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Terwilliger algebras of certain group association schemes

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Abstract

Association schemes are fundamental objects in algebraic combinatorics. An important structure for investigating association schemes is the Terwilliger algebra. Bannai and Munemasa gave a framework for investigating the Terwilliger algebra $T(G)$ of the group association scheme obtained from a finite group G. The main result of this paper gives the explicit structures of the Terwilliger algebra $T(G)$, for $G = GL(2,3)$, the general linear group of invertible 2×2 matrices over the field of order 3, and three other groups. We present the processes involved in obtaining the Terwilliger algebra and illustrate these through the chosen groups. The examples show the varying behavior of $T(G)$ in relation to related algebras for different groups. We use the discrete algebra system GAP to aid computations.

Key words: group association scheme, Terwilliger algebra, $GL(2,3)$, permutation representation, centralizer algebra

MSC 2020: 05E30, 05E16

1 Introduction

The theory of association schemes serves as a unifying framework for the study of codes, designs, graphs with regularity, and other combinatorial objects having large automorphism groups and their classification is far from completion (see [3], [7]). An important tool to study them is the Terwilliger algebra. The algebra was introduced by Paul Terwilliger, who called it the subconstituent algebra, to study certain families of commutative association schemes [9].

Each finite group gives rise to a commutative association scheme called the group association scheme. The Terwilliger algebras of group association schemes are known for certain groups or families of groups (see [1], [2], [4], [8]). Recently, more information has been obtained in the master's thesis of Bastian [5], including data on the Terwilliger algebras of finite groups of order at most 64 obtained using the computer algebra system Magma.

We study the group association scheme of four finite groups, namely: the general linear group $GL(2,3)$ of all 2×2 invertible matrices over the field of order 3; its subgroup $SL(2,3)$ of invertible matrices of determinant 1; the symmetric group S_4 of all permutations on 4 letters, and a non-abelian group of order 21, denoted by 7.3. The dimensions, but not the structures, of the Terwilliger algebras of the last three groups were given with no proof in [4]. In this paper, we will describe the processes in determining explicitly the structure of the Terwilliger algebra from a finite group. The groups were chosen to illustrate the different cases that can occur with regard to the structure of the algebra. Computations were aided using the discrete algebra system GAP [6].

Definition 1 (Association Scheme)**.** Let X be a finite nonempty set and $R_0, R_1, \ldots, R_d$ nonempty subsets of $X \times X$ that satisfy

1. $R_0 = \{(x,x) : x \in X\}$.
2. $R_0 \cup R_1 \cup \cdots \cup R_d = X \times X$ and $R_i \cap R_j = \varnothing$ if $i \neq j$.
3. For all i, $R_i' = R_j$, for some j, where $R_i' = \{(x,y) : (y,x) \in R_i\}$.
4. For all i,j,k, the cardinality p_{ij}^k of the set $\{z \in X : (x,z) \in R_i,\, (z,y) \in R_j\}$ is constant whenever $(x,y) \in R_k$.

Then $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is called an association scheme of class d. The scheme is commutative if $p_{ij}^k = p_{ji}^k$ for all i,j,k.

Every finite group gives rise to an association scheme in the following way.

Definition 2 (Group Association Scheme)**.** Let G be a finite group with conjugacy classes $C_0 = \{e\}, C_1, \ldots, C_d$, where e is the identity of G. Define relations $R_i, 0 \le i \le d$ on G by $(x,y) \in R_i$ if and only if $yx^{-1} \in C_i$. Then $\mathcal{X}(G) = (G, \{R_i\}_{i=0}^d)$ is a commutative association scheme called the group association scheme of G.

Each relation $R_i, 0 \le i \le d$, corresponds to a matrix A_i of size $|G| \times |G|$, called the adjacency matrix of R_i. Let $(x,y) \in G \times G$. The (x,y)-entry $(A_i)_{x,y}$ of A_i is defined as follows:

$$(A_i)_{x,y} = \begin{cases} 1, & \text{if } (x,y) \in R_i \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3 (Bose-Mesner Algebra)**.** Let G be a finite group. The Bose-Mesner algebra $\mathcal{A}$ of the group association scheme $\mathcal{X}(G)$ is the subalgebra of the matrix algebra $\mathrm{Mat}_{|G|}(\mathbb{C})$ defined as $\mathcal{A} = \langle A_0, \ldots, A_d \rangle_{\mathbb{C}}$.

