

Let us now consider the fixed point theorem that we will apply to prove the existence of a solution of problem (5).

Theorem 7 (Schauder's Fixed Point Theorem)**.** *Let X be a Banach Space. Let $K \subseteq X$ be a closed convex set. If T is a continuous function on K into itself such that $T(K)$ is relatively compact, then T has a fixed point in K i.e. $T(x) = x$ for some $x \in K$.*

Theorem 8. *Under assumptions (A1)-(A4), problem (5) admits a unique solution $u = (u_1, u_2)$ in H. Moreover, this solution satisfies the estimate*

$$\|u\|_H \leq C,$$

for some positive constant C.

Proof:
First, we fix $w = (w_1, w_2) \in H$ and consider the problem

$$\begin{cases} \text{Find } u = (u_1, u_2) \in H \text{ such that } \forall\ v = (v_1, v_2) \in H \\ \displaystyle\int_{O_1} A(x, w_1)\nabla u_1 \nabla v_1\ dx + \int_{O_2} A(x, w_2)\nabla u_2 \nabla v_2\ dx \\ \qquad \displaystyle + \int_\Gamma h(x)(u_1 - u_2)(v_1 - v_2)\ ds \\ \displaystyle = \int_{O_1} f(u_1)v_1\ dx + \int_{O_2} f(u_2)v_2\ dx + \int_\Gamma g(x)v_2\ ds. \end{cases} \tag{28}$$

Since $A(x, w_i) \in M(\alpha, \beta, O)$, this problem has unique solution $u = (u_1, u_2) \in H$ by Theorem 4. Let $X = L^2(O_1) \times L^2(O_2)$ and define the function T by

$$T : w = (w_1, w_2) \in X \mapsto T(w) = u = (u_1, u_2) \in X, \tag{29}$$

where $u = (u_1, u_2) \in H$ is the solution of (28). We note here that this solution satisfies the estimate

$$\|u\|_H \leq C,$$

where C is given by (27). For this constant C, define the set

$$K = \{v \in X |\ \|v\|_X \leq C\}, \tag{30}$$

which is closed and convex subset of X. We will show that T is a continuous function on K to K, and that $T(K)$ is relatively compact.

By (30), $T(K) \subseteq K$. Furthermore, if $\{w_n\}$ is a bounded sequence in K, then $\{u_n\}= \{T(w_n)\}$ is also bounded in H and so, the Sobolev embedding theorem guarantees that $\{T(w_n)\}$ is relatively compact in X, and hence in K.

To show the continuity of T, we consider a sequence $\{w_n\} = \{(w_{1_n}, w_{2_n})\}$ such that

$$w_n \to w = (w_1, w_2) \quad \text{strongly in } X.$$

This means that we can extract subsequences $\{w_{1_n}\}$ and $\{w_{2_n}\}$ (denoted by n again) such that

$$\begin{cases} w_{1_n} \to w_1, & \text{in } L^2(O_1) \text{ and a.e. in } O_1, \\ w_{2_n} \to w_2, & \text{in } L^2(O_2) \text{ and a.e. in } O_2. \end{cases} \tag{31}$$

Suppose $u_n = T(w_n)$. We have $\{u_n\} = \{(u_{1_n}, u_{2_n})\} \subseteq K$ satisfies

$$\int_{O_1} A(x, w_{1_n}) \nabla u_{1_n} \nabla v_1 \; dx + \int_{O_2} A(x, w_{2_n})) \nabla u_{2_n} \nabla v_2 \; dx \\ + \int_\Gamma h(x)(u_{1_n} - u_{2_n})(v_1 - v_2) \; ds = \int_{O_1} f(u_{1_n}) v_1 \; dx + \int_{O_2} f(u_{2_n}) v_2 \; dx + \int_\Gamma g(x) v_2 \; ds.$$

