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# On the Placing Probabilities for the Four-Tower Problem Using Recursions Based on Multigraphs

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**Abstract**

The Four-Tower Problem is a four-player gambler's ruin model where two players are involved in an even-money bet during each round. In this problem, the objective is to solve for each player's ruin and final placing probabilities given their initial wealths. Weighted directed multigraphs were constructed to model the transitions between chip states. Linear systems are constructed based on the connections between nodes in these graphs. Solutions for the placing probabilities of each player are obtained from these linear systems. A numerical algorithm is developed to solve the Four-Tower Problem for any positive integer chip total. The solution leads to exact values, and results show that the equities in the this model depend on the number, not just proportion, of chips each player holds.

**Keywords:** Markov chain, graph theory, discrete mathematics, 4-dimensional gambler's ruin, applied probability

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## 1 Introduction

In the classic gambler's ruin scenario, a gambler starts with an initial wealth $A$. Each round, they bet a fixed amount on a game in which their winning probability is $p$. They keep playing the same game until they achieve either success, where their wealth reaches a target amount $S$, or ruin, where they lose all their money.

If the game is fair, that is, the winning probability $p$ is equal to $1/2$, the gambler's probability of success is $\frac{A}{S}$.

We can extend the problem to an $N$-player gambler's ruin problem, where $N > 1$. In particular, consider the four-player gambler's ruin problem. For these type of problems, the main objective is to determine the success and ruin probabilities of each player. Consider a game with four players having initial wealths $S_1$, $S_2$, $S_3$, and $S_4$. The four-player game has several variations based on the players involved in each betting round, and how winners and losers are selected in these rounds.

In this paper, we will focus on the four-player variation called the four-tower game. In a four-tower game, each round has one winner and one loser. Each game involves exactly two of the four players, each with an equal probability of winning the game. The players are paired randomly with equal probabilities, so that each pair has a probability $\frac{1}{6}$ of being selected. We fix the bet size to 1 unit for each game. Suppose $(S_1, S_2, S_3, S_4)$ represents the initial state. The new states after one betting round are shown by the following map:

$$(S_1, S_2, S_3, S_4) \to \begin{cases} (S_1 \pm 1, S_2 \mp 1, S_3, S_4) \\ (S_1 \pm 1, S_2, S_3 \mp 1, S_4) \\ (S_1 \pm 1, S_2, S_3, S_4 \mp 1) \\ (S_1, S_2 \pm 1, S_3 \mp 1, S_4) \\ (S_1, S_2 \pm 1, S_3, S_4 \mp 1) \\ (S_1, S_2, S_3 \pm 1, S_4 \mp 1) \end{cases} . \tag{1}$$

In the past years, several solutions have been used in modeling and solving the $N$-tower problem where $N > 2$. In these problems, the probability of a player finishing first is easily solved by recursion, and is given by the proportion of a player's wealth to the total wealth of the three players involved in the game. Bruss et al (2002) used martingales in giving an asymptotic solution to the three-tower problem [1]. They however described the difficulty they encountered in attempting to generalize their solution for tower problems with more than three players. David (2014) used weighted directed multigraphs and recursions to model and solve the three-tower problem [2]. For a fixed wealth $S$, unique states were generated and transitions between these states were represented as edges in a multigraph. A linear system was constructed based on recurrence equations obtained from this multigraph, which was then solved to obtain the ruin probabilities of each player. After ruin probabilities are solved, placing probabilities of each player can be computed easily. Swan and Bruss (2006) described a matrix-analytic approach in solving the $N$-tower problem [6]. In their methodology, ruin probabilities for each player can be obtained, but placing probabilities for the other remaining players cannot be calculated. This gives a particular disadvantage; equities for each player cannot be completely solved if the game has payouts for the places beyond the first.

Since solving the four-tower problem means finding only the ruin probabilities of each player, finding placing probabilities of each player in the four-tower problem remains an open problem. However, once one player is ruined in the four-tower game, the game is reduced to the three-tower problem for the remaining players. Hence, solutions to the three-tower problem can be used to obtain placing probabilities for the remaining players once one player is ruined in the four-tower game. Solving for the placing probabilities leads to a model for calculating equities in tournaments with players having wealths pitted against each other.

A benchmark for solving the final placing probabilities is the Independent Chip Model or ICM. This model uses an algorithm to calculate equities in a tournament where players have wealths which are pitted against each other. This model uses conditional probabilities in order to calculate the placing probabilities of each player, which can then be used to calculate their equities in the given tournament. This model is credited to Malmuth-Harville [5]. However, there is no proof provided for the algorithm in this model.

The main purpose of this paper is to solve the placing probabilities of each player for the Four-Tower Problem. Specifically, the paper aims to accomplish the following. First, we will construct a weighted directed multigraph for the Four-Tower Problem with prescribed total wealth $S$. From this, we will construct an appropriate linear system that represents the multigraph for the Four-Tower Problem with the prescribed total wealth $S$. Lastly, we will develop a numerical algorithm for solving placing probabilities for the Four-Tower Problem with the prescribed total wealth $S$.

