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L-polar decomposition of the Lorentz group $O(1, n)$

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Abstract

Let $n \geq 2$, and let $L = diag(1, -1, I_{n-1})$ be a diagonal matrix. Let ρ be an involution of the Lorentz group $O(1, n)$ defined by $\rho(A) = LAL$, $A \in O(1, n)$. We show $A = e^X k$ where $k \in O(1, n)$ and X is in the Lie algebra of $O(1, n)$ satisfying $\rho(k) = k$ and $LXL = -X$. We briefly apply the above analysis to other involutions of $O(1, n)$.

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1 Introduction

Let $M_n(\mathbb{R})$ denote the algebra of n-by-n real matrices. Let I_n denote the n-by-n identity matrix. Let $GL_n(\mathbb{R})$ be the multiplicative group of nonsingular matrices in $M_n(\mathbb{R})$. Let $J = 1 \oplus (-I_n) \in GL_{n+1}(\mathbb{R})$ be a diagonal matrix. For $A \in M_n(\mathbb{R})$, let A^T denote the transpose of A. We consider the generalized Lorentz group

$$O(1, n) = \{A \in GL_{n+1}(\mathbb{R}) : A^T JA = J\}.$$

For brevity, we say $O(1, n)$ is a Lorentz group. Let $O(n)$ denote the multiplicative group of real orthogonal matrices $B \in GL_n(\mathbb{R})$ satisfying $B^T B = I_n$. We state some well-known facts. The Lie algebra of the Lie group $O(1, n)$ is

$$\mathfrak{so}(1, n) = \{X \in M_{n+1}(\mathbb{R}) : X^T J + JX = 0\}.$$

Let $L = \mathrm{diag}(1, -1, I_{n-1})$ be a diagonal matrix. We study the involution

$$\rho : O(1, n) \to O(1, n) \tag{1}$$

satisfying $\rho(A) = LAL$, $A \in O(1, n)$. We define

$$\widehat{A} = \rho(A)^{-1}.$$

The differential of ρ at the identity I_{n+1} is a Lie algebra involution

$$d\rho : \mathfrak{so}(1, n) \to \mathfrak{so}(1, n)$$

satisfying $d\rho(X) = LXL$, $X \in \mathfrak{so}(1, n)$. The eigenspaces of $d\rho$ are given by

1. $\mathfrak{p} = \{X \in \mathfrak{so}(1, n) : d\rho(X) = -X\}$

2. $\mathfrak{K} = \{X \in \mathfrak{so}(1,n) : d\rho(X) = X\}$.

We know $\mathfrak{so}(1,n) = \mathfrak{K} \oplus \mathfrak{p}$ is a direct sum of eigenspaces. Consider the subgroup K of $O(1,n)$ that consists of the fixed points of ρ, namely,

$$K = \{A \in O(1,n) : \rho(A) = A\}. \tag{2}$$

Notice, $\mathfrak{K}$ is the Lie algebra of the closed subgroup K of $O(1,n)$. Likewise, $A \in K$ if and only if $A\widehat{A} = I_{n+1}$. We show that if $A \in O(1,n)$, then there exist $X \in \mathfrak{p}$ and $k \in K$ satisfying $A = e^X k$. We say $A = e^X k$ is a L-polar decomposition of A.

The strategy of the proof of the L-polar decomposition can be described briefly. First, we evaluate the matrix exponential e^X of the matrices $X \in \mathfrak{p}$, see Theorem 2.1. Secondly, using Lemma 2.1, we determine the entries of matrix $A\widehat{A}$ as seen in (9). In Section 3, given $A \in O(1,n)$, we determine $X \in \mathfrak{p}$ explicitly such that $e^X = A\widehat{A}$.

There are related works for this paper, and we list some references. The classic polar decomposition for $M_n(\mathbb{R})$ is discussed in [5]. The Cartan involution and polar decomposition of semi-simple Lie groups are discussed in [4]. A generalized polar decomposition for complex matrices is obtained in [3]. The local existence and local uniqueness of generalized polar decompositions on any Lie group with a differentiable involution is proved in [7, Theorem 3.1]. The latter reference gives context to the L-polar decomposition of this paper. Furthermore, generalized polar decomposition of matrix groups defined by indefinite inner-products are studied in [1]. In relation to $O(1,n)$, the exponential mapping of the Lie group $O(m,n)$ is analyzed in [8]. For applications of the Lorentz group to hyperbolic geometry and Möbius transformations, see [2].

2 Preliminaries

By definition, $X \in \mathfrak{p}$ if and only if $LXL = -X$ and $X^T J + JX = 0$. Then the matrices $X \in \mathfrak{p}$ are exactly of the form

$$X = \left[\begin{array}{c|cccc} 0 & x_1 & 0 & \cdots & 0 \\ \hline x_1 & 0 & x_2 & \cdots & x_n \\ 0 & -x_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -x_n & 0 & \cdots & 0 \end{array}\right] \in \mathfrak{p}. \tag{3}$$

where $x_1, \ldots, x_n \in \mathbb{R}$.

