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Colour groups associated with S-orbit colourings of the square and the hexagonal lattices

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Abstract

In this paper, we present a list of colour groups associated with S-orbit colourings of the the square and the hexagonal lattices. Colouring the lattices are done based on the algorithm by Felix and Walo in *Colourings of lattices based on subgroup orbits*. Some results are obtained to determine the generators of the colour groups.

Keywords: square lattice, hexagonal lattice, sublattice, coset, orbit, colour group
2010 MSC: 52C05

1 Introduction

To begin, a(n) (uncoloured) lattice Λ in $\mathbb{R}^2$ is a $\mathbb{Z}$-module of rank 2 and dimension 2. For computational convenience, Λ will be spanned by the two linearly independent vectors $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ over $\mathbb{Z}$ and thus be represented as the 2×2 matrix $\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$. As a group, Λ is isomorphic to $\mathbb{Z}\times\mathbb{Z}$, the free abelian group of rank 2. A subset L of Λ is called a *sublattice* if it is a $\mathbb{Z}$-module of rank 2 and dimension 2 as well. Similarly, L is spanned by the two linearly independent vectors $\left[\begin{smallmatrix}i\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}j\\k\end{smallmatrix}\right]$ over $\mathbb{Z}$ and is represented as $L=\left[\begin{smallmatrix}i&j\\0&k\end{smallmatrix}\right]$, where $i>j\geq0$, $k>0$, and $i,j,k\in\mathbb{Z}$. Note that L has index ik in Λ. This means that L has ik distinct cosets in Λ.

Let u and v be unit translation vectors in $\mathbb{R}^2$, that is, if $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]\in\mathbb{R}^2$, then u and v maps $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix}x+1\\y\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}x\\y+1\end{smallmatrix}\right]$, respectively. The group that u and v generate is called the translation group $T=\langle u,v\rangle$. Clearly, T sends Λ onto itself. If u and v are orthogonal, then Λ is the *square* lattice. If u and v forms an angle of $120°$, then Λ is the *hexagonal* lattice.

Let α and γ be the respective $90°$ and $60°$ counterclockwise rotation about the origin $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$; β be the mirror reflection passing through $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ and parallel to the horizontal axis; and δ be the mirror reflection passing through $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ and inclined at an angle of $60°$ about the positive horizontal axis/ in the direction of uv. The full symmetry group of the square lattice is $G=\langle u,v,\alpha,\beta\rangle$, which is of type $p4m$; while the full symmetry group of the hexagonal lattice is $G = \langle u, v, \gamma, \delta \rangle$, which is of type $p6m$. The *point group* of G, denoted by $P(G)$, consists of all symmetries in G that fix $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$.

The subgroup $T(G)$ of translations of G is generated by the linearly independent translations u and v, that is, $T(G) = \langle u, v \rangle \cong p1$. Similarly, if S is a subgroup of G, then the subgroup of translations of S, denoted by $T(S)$, is also generated by two linearly independent translations and is also of type $p1$. These translation groups form a free abelian group of rank 2 and thus isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Since the lattice Λ and the translation group $T(G)$ are both isomorphic to $\mathbb{Z} \times \mathbb{Z}$, one may identify the translation subgroup $T(G)$ with the lattice Λ. In general, if S is a subgroup of G of finite index and $T(S) = \langle u^i, u^j v^k \rangle$, then we denote the sublattice associated to this translation subgroup by $L[T(S)] = \left[\begin{smallmatrix}i & j\\0 & k\end{smallmatrix}\right]$. Conversely, if L is a sublattice spanned by the linearly independent vectors $\left[\begin{smallmatrix}i\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}j\\k\end{smallmatrix}\right]$ over $\mathbb{Z}$, then we denote the translation subgroup associated to it by $T[L] = \langle u^i, u^j v^k \rangle$. In effect, we have $T(S) = T[L]$.

Let Λ be the square or the hexagonal lattice and let C be a nonempty set of n colours. A *colouring* of Λ is a surjective function $\mathcal{C} : \Lambda \longrightarrow C$. The function $\mathcal{C}$ assigns a colour $c \in C$ to every $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] \in \Lambda$. Whenever we want to emphasise that a colouring uses n colours, we will also call it an n-colouring.