The algebra $\mathcal{A}$ is commutative and semisimple over $\mathbb{C}$ and has a second basis consisting of primitive idempotents $E_0, E_1, \ldots, E_d$ that satisfy

$$E_i \circ E_j = \frac{1}{|G|} \sum_{k=0}^{d} q_{ij}^k E_k,$$

where $\circ$ denotes entry-wise (or Hadamard) multiplication.

The following diagonal matrices E_i^* and A_i^* of size $|G| \times |G|$, where $i = 0, 1, \ldots, d$ will be needed later and are defined as follows:

$$(E_i^*)_{x,x} = \begin{cases} 1, & \text{if } x \in C_i \\ 0, & \text{otherwise} \end{cases}$$

$$(A_i^*)_{x,x} = |G|(E_i)_{e,x}$$

where $x \in G$.

The subalgebra $\mathcal{A}^* = \langle A_0^*, \ldots, A_d^* \rangle_{\mathbb{C}} = \langle E_0^*, \ldots, E_d^* \rangle_{\mathbb{C}}$ of $\mathrm{Mat}_{|G|}(\mathbb{C})$ is called the dual of $\mathcal{A}$. We now define the Terwilliger algebra of a group association scheme.

Definition 4 (Terwilliger Algebra)**.** Let G be a finite group. The Terwilliger algebra, $T(G)$, of the group association scheme $\mathcal{X}(G)$ is the subalgebra of $\mathrm{Mat}_{|G|}(\mathbb{C})$ generated by $\mathcal{A}$ and $\mathcal{A}^*$ over $\mathbb{C}$.

2 Bounds on the Terwilliger Algebra of a Group Association Scheme

We follow the framework of Bannai and Munemasa [4] for our investigation of the Terwilliger algebra of a group association scheme.

Definition 5 (Space of Triple Products)**.** Let G be a finite group and $T(G)$ its Terwilliger algebra. Denote by $T_0(G)$ the subspace of $T(G)$ spanned by the matrix products $E_i^* A_j E_k^*$, for $0 \leq, i, j, k \leq d$ over $\mathbb{C}$.

Definition 6 (Centralizer Algebra)**.** Suppose G acts on itself by conjugation. Identify G with the group of permutation matrices π_g, over all $g \in G$, defined by $(\pi_g)_{x,y} = 1$ if $gxg^{-1} = y$ and 0, otherwise. The centralizer algebra $\widetilde{T}(G)$ consists of all $|G| \times |G|$ matrices over $\mathbb{C}$ that commute with all π_g for all $g \in G$.

Let G be a finite group. In [4], it was shown that that $T(G)$ is bounded below and above by the earlier defined structures $T_0(G)$ and $\widetilde{T}(G)$, respectively. The inclusions are given below:

$$T_0(G) \subseteq T(G) \subseteq \widetilde{T}(G).$$

Thus

$$\dim T_0(G) \leq \dim T(G) \leq \dim \widetilde{T}(G).$$

In [4], a group G is called triply transitive if $T_0(G) = \widetilde{T}(G)$ and triply regular if $T_0(G) = T(G)$. The dihedral and dicyclic groups are known to be triply transitive (see [4] and [2], respectively). For this paper, we have chosen the groups so that the Terwilliger algebras can be equal to either the lower or upper bound or lie strictly between $T_0(G)$ and $\widetilde{T}(G)$.

The dimension of $T_0(G)$ is given by the following.

Lemma 7 ([4])**.** *Let A_i, E_j^* and $T_0(G)$ be as previously defined. Then dim $T_0(G)$ is given by the number of nonzero matrix triple products $E_i^* A_j E_k^*$, where $0 \leq i, j, k \leq d$.*

The dimension of $\widetilde{T}(G)$ is given by the number of orbits of G acting on $G \times G$ by simultaneous conjugation and by Burnside's Lemma, is equal to the average of fixed points. Thus we have the following.