Theorem 2.1. *Let $X \in \mathfrak{p}$ be given as in (3). Let $\alpha = x_1^2 - \sum_{k=2}^n x_k^2$. Then*

(a) $e^X = I_{n+1} + \frac{\sinh(\sqrt{\alpha})}{\sqrt{\alpha}} X + \frac{\cosh(\sqrt{\alpha})-1}{\alpha} X^2, \quad$ *if* $\alpha \neq 0$

(b) $e^X = I_{n+1} + X + \frac{1}{2}X^2,$ *if* $\alpha = 0$.

Proof. We observe

$$X^2 = \left[\begin{array}{c|cccc} x_1^2 & 0 & x_1x_2 & \cdots & x_1x_n \\ \hline 0 & x_1^2 - \sum_{k=2}^n x_k^2 & 0 & \cdots & 0 \\ -x_1x_2 & 0 & -x_2^2 & \cdots & -x_2x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_1x_n & 0 & -x_2x_n & \cdots & -x_n^2 \end{array}\right]. \tag{4}$$

Furthermore, $X^3 = \alpha X$. Inductively, if $k \geq 1$, we find $X^{2k+1} = \alpha^k X$ and $X^{2k} = \alpha^{k-1} X^2$. Let $s_X = \sum_{k=0}^{\infty} \frac{\alpha^k}{(2k+1)!}$ and let $c_X = \sum_{k=1}^{\infty} \frac{\alpha^{k-1}}{(2k)!}$. If $\alpha \neq 0$, then

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!} = I + s_X X + c_X X^2.$$

If $\alpha = 0$, then $e^X = I + X + \frac{1}{2}X^2$. Applying hyperbolic trigonometric functions, if $\alpha \neq 0$, we obtain $s_X = \frac{\sinh(\sqrt{\alpha})}{\sqrt{\alpha}}$ and $c_X = \frac{\cosh(\sqrt{\alpha})-1}{\alpha}$. This proves the theorem. □

In Section 3, we will find useful the following identities.

1. $\lim_{\alpha \to 0} \dfrac{\sinh(\sqrt{\alpha})}{\sqrt{\alpha}} = 1$
2. $\lim_{\alpha \to 0} \dfrac{\cosh(\sqrt{\alpha}) - 1}{\alpha} = \dfrac{1}{2}$

We describe additional properties of matrices $A \in O(1,n)$. Let $a_0 \in \mathbb{R}$, $b, c \in \mathbb{R}^n$, and let $A_0 \in M_n(\mathbb{R})$ satisfy

$$A = \begin{bmatrix} a_0 & b^T \\ c & A_0 \end{bmatrix}. \tag{5}$$

The identity $A^T J A = J$ is equivalent to

$$\begin{bmatrix} a_0^2 - c^T c & a_0 b^T - c^T A_0 \\ a_0 b - A_0^T c & bb^T - A_0^T A_0 \end{bmatrix} = J. \tag{6}$$

Also, we know

$$A^{-1} = JA^T J = \begin{bmatrix} a_0 & -c^T \\ -b & A_0^T \end{bmatrix}.$$

Since $J \in O(1,n)$, $A^T \in O(1,n)$. Applying (6), we obtain $A_0^T A_0 = bb^T + I_n$, $A_0 A_0^T = cc^T + I_n$, $A_0^T c = a_0 b$, $A_0 b = a_0 c$, $b^T b = c^T c$, $a_0^2 - c^T c = 1$, and $|a_0| \geq 1$. Let $J_0 = I_2 \oplus (-I_{n-1})$. Then $LJ = J_0 = JL$. Since $A\widehat{A} = A\rho(A^{-1}) = AL(JA^T J)L$, we find

$$A\widehat{A} \quad = \quad AJ_0 A^T J_0. \tag{7}$$

Let $J_n = 1 \oplus (-I_{n-1})$. Then $AJ_0 = \begin{bmatrix} a_0 & b^T J_n \\ c & A_0 J_n \end{bmatrix}$ and $A^T J_0 = \begin{bmatrix} a_0 & c^T J_n \\ b & A_0^T J_n \end{bmatrix}$. Let $b = (b_1, b_2, \ldots, b_n)^T$, and let $e_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^n$. Thus,

$$A\widehat{A} \quad = \quad \begin{bmatrix} 1 + 2b_1^2 & 2b_1 (J_n A_0 e_1)^T \\ 2b_1 A_0 e_1 & (A_0 A_0^T - I_n) J_n + A_0 J_n A_0^T J_n \end{bmatrix}. \tag{8}$$