An element $h \in G$ is called a *colour symmetry* of a coloured lattice Λ if and only if only those lattice points having the colour c_1 are mapped by h to those lattice points having the colour c_2. The set H of all colour symmetries of the coloured lattice forms a group called the *colour symmetry group*. In symbols, $h \in H$ if and only if $\exists \sigma_h \in S_C$ such that $\forall \left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] \in \Lambda$, $(\mathcal{C} \circ h)\left(\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]\right) = (\sigma_h \circ \mathcal{C})\left(\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]\right)$, where S_C is the group of permutations on the set of colours C and $(\mathcal{C} \circ h)\left(\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]\right) = \mathcal{C}\left(h\left(\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]\right)\right)$.

The mapping $f : H \longrightarrow S_C$ with $h \mapsto \sigma_h$ defines a group homomorphism; and thus the group H acts on C. Consequently, every element of H has a permutation representation in terms of the elements of C, that is, the image of h under f is determined by the way h permutes the colours in C.

An element $k \in H$ is called a *colour-fixing symmetry* of a coloured lattice Λ if and only if it fixes all the colours in the lattice colouring. The set K of all colour-fixing symmetries of the coloured lattice forms a group called the *colour-fixing group*. This group is the kernel of the homomorphism f; and thus K is a normal subgroup of H. Consequently, the group $f(H)$ is isomorphic to the quotient group H/K. This quotient group is called the *colour permutation group*.

Two colourings of lattice Λ are said to be *equivalent* if one of the coloured lattices may be obtained from the other coloured lattice by (1) a bijection from C_1 to C_2 where C_i is the set of colours in the ith coloured lattice for $i = 1, 2$; or (2) a symmetry of the uncoloured lattice; or (3) a combination of (1) and (2).

2 Colourings of a lattice

Let Λ be a lattice. A colouring of Λ by ik colours is achieved by assigning two lattice points of Λ the same colour if and only if they belong to the same coset of a sublattice L of index ik in Λ. In this colouring, the set C_1 of ik colours is identified with the set of ik distinct cosets of L in Λ. In this paper, we will refer to this colouring as *sublattice colouring*. Other names for this colouring are found in [5, 2]. The associated colour symmetry group and colour-fixing group are denoted by H_1 and K_1 respectively.

Illustration 1. Consider a sublattice colouring of the square lattice Λ determined by the sublattice $L = \left[\begin{smallmatrix}8 & 4\\0 & 4\end{smallmatrix}\right]$. The complete set of coset representatives of L in Λ is $\left\{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix}7\\0\end{smallmatrix}\right],\right.$

$\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix}7\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\2\end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix}7\\2\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\3\end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix}7\\3\end{smallmatrix}\right]\}$. The coset $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$ is represented by a_b in the diagram. The colour groups are $H_1 = \langle u, v, \alpha, \beta\rangle \cong p4m$, $K_1 = \langle u, v\rangle \cong p1$, and $H_1/K_1 \cong D_4$.

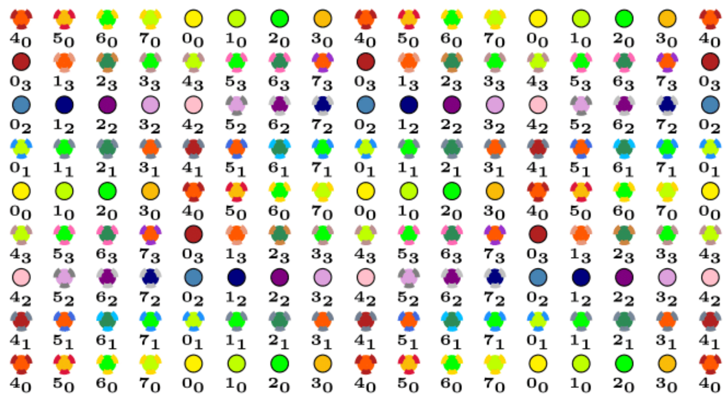
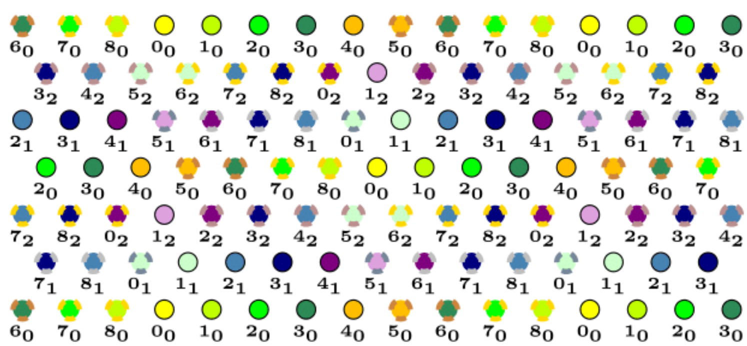