Noting here that $\{u_n\}$ is bounded in X, the Bolzano-Weierstrass Theorem provides the existence of a subsequence $\{u_{n_k}\}= \left\{(u_{1_{n_k}}, u_{2_{n_k}})\right\}$ and $u^*= (u_1^*, u_2^*)$ such that $u_{n_k} \to u^*$ in X. Moreover, since V and $H^1(O_2)$ are reflexive Banach spaces and that the convergent sequences in $L^2(O_i)$ for $i = 1, 2$ have a.e. convergent subsequences and from the Sobolev embedding theorem, one has

$$\begin{cases} u_{1_{n_k}} & \rightharpoonup & u_1^* & \text{weakly in } V, \\ u_{2_{n_k}} & \rightharpoonup & u_2^* & \text{weakly in } H^1(O_2), \\ u_{1_{n_k}} & \to & u_1^* & \text{strongly in } L^2(O_1) \text{ and a.e. in } O_1, \\ u_{2_{n_k}} & \to & u_2^* & \text{strongly in } L^2(O_2) \text{ and a.e. in } O_2. \end{cases} \tag{32}$$

Thus,

$$\nabla u_{i_{n_k}} \rightharpoonup \nabla u_i^* \quad \text{weakly in } \left[L^2(O_i)\right]^N, \tag{33}$$

for $i = 1, 2$. Now, we have to show that

$$u^* = (u_1^*, u_2^*) = T(w). \tag{34}$$

By $(A1)(b)$, we have

$$|A^T(x, w_{i_n}(x)) \nabla v_i| \le \beta |\nabla v_i| \quad \text{a.e. in } O_i,$$

and from (32) and $(A1)(a)$ and $(c.3)$,

$$A^T(x, w_{i_n}(x)) \nabla v_i \to A^T(x, w_i(x)) \nabla v_i \quad \text{a.e. in } O_i,$$

for $i = 1, 2$, and where A^T is the transpose of the matrix field A. Thus, applying Lebesgue Dominated Theorem yields,

$$A^T(x, w_{i_n}(x)) \nabla v_i \to A^T(x, w_i(x)) \nabla v_i \quad \text{in } [L^2(O_i)]^N.$$

This, with the continuity of the trace operator and properties (32) and (33), yields

$$\begin{aligned}
&\lim_{k\to\infty}\left(\int_{O_1} A(x,w_{1_n})\nabla u_{1_{n_k}}\nabla v_1 dx + \int_{O_2} A(x,w_{2_n})\nabla u_{2_{n_k}}\nabla v_2 dx \right.\\
&\qquad \left. + \int_\Gamma h\left(u_{1_{n_k}} - u_{2_{n_k}}\right)(v_1 - v_2) ds\right)\\
&= \lim_{k\to\infty}\left(\int_{O_1} A^T(x,w_{1_n})\nabla v_1 \nabla u_{1_{n_k}} dx + \int_{O_2} A^T(x,w_{2_n})\nabla v_2 \nabla u_{2_{n_k}} dx \right.\\
&\qquad \left. + \int_\Gamma h\left(u_{1_{n_k}} - u_{2_{n_k}}\right)(v_1 - v_2) ds\right)\\
&= \int_{O_1} A^T(x,w_1)\nabla v_1 \nabla u_1^* dx + \int_{O_2} A^T(x,w_2)\nabla v_2 \nabla u_2^* dx + \int_\Gamma h\left(u_1^* - u_2^*\right)(v_1 - v_2) ds\\
&= \int_{O_1} A(x,w_1)\nabla u_1^* \nabla v_1 dx + \int_{O_2} A(x,w_2)\nabla u_2^* \nabla v_2 dx + \int_\Gamma h\left(u_1^* - u_2^*\right)(v_1 - v_2) ds.
\end{aligned} \tag{35}$$

Also, by $(A2)(a)$ and (32) for $i = 1, 2$, one has

$$\lim_{k\to\infty}\int_{O_i} f(u_{n_{ik}}) v_i dx = \int_{O_i} f(u_i^*) v_i \ dx. \tag{36}$$

Hence, by (35) and (36), for every $v = (v_1, v_2) \in H$,

$$\begin{aligned}
&\int_{O_1} A(x,w_1)\nabla u_1^* \nabla v_1 dx + \int_{O_2} A(x,w_2)\nabla u_2^* \nabla v_2 \ dx\\
&\quad + \int_\Gamma h(x)(u_1^* - u_2^*)(v_1 - v_2)\ ds = \int_{O_1} f(u_1^*) v_1 \ dx + \int_{O_2} f(u_2^*) v_2 \ dx + \int_\Gamma g(x) v_2 ds,
\end{aligned}$$

which gives (34) and hence, T is continuous. Therefore, by Schauder's Fixed Point Theorem, T has a fixed point in K. Moreover, in view of (34), (29) and Theorem 4, one obtains that problem (5) has a solution $u \in H$.