# 2 Theoretical Framework

## 2.1 Weighted Directed Multigraphs

We use the following definitions to define a weighted directed multigraph.

1. A *graph* $G(V, E)$ consists of two types of elements, namely *vertices* and *edges*. Every edge has two endpoints in the set of vertices. $V$ is the set of vertices while $E$ is the set of edges.
2. A *weighted graph* is a graph having a weight, or number, associated with each edge.
3. A *directed graph* is a graph where all edges are directed from one vertex to another.
4. A *multigraph* is a graph which may have multiple edges connecting the same pair of vertices.
5. A *weighted directed multigraph* is a graph satisfying items (1) to (4).

Weighted directed multigraphs will be used to model the transitions between states given the players' initial wealths. Variables for the players' placing probabilities will be assigned to each unique state. The system of equations relating these variables will then be solved for these probabilities.

## 2.2 Independent Chip Model

The Independent Chip Model (or ICM) is credited to Malmuth-Harville [5]. This model is used for calculating the placing probabilities of each player in tournaments with players having chip stacks pitted against one another.

In this model, the probability of a player placing first is that player's stack in proportion to the number of chips in all stacks.

Let $N$ be the number of players, $X_i$ be the random variable denoting the placing of Player $i$, and $S_i$ be the current stack size of Player $i$. Then

$$P(X_i = 1) = \frac{S_i}{S} \tag{2}$$

where $S = S_1 + S_2 + \cdots + S_N$.

The probability of placing second, third, etc. is calculated using conditional probabilities. Calculations for $P(X_i = 2)$ and $P(X_i = 3)$ are done using the following equations:

$$\begin{aligned}
P(X_i = 2) &= \sum_{j \neq i} P(X_i = 2 | X_j = 1) P(X_j = 1) \\
&= \sum_{j \neq i} \frac{S_j}{S} \frac{S_i}{S - S_j}; \\
P(X_i = 3) &= \sum_{j \neq i} \sum_{k \neq i,j} P(X_i = 3 | X_k = 2 \cap X_j = 1) P(X_k = 2 | X_j = 1) P(X_j = 1) \\
&= \sum_{j \neq i} \sum_{k \neq i,j} \frac{S_j}{S} \frac{S_k}{S - S_j} \frac{S_i}{S - S_j - S_k}.
\end{aligned}$$

In general, for any $n \in \mathbb{N}$ such that $n \leq N$, we have

$$P(X_i = n) = \sum_{k_2 \neq k_1} \sum_{k_3 \neq k_1, k_2} \cdots \sum_{k_n \neq k_1, \ldots, k_{n-1}} \frac{S_{k_1} S_{k_2} \cdots S_{k_n}}{S(S - S_{k_2}) \cdots (S - S_{k_2} - \cdots - S_{k_n})},$$

where $i = k_1$. The ICM is the commonly used formula for estimating players' placing probabilities in an $N$-player game, specifically in poker tournaments. These placing probabilities are then used to calculate the players' equities.

## 2.3 Absorbing Markov Chains

We use the following definitions to define an absorbing Markov chain [4].

1. Given a set of states $S = \{s_1, s_2, \ldots, s_n\}$, a *Markov chain* is a process satisfying the following conditions:
   (a) The transition probability $p_{ij}$ of moving from an initial state $s_i$ to any next state $s_j$ depends only on the current state $s_i$.
   (b) The sum of all transition probabilities from any state $s_i$ is equal to 1, i.e.,
   $$\sum_{j=1}^{n} p_{ij} = 1 \tag{3}$$
   for all $i \in \{1, 2, \ldots, n\}$.
2. A state $s_i$ of a Markov chain is called an *absorbing state* if it is impossible to leave it, i.e., $p_{ii} = 1$.
3. An *absorbing Markov chain* is a Markov chain satisfying the following conditions:
   (a) It has at least one absorbing state.
   (b) It is possible to go from any state to at least one absorbing state in a finite number of steps.
4. A state $s_i$ of an absorbing Markov chain is called a *transient state* if it is not absorbing.
5. Absorbing Markov chains can be represented using matrices whose sizes depend on the number of transient and absorbing states. Suppose there are $t$ transient states and $r$ absorbing states for an absorbing Markov chain. The transition matrix has the following *canonical form*:
   $$\mathbf{P} = \begin{bmatrix} \mathbf{Q} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{4}$$
   where $\mathbf{Q}$ is a $t \times t$ matrix, $\mathbf{R}$ is a nonzero $t \times r$ matrix, $\mathbf{0}$ is an $r \times t$ zero matrix, and $\mathbf{I}$ is an $r \times r$ identity matrix. The first $t$ states are the transient states while the last $r$ states are the absorbing states.
6. For an absorbing Markov chain $\mathbf{P}$, the matrix $\mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1}$ is called the *fundamental matrix* for $\mathbf{P}$. The entry $n_{ij}$ of $\mathbf{N}$ gives the expected number of times that the process is in the transient state $s_j$ if it started in the transient state $s_i$.