Lemma 8 ([4]). *Let G be a finite group with conjugacy classes $C_0 = \{e\}, C_1, \ldots, C_d$ and $\widetilde{T}(G)$ as above. The dimension of $\widetilde{T}(G)$ is given by*

$$\dim \widetilde{T}(G) = \frac{1}{|G|} \sum_{x \in G} |C_G(x)|^2 = \sum_{i=0}^{d} \frac{|G|}{|C_i|},$$

where $C_G(x)$ is the centralizer of x in G.

Using the above lemmas, we present Table 1 showing the dimensions of $T_0(G)$ and $\widetilde{T}(G)$, computed using GAP [6]. The data on $SL(2,3), S_4$ and 7.3 were given in [4].

Group	dim $T_0(G)$	dim $\widetilde{T}(G)$
$GL(2,3)$	130	136
$SL(2,3)$	75	76
S_4	42	43
7.3	35	41

Table 1: Dimensions of $T_0(G)$ and $\widetilde{T}(G)$

The algebra $\widetilde{T}(G)$ is semisimple and hence can be decomposed into its simple Wedderburn components. The number of simple components corresponds to the number of rows in the character table of the group whose entrywise sums are nonzero. Each simple component is isomorphic to a full matrix algebra of a certain degree. Lemma 3 gives a formula for the degrees of these simple components (see [1]).

Lemma 9. *The degrees d_i of the Wedderburn components of $\widetilde{T}(G)$ are the nonzero row sums in the character table of G. That is,*

$$d_i = \sum_{j=0}^{d} \overline{\chi_i(u_j)}, \text{ where } u_j \in C_j.$$

This yields the following structures of $\widetilde{T}(G)$.

Theorem 10. *Let M_i be the full matrix algebra over $\mathbb{C}$ of degree i and $\widetilde{T}(G)$ be as in Definition 6. Then:*

1. $\widetilde{T}(GL(2,3)) \cong M_2 \bigoplus M_4 \bigoplus M_4 \bigoplus M_6 \bigoplus M_8$
2. $\widetilde{T}(SL(2,3)) \cong M_1 \bigoplus M_1 \bigoplus M_5 \bigoplus M_7$
3. $\widetilde{T}(S_4) \cong M_1 \bigoplus M_2 \bigoplus M_2 \bigoplus M_3 \bigoplus M_5$
4. $\widetilde{T}(7.3) \cong M_2 \bigoplus M_2 \bigoplus M_2 \bigoplus M_2 \bigoplus M_5$

3 The Terwilliger Algebras

We now determine the Terwilliger algebras of the association schemes coming from our four groups. We first give the representatives and ordering of the conjugacy classes of $GL(2,3)$, $SL(2,3)$, S_4, and 7.3 as provided by GAP [6].

$GL(2,3)$ has 8 conjugacy classes, with representatives given as follows.

$$C_0 : \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C_1 : \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \quad C_2 : \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad C_3 : \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$C_4 : \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad C_5 : \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \quad C_6 : \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad C_7 : \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$SL(2,3)$ has 7 conjugacy classes, with representatives given as follows.

$$C_0 : \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C_1 : \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \quad C_2 : \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \quad C_3 : \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$C_4 : \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \quad C_5 : \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad C_6 : \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

S_4 has 5 conjugacy classes, with representatives given as follows.

$$C_0 : (1) \quad C_1 : (1,2) \quad C_2 : (1,2)(3,4) \quad C_3 : (1,2,3) \quad C_4 : (1,2,3,4)$$

The group $7.3 = \langle f_1, f_2 \,|\, f_1^3 = f_2^7 = 1, f_1^{f_2} = f_1^{-1} \rangle$ has 5 conjugacy classes, with representatives given as follows.

$$C_0 : 1 \quad C_1 : f_1 \quad C_2 : f_2 \quad C_3 : f_1^2 \quad C_4 : f_2^3$$

The next two theorems are obtained following the procedures used in [1] and [8]. The proofs are reproduced for completeness and for the reader's convenience.