As for the boundedness of the solution, one can follow exactly the same arguments used in proving the estimate in Theorem 4.

Now suppose that (5) has two bounded solutions in H, namely, $u = (u_1, u_2)$ and $w = (w_1, w_2)$. This says that for all $v = (v_1, v_2) \in H$, we have

$$\begin{aligned}
&\int_{O_1} A(x,u_1)\nabla u_1 \nabla v_1 \ dx + \int_{O_2} A(x,u_2)\nabla u_2 \nabla v_2 \ dx\\
&\qquad + \int_\Gamma h(x)(u_1 - u_2)(v_1 - v_2)\ ds - \int_{O_1} f(u_1) v_1 \ dx - \int_{O_2} f(u_2) v_2 \ dx\\
&= \int_{O_1} A(x,w_1)\nabla w_1 \nabla v_1 \ dx + \int_{O_2} A(x,w_2)\nabla w_2 \nabla v_2 \ dx\\
&\qquad + \int_\Gamma h(x)(w_1 - w_2)(v_1 - v_2)\ ds - \int_{O_1} f(w_1) v_1 \ dx - \int_{O_2} f(w_2) v_2 \ dx.
\end{aligned} \tag{37}$$

We need to show that $u_1 = w_1$ and $u_2 = w_2$ which gives $u = w$. We proceed to this aim by following the arguments used in [10] (see Chapter 11, Section 2).

Define the function F_ξ by,

$$F_\xi(x) = \begin{cases} \displaystyle\int_\xi^x \frac{dt}{(\omega(t))^2}, & \text{if } x \geq \xi, \\ 0, & \text{otherwise,} \end{cases} \tag{38}$$

where $\xi > 0$ is a parameter that tends to 0 and ω is given by $(A1)(c)$. From [10],

$$F_\xi(u_1 - w_1) \in V \quad \text{and} \quad F_\xi(u_2 - w_2) \in H^1(O_2),$$

and that

$$\nabla\left(F_\xi(u_i - w_i)\right) = F'_\xi(u_i - w_i)\nabla(u_i - w_i), \tag{39}$$

for $i = 1, 2$.

Next, we consider the set E defined by

$$E_i = \{x \in O_i : (u_i - w_i)(x) > \xi\}, \tag{40}$$

for $i = 1, 2$. Take $v = (v_1, v_2) \in H$ as a test function in (37) where $v_1 = F_\xi(u_1 - w_1) \in V$ and $v_2 = F_\xi(u_2 - w_2) \in H^1(O_2)$ and then subtract the term

$$\int_{O_1} A(x, u_1)\nabla w_1 \nabla v_1 \; dx + \int_{O_2} A(x, u_2)\nabla w_2 \nabla v_2 \; dx,$$

from both sides of this equation. We have,

$$\begin{aligned}
&\int_{O_1} A(x, u_1)\nabla(u_1 - w_1)\nabla(F_\xi(u_1 - w_1)) \; dx + \int_{O_2} A(x, u_2)\nabla(u_2 - w_2)\nabla(F_\xi(u_2 - w_2)) \; dx \\
&\qquad + \int_\Gamma h(x)(u_1 - u_2)(F_\xi(u_1 - w_1) - F_\xi(u_2 - w_2)) \; ds - \int_{O_1} f(u_1)F_\xi(u_1 - w_1) \; dx \\
&\qquad\qquad - \int_{O_2} f(u_2)F_\xi(u_2 - w_2) \; dx \\
&= \int_{O_1} [A(x, w_1) - A(x, u_1)]\nabla w_1 \nabla(F_\xi(u_1 - w_1)) \; dx \\
&\qquad + \int_{O_2} [A(x, w_2) - A(x, u_2)]\nabla w_2 \nabla(F_\xi(u_2 - w_2)) \; dx \\
&\qquad + \int_\Gamma h(x)(w_1 - w_2)((F_\xi(u_1 - w_1)) - (F_\xi(u_2 - w_2))) \; ds \\
&\qquad - \int_{O_1} f(w_1)(F_\xi(u_1 - w_1)) \; dx - \int_{O_2} f(w_2)(F_\xi(u_2 - w_2)) \; dx.
\end{aligned}$$