Illustration 2. Consider a sublattice colouring of the hexagonal lattice Λ determined by the sublattice $L = \left[\begin{smallmatrix}9 & 6\\0 & 3\end{smallmatrix}\right]$. The complete set of coset representatives of L in Λ is $\{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix}8\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix}8\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\2\end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix}8\\2\end{smallmatrix}\right]\}$. The colour groups are $H_1 = \langle u, v, \gamma, \delta\rangle \cong p6m$, $K_1 = \langle u, v\rangle \cong p1$, and $H_1/K_1 \cong D_6$.



Now, let L be a sublattice of Λ of index ik; and G be the symmetry group of Λ. Suppose S is a plane crystallographic group and a subgroup of G of finite index. Get the orbit of each coset of L in Λ under S. A colouring of Λ using at most ik colours is achieved by assigning two lattice points of Λ the same colour if and only if they belong to the same orbit of a coset of L. This colouring was referred to as S-*orbit colouring* [4]. In this colouring, the set of colours C_2 is the set of distinct orbits of C_1 under S. The associated colour symmetry and colour-fixing groups are denoted by H_2 and K_2 respectively.

Sublattice colouring of Λ is a special case of S-orbit colouring by taking $S = T(S)$, that is, S consists entirely of translations.

Lemma 1 (Felix & Walo, 2011). *Let* $L = \left[\begin{smallmatrix}i & j\\0 & k\end{smallmatrix}\right]$ *be a sublattice of* Λ. *The colour group* H_1 *permutes the cosets of* L *in* Λ *if and only if* H_1 *is the normaliser of* $T[L]$ *in* G.

Consider a sublattice colouring of Λ determined by a sublattice L. Suppose C is the set of cosets of L in Λ. Additionally, consider an S-orbit colouring of Λ determined by a sublattice L and S such that $T(S) = T[L]$. If N is the normaliser of $T[L]$ in G, then by Lemma 1, $N = H_1$. Hence, the normaliser N of $T(S)$ in G is a colour group. Thus, we can talk about the homomorphism $f : H_1 \longrightarrow S_C$, as defined in the introduction.

Theorem 2.1 (**Felix & Walo, 2011**). *Consider an S-orbit colouring of Λ determined by L and S. If C is the set of cosets of L in Λ, $f : H_1 \longrightarrow S_C$ is the homomorphism described above, and $\ker f$ is the kernel of f, then we have the following results:*

1. $S \ker f \leq K_2 \leq H_2 \leq H_1$
2. $\ker f = K_1$
3. $N/\ker f = H_1/K_1$

In [2], de las Peñas and Felix computed the colour groups of sublattice colourings of the square and the hexagonal lattices. Seeing that S-orbit colouring is a generalisation of sublattice colouring, we will also determine the subgroups associated with this colouring. In effect, we will also be studying the subgroup structure of H_1. In our characterisation of colour groups, we take into consideration all colourings, including the equivalent ones, provided they result from distinct sublattices listed in Tables 1 and 2 of [2].

3 Some results

In the following discussions, keep in mind that $i > j \geq 0$, $k > 0$, where $i, j, k \in \mathbb{Z}$. Also, in the succeeding illustrations and examples, we denote the origin $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ by θ.