Theorem 11. *The dimensions of $T(GL(2,3)), T(SL(2,3)), T(S_4)$, and $T(7.3)$ are given as follows.*

1. $\dim T(GL(2,3)) = 130$
2. $\dim T(SL(2,3) = 75$
3. $\dim T(S_4) = 43$
4. $\dim T(7.3) = 37$

Proof: Let $G = GL(2,3)$, $SL(2,3)$, S_4, and 7.3. By definition, $T(G) = \langle E_i^* A_j E_k^* \,|\, 0 \leq i,j,k \leq d \rangle_{\mathbb{C}}$. We determine a set of linearly independent elements from among the $E_i^* A_j E_k^*$ and the products $E_i^* A_j E_k^* \cdot E_k^* A_l E_m^* = E_i^* A_j E_k^* A_l E_m^*$ of these elements. With the aid of GAP computations, we observe that the product $E_i^* A_{i_1} E_j^* \cdot E_j^* A_{i_2} E_k^* \cdot E_k^* A_{i_3} E_l^*$ linearly depends only on the $E_i^* A_{i_4} E_j^*$'s and products of the form $E_i^* A_{i_5} E_j^* \cdot E_j^* A_{i_6} E_k^*$. Therefore, products of more than two blocks of the form $E_i^* A_j E_k^*$ yield no new basis elements. A linearly independent set spanning the products is obtained, thus providing a basis for T.

We summarize the information thus far gathered in the following table:

Group	$\|G\|$	$\|C\|$	dim $T_0(G)$	dim $T(G)$	dim $\widetilde{T}(G)$
$GL(2,3)$	48	8	130	130	136
$SL(2,3)$	24	7	75	75	76
S_4	24	5	42	43	43
7.3	21	5	35	37	41

Table 2: Dimensions of $T_0(G), T(G)$, and $\widetilde{T}(G)$

The rest of this section is devoted to determining explicitly the structures of the Terwilliger algebras, following the procedure in [1]. Specifically, we give the Wedderburn decompositions of these semisimple algebras.

The center of the Terwilliger algebra is the set of all elements that commute with all the other elements. The dimension of the center indicates the number of simple Wedderburn components of $T(G)$. Denote the center of $T(G)$ by $Z(T(G))$.

Theorem 12. *The dimensions of $Z(T(G))$ for $G = GL(2,3)$, $SL(2,3)$, S_4, and 7.3 are given as follows:*

1. $\dim Z(T(GL(2,3))) = 6$
2. $\dim Z(T(SL(2,3))) = 3$
3. $\dim Z(T(S_4)) = 5$
4. $\dim Z(T(7.3)) = 7$.

Proof: Let $y = \sum c_j b_j$, where the $\{b_j\}$ is a basis of $T(G)$ and the c_j are scalars. We solve the system of linear equations $\{x_i y = y x_i\}$ which ranges over all x_i in a basis of $T(G)$. This yields a basis for the center of the Terwilliger algebra.

Let $\{e_i : 1 \leq i \leq s\}$ be a basis of $Z(T(G))$. Then we have $e_i e_j = \sum_{k=1}^{s} t_{ij}^k e_k$. Put $B_i := (t_{ij}^k)$, where $(B_i)_{j,k} = t_{ij}^k$, for $1 \leq i \leq s$. Since these matrices mutually commute, they are simultaneously diagonalizable. Denote by $v_i(i), \ldots, v_s(i)$ the diagonal entries of the diagonalized matrix of B_i and define the matrix M by $M_{ij} := v_i(j)$. Then we get the primitive central idempotents $\{\epsilon_1, \ldots, \epsilon_s\}$ from the relation

$$(e_1, \ldots, e_s) = (\epsilon_1, \ldots, \epsilon_s) M^{-1}.$$

Theorem 13. *The degrees of the irreducible complex representations afforded by the primitive central idempotents are given as follows.*

1. $T(GL(2,3))$:

ϵ_i	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6
$deg\,\epsilon_i$	1	2	3	4	6	8

2. $T(SL(2,3))$:

ϵ_i	ϵ_1	ϵ_2	ϵ_3
$deg\,\epsilon_i$	1	5	7

3. $T(S_4)$:

ϵ_i	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5
$deg\,\epsilon_i$	1	2	2	3	5

4. $T(7.3)$:

ϵ_i	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6	ϵ_7
$deg\,\epsilon_i$	1	1	1	1	2	2	5

Proof: From the structure of semisimple algebras, we know that each simple component of $T(G)$ is afforded by the primitive idempotent ϵ_i and is isomorphic to a full matrix algebra. That is, $T(G)\epsilon_i \cong M_{d_i}$, where d_i is the degree of the irreducible representation afforded by ϵ_i and d_i^2 equals the number of linearly independent elements in the set $\{x_j \epsilon_i\}$, where $\{x_j\}$ is a basis of $T(G)$.