By (39),

$$\begin{aligned}
&\int_{E_1} A(x, u_1)\frac{\nabla(u_1 - w_1)}{\omega^2(u_1 - w_1)}\nabla(u_1 - w_1) \; dx + \int_{E_2} A(x, u_2)\frac{\nabla(u_2 - w_2)}{\omega^2(u_2 - w_2)}\nabla(u_2 - w_2) \; dx \\
&\qquad + \int_\Gamma h((u_1 - w_1) - (u_2 - w_2))(F_\xi(u_1 - w_1) - F_\xi(u_2 - w_2)) \; ds \\
&\qquad + \int_{O_1} (f(w_1) - f(u_1))F_\xi(u_1 - w_1) \; dx + \int_{O_2} (f(w_2) - f(u_2))F_\xi(u_2 - w_2) \; dx \\
&= -\int_{E_1} [A(x, u_1) - A(x, w_1)]\frac{\nabla(u_1 - w_1)}{\omega^2(u_1 - w_1)}\nabla w_1 \; dx \\
&\qquad - \int_{E_2} [A(x, u_2) - A(x, w_2)]\frac{\nabla(u_2 - w_2)}{\omega^2(u_2 - w_2)}\nabla w_2 \; dx.
\end{aligned} \tag{41}$$

Next, we note that

$$F_\xi(u_i - w_i) \geq 0 \text{ for } i = 1,2, \tag{42}$$

by (38), (40) and by the assumption that $\xi > 0$. Thus by (42) and $(A2)(c)$ for $i = 1,2$ we have,

$$(f(w_i) - f(u_i))F_\xi(u_i - w_i) \geq 0. \tag{43}$$

In addition to this, we also have $(A1)(c.1)$ and (38) implying

$$F_\xi(u_i - w_i) \geq F_\xi(u_j - w_j) \quad \text{for} \quad u_i - w_i \geq u_j - w_j, \tag{44}$$

for $i,j = 1,2$. Thus, $(A4)$ and (44) yield,

$$h(x)((u_1 - w_1) - (u_2 - w_2))(F_\xi(u_1 - w_1) - F_\xi(u_2 - w_2)) \geq 0. \tag{45}$$

On the other hand, $(A1)(b)$ and $(c.2)$, (41), (43), (45), Cauchy-Schwarz inequality and the boundedness of the solutions u and w provide

$$\begin{aligned}
&\alpha \left\| \frac{\nabla(u-w)}{\omega(u-w)} \right\|^2_{L^2(E_1)\times L^2(E_2)} = \alpha \left\| \frac{\nabla(u_1-w_1)}{\omega(u_1-w_1)} \right\|^2_{L^2(E_1)} + \alpha \left\| \frac{\nabla(u_2-w_2)}{\omega(u_2-w_2)} \right\|^2_{L^2(E_2)} \\
&\leq \int_{E_1} A(x,u_1)\frac{\nabla(u_1-w_1)}{\omega^2(u_1-w_1)}\nabla(u_1-w_1)\ dx + \int_{E_2} A(x,u_2)\frac{\nabla(u_2-w_2)}{\omega^2(u_2-w_2)}\nabla(u_2-w_2)\ dx \\
&\leq \int_{E_1} A(x,u_1)\frac{\nabla(u_1-w_1)}{\omega^2(u_1-w_1)}\nabla(u_1-w_1)\ dx + \int_{E_2} A(x,u_2)\frac{\nabla(u_2-w_2)}{\omega^2(u_2-w_2)}\nabla(u_2-w_2)\ dx \\
&\quad + \int_\Gamma h(x)((u_1-w_1)-(u_2-w_2))(F_\xi(u_1-w_1) - F_\xi(u_2-w_2))\ ds \\
&\quad + \int_{O_1} (f(w_1) - f(u_1))F_\xi(u_1-w_1)\ dx + \int_{O_2} (f(w_2)-f(u_2))F_\xi(u_2-w_2)\ dx \\
&\leq \int_{E_1} \frac{|A(x,u_1) - A(x,w_1)||\nabla w_1||\nabla(u_1-w_1)|}{\omega^2(u_1-w_1)}\ dx \\
&\quad + \int_{E_2} \frac{|A(x,u_2) - A(x,w_2)||\nabla w_2||\nabla(u_2-w_2)|}{\omega^2(u_2-w_2)}\ dx \\
&\leq \int_{E_1} |\nabla w_1| \frac{|\nabla(u_1-w_1)|}{\omega(u_1-w_1)}\ dx + \int_{E_2} |\nabla w_2| \frac{|\nabla(u_2-w_2)|}{\omega(u_2-w_2)}\ dx \\
&\leq C_1 \left\| \frac{\nabla(u_1-w_1)}{\omega(u_1-w_1)} \right\|_{L^2(E_1)} + C_2 \left\| \frac{\nabla(u_2-w_2)}{\omega(u_2-w_2)} \right\|_{L^2(E_2)} \\
&\leq \max\{C_1, C_2\} \left(\left\| \frac{\nabla(u_1-w_1)}{\omega(u_1-w_1)} \right\|_{L^2(E_1)} + \left\| \frac{\nabla(u_2-w_2)}{\omega(u_2-w_2)} \right\|_{L^2(E_2)} \right).
\end{aligned}$$