Theorem 3.1. *Consider an S-orbit colouring of the lattice Λ (square or hexagonal) determined by a sublattice $L = \left[\begin{smallmatrix} i & j \\ 0 & k \end{smallmatrix}\right]$ and the subgroup $S = \langle u^i, u^j v^k, h\rangle \cong p2$, where h is a $180°$ rotation about a point in L. If*

1. $L = \left[\begin{smallmatrix} odd & odd \\ 0 & odd \end{smallmatrix}\right]$ *or* $\left[\begin{smallmatrix} odd & even \\ 0 & odd \end{smallmatrix}\right]$, *then* $T(H_2) = \langle u^i, u^j v^k\rangle$.
2. $L = \left[\begin{smallmatrix} odd & odd \\ 0 & even \end{smallmatrix}\right]$, *then* $T(H_2) = \langle u^i, u^{(i+j)/2} v^{k/2}\rangle$.
3. $L = \left[\begin{smallmatrix} odd & even \\ 0 & even \end{smallmatrix}\right]$, *then* $T(H_2) = \langle u^i, u^{j/2} v^{k/2}\rangle$.
4. $L = \left[\begin{smallmatrix} even & odd \\ 0 & odd \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} even & odd \\ 0 & even \end{smallmatrix}\right]$, *or* $\left[\begin{smallmatrix} even & even \\ 0 & odd \end{smallmatrix}\right]$, *then* $T(H_2) = \langle u^{i/2}, u^j v^k\rangle$.
5. $L = \left[\begin{smallmatrix} even & even \\ 0 & even \end{smallmatrix}\right]$, *then* $T(H_2) = \langle u^{i/2}, u^{j/2} v^{k/2}\rangle$.

Proof: The complete set of coset representatives of $L = \left[\begin{smallmatrix} i & j \\ 0 & k \end{smallmatrix}\right]$ in Λ is the set $\{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix} i-1 \\ 0 \end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix} 0 \\ k-1 \end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix} i-1 \\ k-1 \end{smallmatrix}\right]\}$. Consider the subgroup $S = \langle u^i, u^j v^k, h\rangle \cong p2$ where h is a $180°$ rotation. Without loss of generality, let the origin θ be the centre of rotation of h. Consider the normaliser N of $T(S)$ in G. From Tables 1 and 2 of [2], $T(N) = \langle u, v\rangle$. Under the homomorphism f, the images of the generators of $T(N)$ under f are

$$\begin{aligned} f(u) =& \left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} i-1 \\ 0 \end{smallmatrix}\right]\right) \cdots \left(\left[\begin{smallmatrix} 0 \\ k-1 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} i-1 \\ k-1 \end{smallmatrix}\right]\right) \\ f(v) =& \left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} 0 \\ k-1 \end{smallmatrix}\right] \left[\begin{smallmatrix} i-j \\ 0 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} i-j \\ k-1 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} j \\ 0 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} j \\ k-1 \end{smallmatrix}\right]\right) \\ & \left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} 1 \\ k-1 \end{smallmatrix}\right] \left[\begin{smallmatrix} i-j+1 \\ 0 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} i-j+1 \\ k-1 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} j+1 \\ 0 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} j+1 \\ k-1 \end{smallmatrix}\right]\right) \cdots \\ & \left(\left[\begin{smallmatrix} i-j-1 \\ 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} i-j-1 \\ 1 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} i-j-1 \\ k-1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 2(i-j)-1 \\ 0 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} 2(i-j)-1 \\ k-1 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} i-2j \\ 0 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} i-2j \\ k-1 \end{smallmatrix}\right]\right). \end{aligned}$$

Similarly, the images of the generators of S under f are $f(u^i) = f(u^j v^k) = \left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\right)$ and

$$\begin{aligned} f(h) = & \left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\right)\left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} i-1 \\ 0 \end{smallmatrix}\right]\right) \cdots \left(\left[\begin{smallmatrix} (i-1)/2 \\ 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} (i+1)/2 \\ 0 \end{smallmatrix}\right]\right) \cdots \left(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \left[\begin{smallmatrix} -a \\ -b \end{smallmatrix}\right]\right) \cdots \\ & \left(\left[\begin{smallmatrix} 0 \\ (k-1)/2 \end{smallmatrix}\right] \left[\begin{smallmatrix} j \\ (k+1)/2 \end{smallmatrix}\right]\right) \cdots \left(\left[\begin{smallmatrix} (i-1)/2 \\ (k-1)/2 \end{smallmatrix}\right] \left[\begin{smallmatrix} j-((i-1)/2) \\ (k+1)/2 \end{smallmatrix}\right]\right). \end{aligned}$$