Let $M = (A_0 A_0^T - I_n) J_n$, and let $D = M + A_0 J_n A_0^T J_n$. For $1 \leq i, j \leq n$, let a_{ij}, m_{ij}, and d_{ij} denote the (i,j)-entries of A_0, M, and D, respectively. We find

$$m_{pq} = \delta_{1q} \left(\left(\sum_{k=1}^{n} a_{pk} a_{1k} \right) - \delta_{p1} \right) + \delta_{pq} - \sum_{k=1}^{n} a_{pk} a_{qk}$$

and

$$J_n A_0^T J_n = \begin{bmatrix} a_{11} & -a_{21} & \cdots & -a_{n1} \\ -a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}.$$

Lemma 2.1. *Let $1 \le p, q \le n$, and let $D = M + A_0 J_n A_0^T J_n$. The entries of D satisfy the following statements.*

1. $d_{11} = 2a_{11}^2 - 1$
2. *If* $2 \le i \le n$, *then* $d_{ii} = 1 - 2a_{i1}^2$
3. *If* $q > p$, *then* $d_{pq} = -2a_{p1}a_{q1}$
4. *If* $p > 1$, *then* $d_{p1} = 2a_{11}a_{p1}$
5. *If* $p > q \neq 1$, *then* $d_{pq} = -2a_{p1}a_{q1}$.

Proof. We prove statement 5. Let $p > q \neq 1$. Note, the (p,q)-entry of $A_0 J_n A_0^T J_n$ is $a_{p1}(-a_{q1}) + \sum_{k=2}^{n} a_{pk}a_{qk}$. Then

$$\begin{aligned} d_{pq} &= m_{pq} + a_{p1}(-a_{q1}) + \sum_{k=2}^{n} a_{pk}a_{qk} \\ &= -\sum_{k=1}^{n} a_{pk}a_{qk} + a_{p1}(-a_{q1}) + \sum_{k=2}^{n} a_{pk}a_{qk} \\ &= -2a_{p1}a_{q1}. \end{aligned}$$

The other statements are proved similarly. $\square$

Applying Lemma 2.1, we can rewrite (8) as follows.

$$A\widehat{A} = \left[\begin{array}{c|ccccc} 1+2b_1^2 & 2b_1a_{11} & -2b_1a_{21} & -2b_1a_{31} & \cdots & -2b_1a_{n1} \\ \hline 2b_1a_{11} & 2a_{11}^2-1 & -2a_{11}a_{21} & -2a_{11}a_{31} & \cdots & -2a_{11}a_{n1} \\ 2b_1a_{21} & 2a_{11}a_{21} & 1-2a_{21}^2 & -2a_{21}a_{31} & \cdots & -2a_{21}a_{n1} \\ 2b_1a_{31} & 2a_{11}a_{31} & -2a_{31}a_{21} & 1-2a_{31}^2 & \cdots & -2a_{31}a_{n1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2b_1a_{n1} & 2a_{11}a_{n1} & -2a_{n1}a_{21} & -2a_{n1}a_{31} & \cdots & 1-2a_{n1}^2 \end{array}\right]. \tag{9}$$

Notice, $A\widehat{A} \in O(n+1)$ is an orthogonal matrix if and only if $b_1 = 0$.

3 Main Result

Let $A \in O(1,n)$ be given as in (5). In this section, we adopt the same notations $a_0 \in \mathbb{R}$, $b, c \in \mathbb{R}^n$, and $A_0 \in GL(n, \mathbb{R})$ that are associated with A. Recall $b = (b_1, b_2, \ldots, b_n)^T$. We construct an $X \in \mathfrak{p}$ such that $e^X = A\widehat{A}$. The cases $b_1 \neq 0$ and $b_1 = 0$ will be treated separately. Note, there is a unique $\beta \ge -\pi^2$ satisfying

$$\cosh(\sqrt{\beta}) = 2a_{11}^2 - 1. \tag{10}$$

If $-\pi^2 \le \beta < 0$, then $\cosh(\sqrt{\beta}) = \cos(\sqrt{|\beta|})$. If $\beta = 0$, we set $\frac{\cosh(\sqrt{\beta})-1}{\beta} = \frac{1}{2}$ since we have a removable discontinuity, see statement 2 after Theorem 2.1. Then $\frac{\cosh(\sqrt{\beta})-1}{\beta} > 0$ for all $\beta \ge -\pi^2$. For each $i \in \{1, \ldots, n\}$, choose $x_i \in \mathbb{R}$ such that