Hence,

$$\left\| \frac{\nabla(u-w)}{\omega(u-w)} \right\|_{L^2(E_1)\times L^2(E_2)} \leq C, \tag{46}$$

where C is independent of ξ.

Now we define the function G_ξ as

$$G_\xi(x) = \begin{cases} \displaystyle\int_\xi^x \frac{ds}{\omega(s)}, & \text{if } x \geq \xi, \\ 0, & \text{otherwise,} \end{cases} \tag{47}$$

with the same assumptions on ξ and ω as for the function F_ξ defined in (38). Again from [10],

$$G_\xi(u_1 - w_1) \in V \quad \text{and} \quad G_\xi(u_2 - w_2) \in H^1(O_2),$$

and that

$$\nabla \left(G_\xi(u_i - w_i)\right) = G'_\xi(u_i - w_i)\nabla(u_i - w_i), \tag{48}$$

for $i = 1, 2$. In view of (2), (48), (47), and (46), we obtain

$$\|G_\xi(u_1 - w_1)\|_V = \|\nabla(G_\xi(u_1 - w_1))\|_{L^2(O_1)} \leq C,$$

and

$$\|G_\xi(u_2 - w_2)\|_{H^1(O_2)} \leq C.$$

We take note that these hold for all $\xi > 0$, and so, there exists a subsequence $\{\xi_k\}$ that approaches 0 as $k \to \infty$ and functions $G_1 \in V$ and $G_2 \in H^1(O_2)$ such that

$$\begin{cases} G_{\xi_k}(u_1 - w_1) \rightharpoonup G_1 & \text{weakly in } V, \\ G_{\xi_k}(u_1 - w_1) \to G_1 & \text{in } L^2(O_1), \text{ a.e. in } O_1, \\ G_{\xi_k}(u_2 - w_2) \rightharpoonup G_2 & \text{weakly in } H^1(O_2), \\ G_{\xi_k}(u_2 - w_2) \to G_2 & \text{in } L^2(O_2), \text{ a.e. in } O_2. \end{cases}$$

These tell us that as $k \to \infty$ and for $i = 1, 2$, one has

$$\lim_{\xi_k \to 0} G_{\xi_k}(u_i - w_i) < \infty \qquad \text{a.e. in } O_i.$$

On the other hand, $(A1)(c)$ means that

$$\lim_{\xi_k \to 0} G_{\xi_k}(u_i - w_i)(x) = \lim_{\xi_k \to 0} \int_{\xi_k}^{(u_i - w_i)(x)} \frac{ds}{\omega(s)} = \infty,$$

almost everywhere in E_i, and this gives a contradiction. Hence,

$$\text{meas}(E_i) = \text{meas}(u_i - w_i > 0) = 0,$$

which implies that $u_i - w_i \leq 0$ or that is, $u_i \leq w_i$. Interchanging the roles of u_i and w_i hands us $u_i \geq w_i$ which yield $u_i = w_i$ for $i = 1, 2$ proving that $u = w$.This ends the proof. □

Remark 9. (a) The well-posedness of problem (1) is a consequence of Theorem 4.