# 3 Methods

This paper aims to calculate placing probabilities for the four players given their initial wealths. Probabilities of placing first are just the proportions of the players' initial wealths to the total wealth in play. What remains is to calculate the probabilities of placing second, third, and fourth for each player. These calculations are done using a numerical algorithm described in this section. We first define the following.

**Definition 1.** A *chip state* is an ordered quadruple $(S_1, S_2, S_3, S_4)$, where $S_1, S_2, S_3, S_4$ are nonnegative integers. The elements of a chip state represent the wealths of each player at a given time.

**Definition 2.** A *chip position* is an element of a chip state $(S_1, S_2, S_3, S_4)$. In this case, the chip positions are $S_1$, $S_2$, $S_3$, and $S_4$. Each position in a chip state has corresponding probabilities of finishing first, second, down to the last place among the four players.

**Definition 3.** A *terminal state* is a chip state where at least one chip position is zero. Placing probabilities for each player for terminal states can be solved based on the solutions to the three-tower problem.

**Definition 4.** A *nonterminal state* is a chip state with four positive chip positions.

Two players are selected randomly using a uniform distribution, and these two players will face each other in an even-money betting round. Without loss of generality, let Player 1 and Player 2 have wealth $x$ and $y$, respectively. The winner of the round is selected randomly, and that player adds to his stack one chip taken from the other player's initial stack. For example, if Player 1 wins over Player 2, their new chip stacks will be $x + 1$ and $y - 1$, respectively. For as long as the states are nonterminal, the process is repeated. Upon reaching a terminal state, placing probabilities for the remaining three players can be obtained from the three-tower problem solutions.

We use the following definition for the mapping for the Four-Tower Problem.

**Definition 5.** Consider a nonterminal state $(S_1, S_2, S_3, S_4)$. The possible states after one round of betting are shown by the map:

$$(S_1, S_2, S_3, S_4) \to \begin{cases} (S_1 \pm 1, S_2 \mp 1, S_3, S_4) \\ (S_1 \pm 1, S_2, S_3 \mp 1, S_4) \\ (S_1 \pm 1, S_2, S_3, S_4 \mp 1) \\ (S_1, S_2 \pm 1, S_3 \mp 1, S_4) \\ (S_1, S_2 \pm 1, S_3, S_4 \mp 1) \\ (S_1, S_2, S_3 \pm 1, S_4 \mp 1) \end{cases}, \tag{5}$$

with each state having probability $\frac{1}{12}$ of being selected.

## 3.1 Construction of Multigraph

Let $S$ be the total wealth of the four players. We construct the weighted directed multigraph using the following algorithm:

1. Unique states $(S_1, S_2, S_3, S_4)$ are generated such that $S = S_1 + S_2 + S_3 + S_4$ and $S_1 \geq S_2 \geq S_3 \geq S_4$, that is, the total wealth of the four players is $S$ and their wealths are arranged in decreasing order. The generated unique states will serve as the nodes in the graph.

2. Terminal states (with at least one zero element) are put on the leftmost column, with the states arranged in decreasing order of their first elements.
3. States with the last element equal to 1 are then put on the next column, with the states also arranged in decreasing order of their first elements.
4. The process of constructing rows and columns of nodes is repeated until all states are exhausted.
5. Given a nonterminal state $(S_1, S_2, S_3, S_4)$, if it is possible for the chip stacks to become $(R_1, R_2, R_3, R_4)$ after one round of betting, an edge directed from $(S_1, S_2, S_3, S_4)$ to $(R_1, R_2, R_3, R_4)$ is constructed, with weight equal to $\frac{1}{12}$, so that the total weight of all outward edges from a nonterminal state must be 1.
6. Loops, which are edges whose initial and final nodes are the same, may be constructed if a state goes to itself (up to permutation of elements) after one betting round. Multiple edges between a pair of nodes may also be constructed if there are multiple possible ways of transitioning between these two states.

## 3.2 Construction of Linear System

We construct the linear system using the following algorithm:

1. *Variable assignment*
   A variable will be associated to each of the unique chip positions from all terminal and nonterminal states in the order they are generated.
2. *Construction of transition matrix* $\mathbf{Q}$
   An $n \times n$ matrix $\mathbf{Q}$ is constructed, where $n$ is the number of unique chip positions from all nonterminal states. Matrix $\mathbf{Q}$ is the matrix representing the transitions between nonterminal chip positions.
3. *Construction of transition matrix* $\mathbf{R}$
   An $n \times m$ matrix $\mathbf{R}$ is also constructed, where $m$ is the number of unique chip positions from all terminal states. This matrix represents the transitions from nonterminal to terminal chip positions.
4. *Computation of entries of matrices* $\mathbf{Q}$ *and* $\mathbf{R}$
   For each of the $n$ variables corresponding to nonterminal positions, we determine where the corresponding chip positions are being moved.
5. *Set up of linear system*
   We set up and solve the linear system
   $$(\mathbf{I} - \mathbf{Q})\mathbf{B} = \mathbf{R}, \tag{6}$$
   where $\mathbf{B}$ is an $n \times m$ matrix containing the probabilities for the $n$ unique nonterminal chip positions of ending up in the $m$ unique terminal chip positions.
6. *Recursion for three-tower problem*
   After solving for matrix $\mathbf{B}$, we construct an $m \times N$ matrix $\mathbf{W}$ containing the placing probabilities for the corresponding terminal positions in the four-tower game. We do this by recursively applying the above algorithm to the three-tower ruin problem until we reach the 2-tower case, which is equivalent to the 2-player gambler's ruin problem.
7. *Computation of placing probabilities*
   We then recursively solve the tower problems up to the four-tower problem to find $\mathbf{BW}$, which contains the final placing probabilities.

The following theorem tells us how absorption probabilities are obtained using the fundamental matrix of an absorbing Markov chain.

**Theorem 1.** *Let $q_{ij}^{(n)}$ denote the entries of matrix $\mathbf{Q}^n$. Let $b_{ij}$ be the probability that an absorbing chain will be absorbed in the absorbing state $s_j$ if it starts in the transient state $s_i$. Let $\mathbf{B}$ be the matrix with entries $b_{ij}$. Then $\mathbf{B}$ is a $t \times r$ matrix, and*

$$\mathbf{B} = \mathbf{NR}, \tag{7}$$

*where $\mathbf{N}$ is the fundamental matrix and $\mathbf{R}$ is as in the canonical form.*

**Proof:** Note that the entry $q_{ij}^{(n)}$ of matrix $\mathbf{Q}^n$ is the probability of being in state $s_j$ after $n$ steps, when the chain starts at state $s_i$. We have

$$\mathbf{B}_{ij} = \sum_{n=1}^{\infty} \sum_{k=1}^{t} q_{ik}^{(n)} r_{kj} = \sum_{k=1}^{t} \sum_{n=1}^{\infty} q_{ik}^{(n)} r_{kj} = \sum_{k=1}^{t} n_{ik} r_{kj} = (\mathbf{NR})_{ij}. \tag{8}$$

Hence, $\mathbf{B} = \mathbf{NR}$.

# 4 Results

In general, for any positive integer $S$, the system can be modeled as a multigraph with loops where the vertices are the unique states up to permutations where the directed edges represent the transitions between states. The following theorem gives us the number of directed edges from a nonterminal state $(S_1, S_2, S_3, S_4)$ to terminal and nonterminal states.

**Theorem 2.** *Consider a nonterminal state $(S_1, S_2, S_3, S_4)$, where $S_1 \geq S_2 \geq S_3 \geq S_4 \geq 1$. Let $1 \leq m \leq 4$ such that $S_i > 1$ if $i \leq m$ and $S_i = 1$ if $i > m$.*

*(i) There are $12 - 3m$ directed edges from $(S_1, S_2, S_3, S_4)$ to terminal states.*

*(ii) There are $3m$ directed edges from $(S_1, S_2, S_3, S_4)$ to nonterminal states.*

**Proof:** Consider a nonterminal state $(S_1, S_2, S_3, S_4)$ satisfying the given conditions. Then the first $m$ positions of this state are greater than 1 while the last $4 - m$ positions are equal to 1, if $m < 4$. In selecting pairs of players for each round, suppose the player selected first loses while the second player wins that round. There are $4 - m$ ways of selecting one of the players with wealth 1 first, and three ways of selecting one of the remaining players second. In each of these cases, the new state will be a terminal state since the losing player will be ruined after the round. Hence, there are $3(4 - m) = 12 - 3m$ directed edges to terminal states. In the remaining cases, we select first one of the $m$ players with wealth greater than 1, and any of the remaining $N - 1$ players second. In any of these cases, since the losing player has wealth greater than 1 at the beginning of the round, the new state will still be a nonterminal state. Hence, there are $3m$ directed edges to nonterminal states.

The next theorem describes the connections between a nonterminal state $(S_1, S_2, S_3, S_4)$ and some terminal states.