From the above, we have our main structure theorem.

Theorem 14. *Let M_i be the full matrix algebra over $\mathbb{C}$ of degree i. The Terwilliger algebras of the group association schemes of $GL(2,3)$, $SL(2,3)$, S_4, and 7.3 are given as follows.*

1. $T(GL(2,3)) \cong M_1 \bigoplus M_2 \bigoplus M_3 \bigoplus M_4 \bigoplus M_6 \bigoplus M_8$
2. $T(SL(2,3)) \cong M_1 \bigoplus M_5 \bigoplus M_7$
3. $T(S_4) \cong M_1 \bigoplus M_2 \bigoplus M_2 \bigoplus M_3 \bigoplus M_5$
4. $T(7.3) \cong M_1 \bigoplus M_1 \bigoplus M_1 \bigoplus M_1 \bigoplus M_2 \bigoplus M_2 \bigoplus M_5$

We note that for the group $GL(2,3)$, one component M_4 of $\widetilde{T}(GL(2,3))$ decomposes in $T(GL(2,3))$ into two components $M_1 \bigoplus M_3$. For the group 7.3, the sum $M_2 \bigoplus M_2$ in $\widetilde{T}(7.3)$ decomposes into $M_1 \bigoplus M_1 \bigoplus M_1 \bigoplus M_1$ in $T(7.3)$. Here, $M_1 \cong \mathbb{C}$. Combining Theorem 2 and Theorem 5, we can see that the relation $\dim T(G) = \sum_i d_i^2$ is satisfied, as should be.

4 Distribution of Basis Elements

We exhibit the behavior of the basis elements for the vector spaces $T_0(G)$, $T(G)$, and $\widetilde{T}(G)$ using the square tables of size $d+1$ below. Indexed by the conjugacy classes, $C_0, C_1, \ldots, C_d$, these tables indicate how the basis elements are distributed. An entry n in the (C_iC_k)–position indicates that there are n nonzero basis elements. Thus, in $T_0(G)$, the (C_iC_k)–entry gives the number of nonzero products $E_i^* A_j E_k^*$. As these tables are symmetric, we omit the entries below the diagonal. We will simply write T_0, T and $\widetilde{T}$ when the reference to the group is clear.

For $GL(2,3)$, $T_0 = T$. The $T_0 = T$ entry in position (C_7C_7) is 5 while the entry for $\widetilde{T}$ in the same position is written as 3+0+4. This means that three of the five basis elements in $T_0 = T$ are basis elements in $\widetilde{T}$ but the two others split into two each, for a total of 3+0+4 = 7 basis elements in $\widetilde{T}$. The entries for the other groups are understood in the same manner.

$$T_0 = T : \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 4 & 1 & 4 & 2 & 2 & 2 & 3 \\ & & 1 & 1 & 1 & 1 & 1 & 1 \\ & & & 4 & 2 & 2 & 2 & 3 \\ & & & & 3 & 3 & 3 & 3 \\ & & & & & 3 & 3 & 3 \\ & & & & & & 3 & 3 \\ & & & & & & & 5 \end{bmatrix}; \quad \widetilde{T} : \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 4 & 1 & 4 & 2 & 2 & 2 & 2+0+2 \\ & & 1 & 1 & 1 & 1 & 1 & 1 \\ & & & 4 & 2 & 2 & 2 & 2+0+2 \\ & & & & 3 & 3 & 3 & 3 \\ & & & & & 3 & 3 & 3 \\ & & & & & & 3 & 3 \\ & & & & & & & 3+0+4 \end{bmatrix}$$

$SL(2,3)$ is a similar to $GL(2,3)$ in that $T_0 = T$. The entries of $T_0 = T$ and $\widetilde{T}$ differ only in the (C_6C_6)–position.

$$T_0 = T : \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 2 & 2 & 1 & 2 & 2 & 2 \\ & & 2 & 1 & 2 & 2 & 2 \\ & & & 1 & 1 & 1 & 1 \\ & & & & 2 & 2 & 2 \\ & & & & & 2 & 2 \\ & & & & & & 3 \end{bmatrix}; \quad \widetilde{T} : \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 2 & 2 & 1 & 2 & 2 & 2 \\ & & 2 & 1 & 2 & 2 & 2 \\ & & & 1 & 1 & 1 & 1 \\ & & & & 2 & 2 & 2 \\ & & & & & 2 & 2 \\ & & & & & & 2+0+2 \end{bmatrix}$$