1. $\frac{\cosh(\sqrt{\beta})-1}{\beta} x_1^2 = 2b_1^2$

2. $\frac{\cosh(\sqrt{\beta})-1}{\beta}x_i^2 = 2a_{i1}^2, i \geq 2$

Then
$$\frac{\cosh(\sqrt{\beta})-1}{\beta}\left(x_1^2 - \sum_{i=2}^{n} x_i^2\right) = 2\left(b_1^2 - \sum_{i=2}^{n} a_{i1}^2\right).$$
Since $A \in O(1,n)$, the entries in the second column of A satisfy $b_1^2 - \sum_{i=1}^{n} a_{i1}^2 = -1$. Thus, we find
$$\begin{aligned}\frac{\cosh(\sqrt{\beta})-1}{\beta}\left(x_1^2 - \sum_{i=2}^{n} x_i^2\right) &= 2\left(a_{11}^2 - 1\right)\\ &= \cosh(\sqrt{\beta}) - 1.\end{aligned}$$
Then $x_1^2 - \sum_{i=2}^{n} x_i^2 = \beta$. Now, construct the matrix X in (3) with the chosen values x_i satisfying statements 1 and 2. We apply Theorem 2.1 and its notation α, and we find $\alpha = \beta$.

Lemma 3.1. *The corresponding diagonal entries in (9) and e^X are equal.*

Proof. We apply (10) and the recent statements 1 and 2. From Theorem 2.1 , we see the $(1,1)$-entry of e^X is $1 + \frac{\cosh(\sqrt{\alpha})-1}{\alpha}x_1^2$. Since $\alpha = \beta$, the latter is equivalent to $1 + 2b_1^2$. Then the $(1,1)$-entries of e^X and (9) are equal.

Next, the $(2,2)$-entry of e^X is $1+\frac{\cosh(\sqrt{\alpha})-1}{\alpha}(x_1^2 - \sum_{k=2}^{n} x_k^2)$. Since $\alpha = \beta$, the $(2,2)$-entry is equal to $\cosh(\sqrt{\beta})$, or equivalently $2a_{11}^2 - 1$. Then the $(2,2)$-entries of e^X and (9) are equal.

Let $i \in \{2, \ldots, n\}$. The (i,i)-entry of e^X is $1 - \frac{\cosh(\sqrt{\beta})-1}{\beta}x_i^2$. By statement 2, the latter expression is equivalent to $1 - 2a_{i1}^2$. Thus, the (i,i)-entries of e^X and (9) are equal. □

Lemma 3.2. *Let $i \in \{2, \ldots, n\}$. Then*

(a) $2b_1a_{11} = \frac{\sinh(\sqrt{\alpha})}{\sqrt{\alpha}}x_1$ *or* $2b_1a_{11} = \frac{\sinh(\sqrt{\alpha})}{\sqrt{\alpha}}(-x_1)$

(b) $-2b_1a_{i1} = \frac{\cosh(\sqrt{\alpha})-1}{\alpha}x_1x_i$ *or* $-2b_1a_{i1} = \frac{\cosh(\sqrt{\alpha})-1}{\alpha}x_1(-x_i)$

Proof. To prove statement (a), it is sufficient to show that the square of the left side is equal to the square of the right side of either equation. We show $4b_1^2a_{11}^2 = \frac{\sinh^2(\sqrt{\alpha})}{\alpha}x_1^2$. Equivalently, we show
$$4b_1^2\Big(\frac{\cosh(\sqrt{\alpha})+1}{2}\Big) = \frac{\cosh^2(\sqrt{\alpha})-1}{\alpha}\Big(2b_1^2\Big(\frac{\cosh(\sqrt{\alpha})-1}{\alpha}\Big)^{-1}\Big).$$
The above statement is an identity since $\alpha = \beta \geq -\pi^2$. This proves (a).

To prove (b), we show $4b_1^2a_{i1}^2 = \frac{(\cosh(\sqrt{\alpha})-1)^2}{\alpha^2}x_1^2x_i^2$. Equivalently, we show
$$4b_1^2a_{i1}^2 = \frac{(\cosh(\sqrt{\alpha})-1)^2}{\alpha^2}\Big(2b_1^2\Big(\frac{\cosh(\sqrt{\alpha})-1}{\alpha}\Big)^{-1}\Big)\Big(2a_{i1}^2\Big(\frac{\cosh(\sqrt{\alpha})-1}{\alpha}\Big)^{-1}\Big).$$
The above equation is an identity. This proves (b). □

In statements 1 and 2 after (10), for each $i \geq 1$, we may replace x_i by $-x_i$.

Corollary 3.1. *We can assume the chosen values $x_i \in \mathbb{R}$, $i \geq 1$, in statements 1 and 2 after (10) further satisfy*

1. $2b_1a_{11} = \frac{\sinh(\sqrt{\alpha})}{\sqrt{\alpha}}x_1$

2. $-2b_1a_{i1} = \frac{\cosh(\sqrt{\alpha})-1}{\alpha}x_1x_i$, *if* $i \geq 2$.