**Theorem 3.** *Given a nonterminal state $(S_1, S_2, S_3, S_4)$, where $S_1 \geq S_2 \geq S_3 \geq S_4 \geq 1$, it is connected to the following terminal states, with edge weights $\frac{1}{12}$:*

*(i) $(S_1 + 1, S_2, S_3, 0)$, if $S_4 = 1$*

*(ii)* $(S_1, S_2+1, S_3, 0)$, *if* $S_4 = 1$

*(iii)* $(S_1, S_2, S_3+1, 0)$, *if* $S_4 = 1$

*(iv)* $(S_1, \ldots, S_i+1, \ldots, S_j = 0, \ldots, S_4)$, *for each* $i \in \{1, 2, 3, 4\}$ *such that* $i \neq j$ *and for each* $j < 4$ *such that* $S_j = 1$

**Proof:** We prove $(i)$. Let $(S_1, S_2, S_3, S_4)$, where $S_1 \geq S_2 \geq S_3 \geq S_4 = 1$, be a nonterminal state. Since players are selected randomly using a uniform distribution, there are $\binom{4}{2} = 6$ ways of selecting a pair. Player 1 and Player 4 are paired with probability $\frac{1}{6}$. In this pairing, each player involved has probability $\frac{1}{2}$ of winning one chip from the other player. Hence, the probability of $(S_1, S_2, S_3, S_4)$ transitioning to $(S_1+1, S_2, S_3, 0)$ is $\frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$. The proofs for $(ii)$ to $(iv)$ follow similarly.

The following theorem describes the connections between a nonterminal state $(S_1, S_2, S_3, S_4)$ and some nonterminal states.

**Theorem 4.** *Given a nonterminal state* $(S_1, S_2, S_3, S_4)$, *where* $S_1 \geq S_2 \geq S_3 \geq S_4 \geq 1$, *it is connected with edge weights* $\frac{1}{12}$ *to nonterminal states of the form*

*(i)* $(S_1 \pm 1, S_2 \mp 1, S_3, S_4)$

*(ii)* $(S_1 \pm 1, S_2, S_3 \mp 1, S_4)$

*(iii)* $(S_1 \pm 1, S_2, S_3, S_4 \mp 1)$

*(iv)* $(S_1, S_2 \pm 1, S_3 \mp 1, S_4)$

*(v)* $(S_1, S_2 \pm 1, S_3, S_4 \mp 1)$

*(vi)* $(S_1, S_2, S_3 \pm 1, S_4 \mp 1)$

*provided that each chip position in the new state is nonzero. The chip positions are rearranged in decreasing order to get a valid chip state.*

**Proof:** Let $(S_1, S_2, S_3, S_4)$ be a nonterminal state. Since players are selected randomly using a uniform distribution and there is a winner and a loser in each round, there are four ways of selecting a winner and three ways of selecting a loser from the remaining players. In each round, the winner gains one chip from the loser. Suppose Player $i$ is selected as the winner, while Player $j$ is selected as the loser. Then from Definition 5, the transition after one round is

$$(S_1, \ldots, S_i, \ldots, S_j, \ldots, S_4) \rightarrow (S_1, \ldots, S_i+1, \ldots, S_j-1, \ldots, S_4).$$

Since there are 12 ways of selecting these pairs of players, each with equal probability, then each outward edge from this nonterminal state has weight $\frac{1}{12}$.

The following corollary describes the existence of loops in the multigraph given certain conditions.

**Corollary 5.** *Given a nonterminal state* $(S_1, S_2, S_3, S_4)$, *where* $S_1 \geq S_2 \geq S_3 \geq S_4 \geq 1$, *for each* $i \in \{1, 2, 3, 4\}$ *and for each* $j$ *such that* $i < j \leq 4$ *and* $S_i = S_j + 1$, *it has a loop with edge weight* $\frac{1}{12}$. *The transition of positions along the loop is* $(S_i, S_j) \mapsto (S_j, S_i)$.

**Proof:** Let $(S_1, S_2, S_3, S_4)$ be a nonterminal state. Let $i \in \{1, 2, 3\}$, and take $j \in \{i, i+1, \ldots, 4\}$ such that $S_i = S_j + 1$, if such $j$ exists. With probability $\frac{1}{12}$, Player $j$ is selected as the winner, while Player $i$ is selected as the loser. After one round, Player $i$ has wealth $S_i - 1$ or $S_j$ while Player $j$ has wealth $S_j + 1$ or $S_i$, while the wealths of the remaining players remain unchanged. Then the transition after one round is

$$(S_1, \ldots, S_i, \ldots, S_j, \ldots, S_4) \to (S_1, \ldots, S_j, \ldots, S_i, \ldots, S_N).$$

Rearranging the chip positions in decreasing order yields the original state. Hence, there is a loop whose initial and final states after one round is the chosen nonterminal state.

**Example 1.** In this example, we take a more detailed look at the four-tower results for $S = 6$. We first look at the graph for the four-tower game with total wealth $S = 6$, as shown in Figure 1.