For S_4, $T = \widetilde{T}$. Here the (C_3C_3)–entry in T_0 is 3 while in $T = \widetilde{T}$, it is 3+1. Thus, an additional element in $T = \widetilde{T}$ was added to the 3 basis elements in T_0.

$$T_0 : \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 3 & 2 & 2 & 2 \\ & & 2 & 1 & 2 \\ & & & 3 & 2 \\ & & & & 3 \end{bmatrix}; \qquad T = \widetilde{T} : \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 3 & 2 & 2 & 2 \\ & & 2 & 1 & 2 \\ & & & 3+1 & 2 \\ & & & & 3 \end{bmatrix}$$

For 7.3, T is properly contained between T_0 and $\widetilde{T}$. Consider the entry 1 in the (C_1C_3)–positions of T_0 and T. The corresponding entry in $\widetilde{T}$ is 0+0+3. This indicates that that basis element in T_0 remains as a basis element in T but splits into three elements in $\widetilde{T}$.

$$T_0 : \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 3 & 1 & 1 & 1 \\ & & 3 & 1 & 2 \\ & & & 3 & 1 \\ & & & & 3 \end{bmatrix}; T : \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 3 & 1 & 1 & 1 \\ & & 3 & 1 & 2+1 \\ & & & 3 & 1 \\ & & & & 3 \end{bmatrix}; \widetilde{T} : \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 3 & 1 & 0+0+3 & 1 \\ & & 3 & 1 & 2+1+0 \\ & & & 3 & 1 \\ & & & & 3 \end{bmatrix}$$

Finally, the (C_2C_4)–entry in T_0 entry is 2; the (C_2C_4)–entry in T entry 2+1; and (C_2C_4)–entry in $\widetilde{T}$ entry is 2+1+0. Thus, a new basis element in T was added to that of T_0 and those 2+1 elements remain as basis elements in $\widetilde{T}$.

5 Some GAP Codes

We append some GAP codes to aid the reader in setting up and reconstructing some computations.

Let `G` be a group with `g` number of elements. Using GAP, define the following:

```
> C := ConjugacyClasses(G);
> GG := Cartesian(G, G);
> Or := OrbitsDomain[G, GG, OnPairs];
```

Store the elements of G thus:

```
> El := [ ]
> for i in [0..n] do;
> Append(El, Elements(C[i + 1]));
> od;
```

where `n` is the number of conjugacy classes.

We can then run the following code for $\widetilde{T}$.

```
> tilde := function(i)
> local j, k, M;
> M := NullMat(g, g);
> for j in [1..g] do
> for k in [1..g] do
> if \in([El[j], El[k]], Or[i]) then
> M[j][k] := 1;
> else
> M[j][k] := 0;
> fi;
> od;
> od;
```

```
> return M;
> end;
```

GAP has the package GRAPE with the command `AdjacencyMatrices(G)` that returns the adjacency matrices A_i of the association scheme. However, since we want to preserve the ordering of the C_i's obtained earlier, an alternative code is:

```
> adjmat := function(i)
> local j, k, M;
> M := NullMat(g);
> for j in [1..g] do
> for k in [1..g] do
> if \in(LeftQuotient(Inverse(El[j]), Inverse(El[k])), C[i + 1]) then
> M[j][k] := 1;
> else
> M[j][k] := 0;
> fi;
> od;
> od;
> return M;
> end;
```

where `El` is the list of the elements of the group; `g`, the order of the group; and `n`, the number of conjugacy classes.

To obtain the diagonal matrices E_i^*, the following is an intuitive GAP code:

```
> estar := function(i)
> local j, M;
> M := NullMat(g);
> for j in [1..g] do
> if \in(L[j], C[i + 1]) then
> M[j][j] := 1;
> else
> M[j][j] := 0;
> fi;
> od;
> return M;
> end;
```

where `El` is the list of the elements of the group; `g`, the order of the group; and `n`, the number of conjugacy classes.

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