Consequently, the first rows of (9) and e^X *are equal. The first columns of (9) and* e^X *are equal. Furthermore, the diagonal entries of (9) and* e^X *are equal, respectively.*

Lemma 3.3. *Let* $b_1 \neq 0$, *and* $i \geq 2$. *Then* $-2a_{11}a_{i1} = \frac{\sinh(\sqrt{\alpha})}{\sqrt{\alpha}}x_i$. *In particular, the second rows of (9) and* e^X *are equal. Also, the second columns of (9) and* e^X *are equal.*

Proof. From statement 1 after (10), since $b_1 \neq 0$ we find $x_1 \neq 0$. From statement 1 before this lemma, we find $a_{11} \neq 0$ iff $\frac{\sinh(\sqrt{\alpha})}{\sqrt{\alpha}} \neq 0$. We may assume $a_{11} \neq 0$ otherwise the lemma is trivially true. Then

$$\begin{aligned} x_i &= -2b_1a_{i1}\Big(\frac{\cosh(\sqrt{\alpha})-1}{\alpha}x_1\Big)^{-1} \\ &= -2b_1a_{i1}\Big(\frac{\cosh(\sqrt{\alpha})-1}{\alpha}\Big)^{-1}(2b_1a_{11})^{-1}\frac{\sinh(\sqrt{\alpha})}{\sqrt{\alpha}} \\ &= -a_{i1}a_{11}^{-1}\Big(\frac{\cosh(\sqrt{\alpha})-1}{\alpha}\Big)^{-1}\frac{\sinh(\sqrt{\alpha})}{\sqrt{\alpha}}. \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{\sinh(\sqrt{\alpha})}{\sqrt{\alpha}}x_i &= -a_{i1}a_{11}^{-1}\Big(\frac{\cosh(\sqrt{\alpha})-1}{\alpha}\Big)^{-1}\frac{\cosh^2(\sqrt{\alpha})-1}{\alpha} \\ &= -2a_{i1}a_{11}^{-1}\Big(\frac{\cosh(\sqrt{\alpha})+1}{2}\Big). \end{aligned}$$

However, $a_{11}^2 = \frac{\cosh(\sqrt{\alpha})+1}{2}$ by (10). Hence, $\frac{\sinh(\sqrt{\alpha})}{\sqrt{\alpha}}x_i = -2a_{11}a_{i1}$. □

Lemma 3.4. *Let* $b_1 \neq 0$, *and let* $j > i \neq 1$. *Then* $2a_{i1}a_{j1} = \frac{\cosh(\sqrt{\alpha})-1}{\alpha}x_ix_j$. *In particular, the corresponding entries of (9) and* e^X *that do not lie in either the first two rows or first two columns are equal.*

Proof. We apply statement 2 before Lemma 3.3, and statement 1 after (10). Then

$$\begin{aligned} x_ix_j &= 4b_1^2a_{i1}a_{j1}x_1^{-2}Big(\frac{\cosh(\sqrt{\alpha})-1}{\alpha}\Big)^{-2} \\ &= 4b_1^2a_{i1}a_{j1}(2b_1^2)^{-1}\Big(\frac{\cosh(\sqrt{\alpha})-1}{\alpha}\Big)^{-1} \\ &= 2a_{i1}a_{j1}\Big(\frac{\cosh(\sqrt{\alpha})-1}{\alpha}\Big)^{-1}. \end{aligned}$$

This proves the lemma. □

Now, we combine Corollary 3.1, Lemma 3.3, and Lemma 3.4. We have the following result.

Corollary 3.2. *If* $A \in O(1,n)$ *and* $b_1 \neq 0$, *then there exists* $X \in \mathfrak{p}$ *such that* $A\widehat{A} = e^X$.

Next, suppose $A \in O(1,n)$ and $b_1 = 0$. Choose β as in (10), and let $x_1, \ldots, x_n \in \mathbb{R}$ satisfy statements 1 and 2 after (10). Then $x_1 = 0$, see statement 1 after (10). Applying Lemma 3.1, we see that the corresponding diagonal entries of (9) and e^X are equal. Applying Theorem 2.1, we find that all entries in the first row and first column of (9) and e^X are all zero except for the $(1,1)$-entry.

Lemma 3.5. *Let $b_1 = 0$. For each $i \geq 2$, then either $-2a_{11}a_{i1} = \frac{\sinh(\sqrt{\alpha})}{\sqrt{\alpha}} x_i$ or $-2a_{11}a_{i1} = \frac{\sinh(\sqrt{\alpha})}{\sqrt{\alpha}}(-x_i)$*

Proof. It is sufficient to show that the square of the left side is equal to the square of the right side of either equation. We apply statement 2 after (10). We find

$$\begin{aligned}\frac{\sinh^2(\sqrt{\alpha})}{\alpha} x_i^2 &= 2\frac{\cosh(\sqrt{\alpha})+1}{2}\frac{\cosh(\sqrt{\alpha})-1}{\alpha}(2a_{i1}^2)\Big(\frac{\cosh(\sqrt{\alpha})-1}{\alpha}\Big)^{-1} \\ &= 4a_{i1}^2\frac{\cosh(\sqrt{\alpha})+1}{2}.\end{aligned}$$

Applying identity (10), we obtain $\frac{\sinh^2(\sqrt{\alpha})}{\alpha} x_i^2 = 4a_{i1}^2 a_{11}^2$. This proves the lemma. □

Notice, for any $i \geq 2$, if we replace x_i by $-x_i$, the corresponding diagonal entries of (9) and e^X are still equal. Furthermore, the entries in the first row and the first column of (9) and e^X are still zero except for the $(1,1)$-entry.