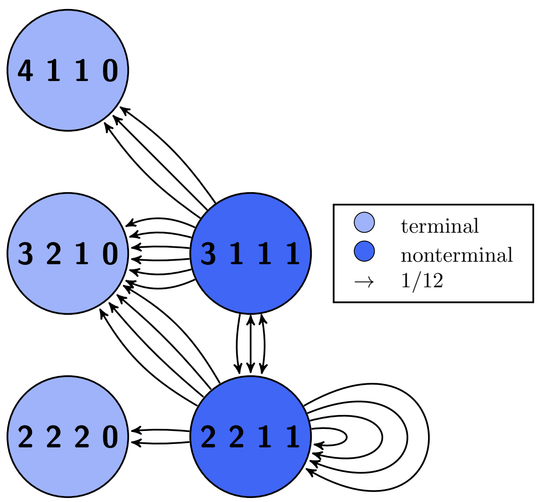


*Figure 1:* Lattice for $N = 4$, $S = 6$

Here, the unique terminal states are $(4, 1, 1, 0)$, $(3, 2, 1, 0)$ and $(2, 2, 2, 0)$, denoted by $P_1$, $P_2$, and $P_3$, respectively. The unique nonterminal states are $(3, 1, 1, 1)$ and $(2, 2, 1, 1)$, denoted by $P_4$, and $P_5$, respectively. From these states, we observe that there are four unique chip positions in nonterminal states and 9 unique chip positions in terminal states. Also, notice that there are 12 outward edges from each nonterminal state, with each edge having weight $\frac{1}{12}$. Four loops are present for the state $P_5$ since it can transition to itself if Player 3 or 4 beats Player 1 or 2 for one chip.

After the construction of a multigraph for a given wealth total $S$, a linear system representing the transitions between the states is constructed, as detailed in Section 3.2. For example, if $S = 6$, we define four variables $v_1$, $v_2$, $v_3$, and $v_4$ corresponding respectively to the unique nonterminal chip positions 3 and 1 in $P_4$, and positions 2 and 1 in $P_5$. We also define nine variables $w_1$, $w_2$, up to $w_9$ for terminal states, corresponding respectively to the chip positions 4, 1, and 0 in $P_1$, positions 3, 2, 1, and 0 in $P_2$, and positions 2 and 0 in $P_3$. We get the linear system by setting up the recurrence:

$$\left( \mathbf{I}_4 - \frac{1}{12} \begin{bmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix} \right) \mathbf{B} = \frac{1}{12} \begin{bmatrix} 3 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 \end{bmatrix} \tag{9}$$

The solutions to this system is an $n \times m$ matrix that gives the absorption probabilities for the $n$ unique nonterminal chip positions into the $m$ unique terminal chip positions. This system has the following solution:

$$\mathbf{B} = \begin{bmatrix} 0.2568 & 0.0066 & 0.0033 & 0.5681 & 0.061 & 0.0188 & 0.0188 & 0.0606 & 0.0061 \\ 0.0033 & 0.1756 & 0.0878 & 0.0329 & 0.2019 & 0.216 & 0.216 & 0.0465 & 0.0202 \\ 0.0272 & 0.0263 & 0.0131 & 0.2723 & 0.2441 & 0.0751 & 0.0751 & 0.2423 & 0.0244 \\ 0.0061 & 0.0404 & 0.0202 & 0.061 & 0.0892 & 0.2582 & 0.2582 & 0.1577 & 0.1089 \end{bmatrix} \tag{10}$$

which means that for $v_1$ (3 in $P_4$), its probability of ending up in the terminal chip position $w_1$ (4 in $P_1$) is 25.68%, and so on. Note that $w_3$, $w_7$ and $w_9$ represent ruined chip positions for the four-tower game. This implies that $v_1$'s ruin probability is given by the sum

$$B_{1,3} + B_{1,7} + B_{1,9} = 2.82\%. \tag{11}$$

Recall that each player's probability of finishing first is equal to the proportion of that player's wealth to the total wealth of all players. To calculate the players' probabilities of finishing second or third, we need the solutions to the three-tower problem for $S = 6$. The graph for the three-tower game with total wealth $S = 6$ is shown in Figure 2.

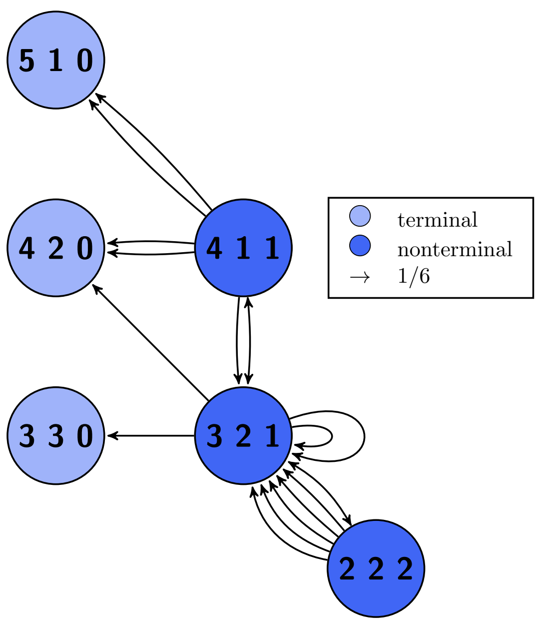


*Figure 2:* Lattice for $N = 3$, $S = 6$

A linear system representing the transitions between the states is constructed. We define six variables $v_1$ to $v_6$ corresponding to the unique chip positions from the nonterminal states $(4, 1, 1)$, $(3, 2, 1)$ and $(2, 2, 2)$, and eight variables $w_1$ to $w_8$ corresponding to the unique chip positions from the terminal states $(5, 1, 0)$, $(4, 2, 0)$ and $(3, 3, 0)$. We get the linear system by setting up the recurrence:

$$\left(\mathbf{I}_6 - \frac{1}{6}\begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 & 0 \end{bmatrix}\right)\mathbf{C} = \frac{1}{6}\begin{bmatrix} 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{12}$$