If $L = \left[\begin{smallmatrix}\text{odd} & \text{odd}\\ 0 & \text{odd}\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}\text{odd} & \text{even}\\ 0 & \text{odd}\end{smallmatrix}\right]$, the $f(S)$-orbits are $\{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\}$, $\{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}i-1\\0\end{smallmatrix}\right]\}$, $\ldots$, $\{\left[\begin{smallmatrix}(i-1)/2\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}(i+1)/2\\0\end{smallmatrix}\right]\}$, $\ldots$, $\{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right], \left[\begin{smallmatrix}-a\\-b\end{smallmatrix}\right]\}$, $\ldots$, $\{\left[\begin{smallmatrix}0\\(k-1)/2\end{smallmatrix}\right], \left[\begin{smallmatrix}j\\(k+1)/2\end{smallmatrix}\right]\}$, $\ldots$, $\{\left[\begin{smallmatrix}(i-1)/2\\(k-1)/2\end{smallmatrix}\right], \left[\begin{smallmatrix}j-((i-1)/2)\\(k+1)/2\end{smallmatrix}\right]\}$. The modified permutation representations of $f(u)$ and $f(v)$ with respect to the $f(S)$-orbits are

$$\begin{aligned} f(u) &= (\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\ \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\ \left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}i-1\\0\end{smallmatrix}\right])\cdots(\left[\begin{smallmatrix}0\\k-1\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}i-1\\k-1\end{smallmatrix}\right]) \\ &\to (\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\ \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\ \left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])\cdots(\left[\begin{smallmatrix}j\\1\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}j-i-1\\1\end{smallmatrix}\right]) \end{aligned}$$

$$\begin{aligned} f(v) &= (\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\ \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}0\\a\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}0\\k-1\end{smallmatrix}\right]\ \left[\begin{smallmatrix}i-j\\0\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}i-j\\k-1\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}j\\0\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}j\\k-a\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}j\\k-1\end{smallmatrix}\right])\cdots \\ &\to (\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\ \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}0\\a\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}0\\k-1\end{smallmatrix}\right]\ \left[\begin{smallmatrix}i-j\\0\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}i-j\\k-1\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}i-j\\0\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}0\\a\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right])\cdots \end{aligned}$$

The element $f(u)$ does not effect any permutation of the $f(S)$-orbits since $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ is sent to both $\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$. In fact, for every integer $0 < a < i$, let $d = gcd(i, a)$. The multiple

$$\begin{aligned} f(u)^a &= (\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\ \left[\begin{smallmatrix}a\\0\end{smallmatrix}\right]\ \left[\begin{smallmatrix}2a\\0\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}(i/d)a\\0\end{smallmatrix}\right])\ \cdots \\ &\to (\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\ \left[\begin{smallmatrix}a\\0\end{smallmatrix}\right]\ \left[\begin{smallmatrix}2a\\0\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}a\\0\end{smallmatrix}\right])\ \cdots \end{aligned}$$

of $f(u)$ does not effect any permutatution of the $f(S)$-orbits since the order of $f(u)$ is i, which is odd; and so the length of the cycles in $f(u)^a$ is also odd. It follows that the coset $\left[\begin{smallmatrix}2a\\0\end{smallmatrix}\right] \neq \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ and that $\left[\begin{smallmatrix}a\\0\end{smallmatrix}\right]$ is sent to both $\left[\begin{smallmatrix}2a\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$.

Similarly, the element $f(v)$ does not effect any permutation of the $f(S)$-orbits since $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ is sent to both $\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$. Suppose $d_1 = gcd(i, j)$. For every integer $0 < a < ik/d_1$, let $d_2 = gcd(a, ik/d_1)$. The multiple

$$\begin{aligned} f(v)^a &= (\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\ \left[\begin{smallmatrix}0\\a\end{smallmatrix}\right]\ \left[\begin{smallmatrix}0\\2a\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}0\\(ik/d_1d_2)a\end{smallmatrix}\right])\ \cdots \\ &\to (\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\ \left[\begin{smallmatrix}0\\a\end{smallmatrix}\right]\ \left[\begin{smallmatrix}0\\2a\end{smallmatrix}\right]\ \cdots\ \left[\begin{smallmatrix}0\\a\end{smallmatrix}\right])\ \cdots \end{aligned}$$

of $f(v)$ does not effect any permutatution of the $f(S)$-orbits since the order of $f(v)$ is ik/d, which is odd; and so the length of the cycles in $f(v)^a$ is also odd. It follows that the coset $\left[\begin{smallmatrix}0\\2a\end{smallmatrix}\right] \neq \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ and that $\left[\begin{smallmatrix}0\\a\end{smallmatrix}\right]$ is sent to both $\left[\begin{smallmatrix}0\\2a\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$. Therefore $T(H_2) = \langle u^i, v^{ik/d}, u^jv^k\rangle = \langle u^i, u^jv^k\rangle$.