Corollary 3.3. *Let $b_1 = 0$, and let $i \geq 2$. We can assume the chosen values $x_i \in \mathbb{R}$ in statements 1 and 2 after (10) satisfy $-2a_{11}a_{i1} = \frac{\sinh(\sqrt{\alpha})}{\sqrt{\alpha}} x_i$. In particular, the second rows of (9) and e^X are equal. Also, the second columns of (9) and e^X are equal.*

Lemma 3.6. *Let $b_1 = 0$, and let $j > i \neq 1$, then $2a_{i1}a_{j1} = \frac{\cosh(\sqrt{\alpha})-1}{\alpha} x_i x_j$. In particular, the corresponding entries of (9) and e^X that do not lie in either the first two rows or first two columns are equal.*

Proof. From (10), we find $a_{11} = 0$ if and only if $\beta = -\pi^2$. Recall, $\alpha = \beta$. Suppose $\alpha \neq -\pi^2$. We apply Corollary (3.3). Then

$$\frac{\cosh(\sqrt{\alpha})-1}{\alpha} x_i x_j = \frac{\cosh(\sqrt{\alpha})-1}{\alpha}(4a_{11}^2)a_{i1}a_{j1}\Big(\frac{\sinh(\sqrt{\alpha})}{\sqrt{\alpha}}\Big)^{-2}.$$

Applying identity (10), the above right side reduces to

$$4\frac{\cosh(\sqrt{\alpha})-1}{\alpha}\Big(\frac{\cosh(\sqrt{\alpha})+1}{2}\Big)a_{i1}a_{j1}\Big(\frac{\cosh^2(\sqrt{\alpha})-1}{\alpha}\Big)^{-1} = 2a_{i1}a_{j1}.$$

Suppose $\alpha = -\pi^2$. Then $a_{11} = 0 = x_1$, see (10) and the statement 1 that follows. Also, $\frac{\sinh(\sqrt{\alpha})}{\sqrt{\alpha}} = 0$. We apply Corollary 3.3. Notice, the entries in the second row and the second column of (9) and e^X are all zero except for the $(2,2)$-entry. Then for any $i \geq 2$, we may replace x_i by $-x_i$, and the diagonal entries of (9) and e^X remain equal, respectively, and the entries in the first and second rows and columns of (9) and e^X are still zero except for the $(1,1)$- and $(2,2)$-entries.

Now, for all $i \geq 2$, we choose the sign of x_i to be the same as the sign of a_{i1}, and where

$$\sqrt{\frac{\cosh(\sqrt{\alpha})-1}{\alpha}}\, x_i = \sqrt{2}a_{i1}.$$

Notice, each x_i satisfies statement 2 after (10). Clearly, $2a_{i1}a_{j1} = \frac{\cosh(\sqrt{\alpha})-1}{\alpha} x_i x_j$. $\square$

Now, we combine the above discussion after Corollary 3.2.

Corollary 3.4. *If $A \in O(1,n)$ and $b_1 = 0$, then there exists $X \in \mathfrak{p}$ such that $A\widehat{A} = e^X$.*

Corollary 3.5. *Let $\mathcal{S} = \{A\widehat{A} : A \in O(1,n)\}$, and let $\mathcal{T} = \{e^X : X \in \mathfrak{p}\}$. Then $\mathcal{S} = \mathcal{T}$.*

Proof. Applying corollaries 3.2 and 3.4, we find $\mathcal{S} \subseteq \mathcal{T}$. Conversely, let $X \in \mathfrak{p}$. Since $X/2 \in \mathfrak{p}$, we find $\widehat{e^{X/2}} = e^{X/2}$. Clearly, $e^X = e^{X/2}\widehat{e^{X/2}} \in \mathcal{S}$. Then $\mathcal{T} \subseteq \mathcal{S}$. $\square$

The following is the main result.

Theorem 3.1. *Let $\phi : \mathfrak{p} \times K \to O(1,n)$ be defined by $\phi(X,k) = e^X k$, $X \in \mathfrak{p}$, $k \in K$. Then ϕ is surjective. However, ϕ is not injective.*

Proof. Let $A \in O(1,n)$. We apply Corollary 3.5, and recall that $\mathfrak{p}$ is a real subspace of $\mathfrak{so}(1,n)$. Then $A\widehat{A} = e^{2X}$ for some $X \in \mathfrak{p}$. Note, $\widehat{e^{-X}} = e^{-X}$. Let $A = e^X k$ where $k \in O(1,n)$. Then $k\widehat{k} = e^{-X}A\widehat{A}\widehat{e^{-X}} = I$. Thus, $k \in K$ and $\phi(X,k) = A$. Hence, ϕ is surjective.