This system has the following solution:

$$\mathbf{C} = \begin{bmatrix} 0.3619 & 0.0066 & 0.0066 & 0.4475 & 0.0313 & 0.0212 & 0.1104 & 0.0146 \\ 0.0066 & 0.1842 & 0.1842 & 0.0262 & 0.2343 & 0.2394 & 0.0698 & 0.0552 \\ 0.0857 & 0.0197 & 0.0197 & 0.3426 & 0.0939 & 0.0635 & 0.3312 & 0.0438 \\ 0.0247 & 0.0501 & 0.0501 & 0.099 & 0.2716 & 0.1294 & 0.2957 & 0.0793 \\ 0.0146 & 0.0552 & 0.0552 & 0.0584 & 0.1345 & 0.3071 & 0.1231 & 0.2519 \\ 0.0417 & 0.0417 & 0.0417 & 0.1667 & 0.1667 & 0.1667 & 0.25 & 0.125 \end{bmatrix} \tag{13}$$

which gives the absorption probabilities given an initial position $v_i$ and final position $w_j$ for $1 \leq i \leq 6$ and $1 \leq j \leq 8$. To calculate the players' placing probabilities, we need the solutions to the two-player game for $S = 6$. In this scenario, the players' winning probabilities are equal to the proportion of their wealths to the total wealth of all players. We have the following matrix $\mathbf{X}$ whose entries are the placing probabilities for each nonterminal chip position in the three-tower game:

$$\mathbf{X} = \begin{array}{c} \\ 5 \\ 1 \\ 0 \\ 4 \\ 2 \\ 0 \\ 3 \\ 0 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 5/6 & 1/6 & 0 \\ 1/6 & 5/6 & 0 \\ 0 & 0 & 1 \\ 2/3 & 1/3 & 0 \\ 1/3 & 2/3 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}. \tag{14}$$

Using $\mathbf{C}$ in (13) and $\mathbf{X}$ in (14), we get matrix $\mathbf{CX}$ shown in (15), whose entries are the placing probabilities for each nonterminal chip position in the three-tower game:

$$\mathbf{CX} = \begin{array}{c} \\ 4 \\ 1 \\ 3 \\ 2 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.6667 & 0.2910 & 0.0423 \\ 0.1667 & 0.3545 & 0.4788 \\ 0.5000 & 0.3731 & 0.1269 \\ 0.3333 & 0.4078 & 0.2589 \\ 0.1667 & 0.2191 & 0.6142 \\ 0.3333 & 0.3333 & 0.3333 \end{bmatrix} \end{array}. \tag{15}$$

From this matrix, we get matrix $\mathbf{W}$ for the four-tower game by adding rows for zero positions and a column for 4th place probabilities, as shown in (16):

$$\mathbf{W} = \begin{array}{c} \\ 4 \\ 1 \\ 0 \\ 3 \\ 2 \\ 1 \\ 0 \\ 2 \\ 0 \end{array} \begin{array}{c} \begin{array}{cccc} \quad 1 & \quad\quad 2 & \quad\quad 3 & \quad 4 \end{array} \\ \begin{bmatrix} 0.6667 & 0.2910 & 0.0423 & 0 \\ 0.1667 & 0.3545 & 0.4788 & 0 \\ 0 & 0 & 0 & 1 \\ 0.5000 & 0.3731 & 0.1269 & 0 \\ 0.3333 & 0.4078 & 0.2589 & 0 \\ 0.1667 & 0.2191 & 0.6142 & 0 \\ 0 & 0 & 0 & 1 \\ 0.3333 & 0.3333 & 0.3333 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}. \tag{16}$$

Finally, using **B** in (10) and **W** in (16), we get matrix **BW** shown below, whose entries are the placing probabilities for each nonterminal chip position in the four-tower game:

$$\mathbf{BW} = \begin{array}{c} \\ 3 \\ 1 \\ 2 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} \quad 1 & \quad\quad 2 & \quad\quad 3 & \quad\quad 4 \end{array} \\ \begin{bmatrix} 0.5000 & 0.3382 & 0.1336 & 0.0282 \\ 0.1667 & 0.2206 & 0.2888 & 0.3239 \\ 0.3333 & 0.3156 & 0.2384 & 0.1127 \\ 0.1667 & 0.1844 & 0.2616 & 0.3873 \end{bmatrix} \end{array}. \tag{17}$$

**Example 2.** In this example, we compare the results obtained for the placing probabilities for some states of the form $(4k, 3k, 2k, k)$ using the multigraph model and ICM. Table 1 shows these placing probabilities. First place probabilities are the same for both models since these depend only on the proportion of the chip stacks. We observe that the solutions obtained from the multigraph model depend on the actual number of chips of each player. Note that placing probabilities using the ICM depend only on the proportion of the chip stacks, which means that all states of the form $(4k, 3k, 2k, k)$ where $k \in \mathbb{N}$ have the same solution using the ICM. These subtle differences in placing probabilities for states having a constant chip ratio would have been difficult or impossible to detect using other models involving simulations.