(b) Another way how to apply the Schauder's Fixed Point Theorem is by considering the problem

$$\begin{cases} \text{Find } u = (u_1, u_2) \in H \text{ such that } \forall\, v = (v_1, v_2) \in H \\ \displaystyle\int_{O_1} A(w_1)\nabla u_1 \nabla v_1 \, dx + \int_{O_2} A(w_2)\nabla u_2 \nabla v_2 \, dx \\ \qquad + \displaystyle\int_\Gamma h(x)(u_1 - u_2)(v_1 - v_2) \, ds \\ = \displaystyle\int_{O_1} f(w_1) v_1 \, dx + \int_{O_2} f(w_2) v_2 \, dx + \int_\Gamma g(x) v_2 \, ds. \end{cases}$$

where $(w_1, w_2) \in X$ is fixed, instead of problem (28). One can initially show that this admits a unique solution and then proceed as what was done above.

The homogenization of problem (1) is currently being worked on by the authors.

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5 References

[1] J.L. Auriault and H. Ene. *Macroscopic modelling of heat transfer in composites with interfacial thermal barrier*, Int. J. of Heat and Mass Transfer, **37** (1994):2885-2892.

[2] N. André and M. Chipot. *A remark on uniqueness for quasilinear elliptic equations*, Singularities and Differential Equations (Warsaw, 1993), Banach Center Publ., **33**, Polish Acad. Sci., Warsaw (1996), 9-18.

[3] R. Beltran, B. Cabarrubias and M. Roque. *On the solvability of a class of a quasilinear elliptic partial differential equation*, Philippine Journal of Science, **146**(2) (2017), 139-145.

[4] S. Bendib. *Homogénéisation d'une classe de problèmes non linéaires avec des con ditions de Fourier dans des ouverts perforés*, Thèse, Institut National Polytechnique de Lorraine, France, 2004.

[5] S. Bendib and R.L. Tcheugoué Tébou. *Homogénéisation d'une classe de problèmes non linéaires dans des domaines perforés*, C. R. Acad. Sci. Paris Sr. I Math. **328** (1999), No. 12, 1145–1149.

[6] R. Bunoiu and C. Timofte. *On the homogenization of a two-conductivity problem with flux jump*, Commun. Math. Sci., Vol. **15**, No. 3, 745-763.

[7] B. Cabarrubias. *Existence, uniqueness and homogenization results for a class of non-linear PDE in perforated domains*, Doctoral Dissertation, University of the Philippines Diliman, Philippines and University of Rouen, France, 2012.

[8] B. Cabarrubias and P. Donato. *Existence and Uniqueness for a Quasilinear Elliptic Problem with Nonlinear Robin Conditions*, Carpathian Journal, **27** (2) (2011), 173-184.

[9] H.S. Carslaw and J.C. Jaeger. Conduction of Heat Solids, Oxford, At the Clarendon Press, 1947.

[10] M. Chipot. Elliptic Equations: An Introductory Course, Birkhuser Verlag AG, Germany, 2009.

[11] D. Cioranescu, P. Donato and R. Zaki. *Asymptotic behavior of elliptic problems in perforated domains with nonlinear boundary conditions*, Asymptotic Analysis **53** (2007) 209-235.

[12] C. Conca. *On the application of the homogenization theory to a class of problems arising in fluid mechanics*, J. Math. Pures et Appl., 64 (1985), 31-75.

[13] P. Donato, K.H. Le Nguyen, and R. Tardieu. *The periodic unfolding method for a class of imperfect transmission problems*, Journal of Mathematical Sciences, **176** (6) (2011), 891- 927.

[14] P. Donato, L. Faella and S. Monsurró. *Homogenization of the wave equation in composites with imperfect interface: A memory effect*. J. Math. Pures Appl. **87** (2007) 119-143.

[15] P. Donato and S. Monsurró. *Homogenization of two heat conductors with and interfacial contact resistance*. Analysis and Application, Vol. **2**, No. 3 (2004), 247-273.

[16] E. Jose. *Homogenization of a parabolic problem with an imperfect interface*, Rev. Roumaine Math. Pures Appl., **54** (2009), 3, 189-222.

[17] I.C. Lomerio and E. Jose. *Asymptotic analysis of a certain class of semilinear parabolic problem with interfacial contact resistance*, Bull. Malays. Math. Sci. Soc., Vol. **42**, Issue 3, (2019) 1011-1054.

[18] S. Monsurró. *Homogenization of a two-component composite with interfacial thermal barrier*, Adv. in Math. Sci. Appl., **13**(2003), 43-63.

[19] Z. Yang. *Homogenization and correctors for the hyperbolic problems with imperfect interfaces via the periodic unfolding method*, Communications on Pure and Applied Analysis, Vol. **13**, No. 1 (2014), 249-272.

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