To see that ϕ is not injective. As in Theorem 2.1, choose $X \in \mathfrak{p}$ such that $\alpha = -4l^2\pi^2$ where $l \in \mathbb{Z}\setminus\{0\}$ is a nonnegative integer. For example, let $x_1 = 0$, $x_2 = 2l\pi$, and let $x_i = 0$ for $i \geq 3$. Then $\phi(X, I_{n+1}) = e^X = I_{n+1}$. Thus, ϕ is not injective. $\square$

We state a special case of Corollary 3.2.

Corollary 3.6. *Let $A \in O(1,n)$. If $A\widehat{A}$ is positive-definite, then there exists $x_1 \in \mathbb{R}$ such that*

$$A\widehat{A} = \begin{bmatrix} \cosh(x_1) & \sinh(x_1) \\ \sinh(x_1) & \cosh(x_1) \end{bmatrix} \oplus I_{n-1}.$$

Proof. If $A\widehat{A}$ is a positive-definite orthogonal matrix, then $A\widehat{A} = I_{n+1}$ and we choose $x_1 = 0$. Suppose $A\widehat{A}$ is positive-definite but not an orthogonal matrix. Applying the remark after (9), we find $b_1 \neq 0$. Then $x_1 \neq 0$. Since $A\widehat{A}$ is symmetric, the first row and first column should have the same entries. Applying (8), we find $A_0 e_1 = (a_{11}, 0, \ldots, 0)^T$, i.e., $a_{i1} = 0$ for $i \geq 2$. Applying statement 2 after (10), we find $x_i = 0$ for $i \geq 2$. Then $\alpha = x_1^2 > 0$. Notice, $\alpha \neq -\pi^2$ and $\sinh(\sqrt{\alpha}) > 0$. Let x_1 be defined by statement 1 in Corollary 3.1. Then the matrix X in (3) has the form $X = \begin{bmatrix} 0 & x_1 \\ x_1 & 0 \end{bmatrix} \oplus 0_{n-1}$ where $0_{n-1} \in M_{n-1}(\mathbb{R})$ is a zero matrix. The theorem follows from Corollary 3.2. $\square$

Let $A \in O(1,n)$ be given as in (5). Recall, a_{ij} is the (i,j)-entry of A_0. In Theorem 3.1, $A = e^X k$ is the L-polar decomposition of A where $X \in \mathfrak{p}$ and $k \in K$. If $a_{11}^2 > 1$, we claim A has a unique L-polar decomposition. Suppose $A = e^{X_1}k_1 = e^{X_2}k_2$ where $X_1, X_2 \in \mathfrak{p}$ and $k_1, k_2 \in K$. Then $A\widehat{A} = e^{2X_1} = e^{2X_2}$. For each $i \in \{1,2\}$, choose $\alpha_i \in \mathbb{R}$ as in Theorem 2.1 satisfying

$$e^{2X_i} = I_{n+1} + \frac{\sinh(\sqrt{\alpha_i})}{\sqrt{\alpha_i}}(2X_i) + \frac{\cosh(\sqrt{\alpha_i}) - 1}{\alpha_i}(4X_i^2).$$

Recall, if $\alpha_i = 0$, we set $\frac{\sinh(\sqrt{\alpha_i})}{\sqrt{\alpha_i}} = 1$ and $\frac{\cosh(\sqrt{\alpha_i})-1}{\alpha_i} = \frac{1}{2}$. In particular, the $(2,2)$-entries of $A\widehat{A}$ and e^{2X_i} are equal. Using the (2,2)-entry of $A\widehat{A}$ in (9), we obtain $\frac{\cosh(\sqrt{\alpha_i})-1}{\alpha_i}\alpha_i + 1 = 2a_{11}^2 - 1$. Equivalently, we obtain $\cosh(\sqrt{\alpha_i}) = 2a_{11}^2 - 1$. If $a_{11}^2 > 1$, we find $\cosh(\sqrt{\alpha_i}) > 1$. In particular, $\alpha_i > 0$. Since $\cosh(\sqrt{\alpha_1}) = \cosh(\sqrt{\alpha_2})$, we find $\alpha_1 = \alpha_2$. Since $\frac{\sinh(\sqrt{\alpha_i})}{\sqrt{\alpha_i}} > 0$ and $e^{2X_1} = e^{2X_2}$, we find $X_1 = X_2$. Consequently, $k_1 = k_2$. Hence, the L-polar decomposition of A is unique if $a_{11}^2 > 1$.