# 5 Conclusions

In this paper, a method of solving player equities in the Four-Tower Problem was presented. The assumptions for the problem were that betting was even-money with no draw, bet size is fixed at 1 unit, and the participating players for each round were selected randomly following a uniform distribution. The method used recursions with players having integer wealths at the beginning of each round. A multigraph with nodes representing the different states for a fixed total wealth $S$ was constructed, and a linear system was constructed to represent the transition between these states. Solutions of this linear system give the absorption probabilities for each player in all possible states, and placing probabilities can be obtained by applying the algorithm recursively. An advantage of this method is that it gives exact solutions to the problem, as opposed to Bruss's numerical method that only gives approximations of solutions to a related problem [1]. Subtle differences in equities can be observed, with one result showing that different chip states having the same chip ratio result in slightly different player equities. In contrast, methods such as the ICM or models using Brownian motion give equities which are independent of any scaling factor.

The model may be extended to solve a general $N$-tower problem. The model may also be extended to one wherein the bet size may vary depending on the wealth of the

| State | Position | $P(X_i = 1)$ | $P(X_i = 2)$ | $P(X_i = 3)$ | $P(X_i = 4)$ |
|---|---|---|---|---|---|
| $(4, 3, 2, 1)$ | 4 | 0.4000 | 0.3407 | 0.2054 | 0.0539 |
| $(8, 6, 4, 2)$ | 8 | 0.4000 | 0.3406 | 0.2062 | 0.0531 |
| $(12, 9, 6, 3)$ | 12 | 0.4000 | 0.3406 | 0.2064 | 0.0530 |
| $(16, 12, 8, 4)$ | 16 | 0.4000 | 0.3406 | 0.2064 | 0.0530 |
| $(20, 15, 10, 5)$ | 20 | 0.4000 | 0.3406 | 0.2065 | 0.0529 |
| ICM | $4k$ | 0.4000 | 0.3159 | 0.2063 | 0.0778 |
| $(4, 3, 2, 1)$ | 3 | 0.3000 | 0.3160 | 0.2753 | 0.1087 |
| $(8, 6, 4, 2)$ | 6 | 0.3000 | 0.3160 | 0.2763 | 0.1077 |
| $(12, 9, 6, 3)$ | 9 | 0.3000 | 0.3160 | 0.2765 | 0.1076 |
| $(16, 12, 8, 4)$ | 12 | 0.3000 | 0.3160 | 0.2765 | 0.1075 |
| $(20, 15, 10, 5)$ | 15 | 0.3000 | 0.3160 | 0.2766 | 0.1075 |
| ICM | $3k$ | 0.3000 | 0.3083 | 0.2619 | 0.1298 |
| $(4, 3, 2, 1)$ | 2 | 0.2000 | 0.2274 | 0.3353 | 0.2373 |
| $(8, 6, 4, 2)$ | 4 | 0.2000 | 0.2274 | 0.3360 | 0.2366 |
| $(12, 9, 6, 3)$ | 6 | 0.2000 | 0.2274 | 0.3361 | 0.2364 |
| $(16, 12, 8, 4)$ | 8 | 0.2000 | 0.2274 | 0.3362 | 0.2364 |
| $(20, 15, 10, 5)$ | 10 | 0.2000 | 0.2274 | 0.3362 | 0.2364 |
| ICM | $2k$ | 0.2000 | 0.2413 | 0.3175 | 0.2413 |
| $(4, 3, 2, 1)$ | 1 | 0.1000 | 0.1159 | 0.1841 | 0.6000 |
| $(8, 6, 4, 2)$ | 2 | 0.1000 | 0.1160 | 0.1815 | 0.6026 |
| $(12, 9, 6, 3)$ | 3 | 0.1000 | 0.1160 | 0.1810 | 0.6030 |
| $(16, 12, 8, 4)$ | 4 | 0.1000 | 0.1160 | 0.1809 | 0.6031 |
| $(20, 15, 10, 5)$ | 5 | 0.1000 | 0.1160 | 0.1808 | 0.6032 |
| ICM | $1k$ | 0.1000 | 0.1345 | 0.2143 | 0.5512 |

*Table 1:* Solutions for Some States of the Form $(4k, 3k, 2k, k)$

pair of players involved in each round, with the possibility of bet size selection following a distribution other than the uniform distribution. This may be a better approximation of most poker tournaments since bet size between players usually belongs to the lower range of permissible bet sizes.

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