4 Some related involutions

We discuss an extension of Theorem 3.1. Recall, $J = 1 \oplus (-I_n)$. Let $w \in \mathbb{R}^{n+1}$ satisfy $w^T Jw \neq 0$. Let $S_w = I_{n+1} - 2(w^T Jw)^{-1} ww^T J$. The matrix S_w is a J-Householder matrix, see [6]. Then $S_w^2 = I_{n+1}$ and $S_{tw} = S_w$ for all nonzero $t \in \mathbb{R} \setminus \{0\}$. Also, $S_w \in O(1,n)$. Suppose $v \in \mathbb{R}^{n+1}$ and $v^T Jv = w^T Jw \neq 0$. Then either $(v+w)^T J(v+w) \neq 0$ or $(v-w)^T J(v-w) \neq 0$. If $(v+w)^T J(v+w) \neq 0$, then $S_{v+w}(v) = -w$. Similarly, if $(v-w)^T J(v-w) \neq 0$, then $S_{v-w}(v) = w$. In any case, there is a J-Householder matrix D satisfying $Dv = -w$ or $Dv = w$. The construction of D is known, see Lemma 12 in [6], but we provided some details for the sake of completeness. Then $DS_v D^{-1} = S_{Dv} = S_w$.

We consider special cases for w and v. Recall, $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$ and $L = diag(1, -1, I_{n-1}) \in O(1,n)$. If $w = 0 \oplus e_1 \in \mathbb{R}^{n+1}$, then $w^T Jw = -1$ and $S_w = L$. Let $v \in \mathbb{R}^{n+1}$ satisfy $v^T Jv = -1$ Define an involution $\rho_v : O(1,n) \to O(1,n)$ by letting $\rho_v(A) = S_v A S_v$, $A \in O(1,n)$. Let $d\rho_v$ be the differential ρ_v at I_n. Then $d\rho_v$ is an involution of the Lie algebra $\mathfrak{so}(1,n)$ where $d\rho_v(X) = S_v X S_v$, $X \in \mathfrak{so}(1,n)$. Consider the eigenspaces of $d\rho_v$, namely, $\mathfrak{p}_v = \{X \in \mathfrak{so}(1,n) : d\rho_v(X) = -X\}$ and $\mathfrak{K}_v = \{X \in \mathfrak{so}(1,n) : d\rho_v(X) = X\}$. Let K_v be the closed subgroup of $O(1,n)$ consisting of the fixed points of ρ_v.

Let $\xi : O(1,n) \to O(1,n)$ be an inner-automorphism satisfying $\xi(A) = DAD$. Since $DS_w = S_v D$, we find $\xi \circ \rho = \rho_v \circ \xi$. Now, we apply Theorem 3.1. Then $\xi^{-1}(A) = e^X k$ where $\rho(k) = k$ and $d\rho(X) = -X$. Thus, $A = e^{d\xi(X)} \xi(k)$, $\rho_v(\xi(k)) = \xi(k)$ and $d\rho_v(d\xi(X)) = -d\xi(X)$. Now, we have the following result.

Theorem 4.1. *Let $v \in \mathbb{R}^{n+1}$ satisfy $v^T Jv = -1$. Let $S_v = I_{n+1} + 2vv^T J$ be the corresponding J-Householder matrix. If $A \in O(1,n)$, then there exist $Y \in \mathfrak{so}(1,n)$ and $m \in O(1,n)$ satisfying $A = e^Y m$, $S_v m S_v = m$, and $S_v Y S_v = -Y$. The factorization $A = e^Y m$ is not unique.*

Lastly, consider the case when $v^T Jv = 1$ and its corresponding involution ρ_v. Similarly, by constructing an inner-automorphism ξ as defined before Theorem 16, we may assume $v = (1, 0, \ldots, 0)^T \in \mathbb{R}^{n+1}$. Then the associated J-Householder matrix satisfies $S_v = I_{n+1} - 2vv^T J = -J$. The corresponding involution ρ_v of $O(1,n)$ satisfies $\rho_v(A) = S_v A S_v = JAJ$, $A \in O(1,n)$. Since $A^T JA = J$, we find $\rho_v(A) = A^{-T}$ and $d\rho_v(X) = -X^T$, $X \in \mathfrak{so}(1,n)$. Now, $d\rho_v$ is a well-known Cartan involution of $\mathfrak{so}(1,n)$, see [4]. Furthermore, it is known that each $A \in O(1,n)$ may be expressed uniquely as $A = e^Y m$ for some symmetric matrix Y in $\mathfrak{so}(1,n)$, and an orthogonal matrix m in $O(1,n)$. Summarizing, the J-Householder matrices S_v where $v^T Jv = -1$ and the possible non-uniqueness of the factorization $A = e^Y m$ is treated in Theorem 16. While the J-Householder matrices S_v where $v^T Jv = 1$, and the uniqueness of the factorization $A = e^Y m$ is discussed after Theorem 16.

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5 References

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