If $L = \left[\begin{smallmatrix}\text{odd} & \text{odd}\\ 0 & \text{even}\end{smallmatrix}\right]$, the $f(S)$-orbits are $\{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\}$, $\{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}i-1\\0\end{smallmatrix}\right]\}$, $\ldots$, $\{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right], \left[\begin{smallmatrix}-a\\-b\end{smallmatrix}\right]\}$, $\ldots$, $\{\left[\begin{smallmatrix}(i+j)/2\\k/2\end{smallmatrix}\right]\}$. The modified permutation representations of $f(u)$ and $f(v)$ with respect to the $f(S)$-orbits are computed similarly like in the previous case. On the other hand, the modified permutation representation of $f(u^{i+j/2}v^{k/2})$ is

$$\begin{aligned} f(u^{i+j/2}v^{k/2}) &= (\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\ \left[\begin{smallmatrix}(i+j)/2\\k/2\end{smallmatrix}\right])\ \cdots\ (\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]\ \left[\begin{smallmatrix}(2a+i+j)/2\\(2b+k)/2\end{smallmatrix}\right])\cdots(\left[\begin{smallmatrix}-a\\-b\end{smallmatrix}\right]\ \left[\begin{smallmatrix}(-2a+i+j)/2\\(-2b+k)/2\end{smallmatrix}\right])\ \cdots \\ &\quad (\left[\begin{smallmatrix}(i-j-2)/2\\(k-2)/2\end{smallmatrix}\right]\ \left[\begin{smallmatrix}i-1\\k-1\end{smallmatrix}\right]) \\ &\to (\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\ \left[\begin{smallmatrix}(i+j)/2\\k/2\end{smallmatrix}\right])\ \cdots\ (\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]\ \left[\begin{smallmatrix}(2a+i+j)/2\\(2b+k)/2\end{smallmatrix}\right])\ \cdots\ (\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]\ \left[\begin{smallmatrix}(2a+i+j)/2\\(2b+k)/2\end{smallmatrix}\right])\ \cdots \\ &\quad (\left[\begin{smallmatrix}(i-j-2)/2\\(k-2)/2\end{smallmatrix}\right]\ \left[\begin{smallmatrix}i-1\\k-1\end{smallmatrix}\right]) \end{aligned}$$

The $f(S)$-orbit of $\left[\begin{smallmatrix}(2a+i+j)/2\\(2b+k)/2\end{smallmatrix}\right]$ is $\{\left[\begin{smallmatrix}(2a+i+j)/2\\(2b+k)/2\end{smallmatrix}\right], \left[\begin{smallmatrix}(-2a-i-j)/2\\(-2b-k)/2\end{smallmatrix}\right]\}$. The coset $\left[\begin{smallmatrix}(-2a-i-j)/2\\(-2b-k)/2\end{smallmatrix}\right]$ is clearly equal to $\left[\begin{smallmatrix}(-2a+i+j)/2\\(-2b+k)/2\end{smallmatrix}\right]$. Therefore $T(H_2) = \langle u^i, v^{ik/d}, u^{i+j/2}v^{k/2}\rangle = \langle u^i, u^{i+j/2}v^{k/2}\rangle$. The proofs of the remaining cases are constructed similarly.

Illustration. In this S-orbit colouring of the square lattice determined by $L = \left[\begin{smallmatrix} 4 & 2 \\ 0 & 3 \end{smallmatrix}\right]$ and $S = \langle u^4, u^2v^3, \alpha^2 \rangle \cong p2$, we have $H_2 = \langle u^2, v^3, \beta, \alpha^2\beta \rangle \cong pmm$, $K_2 = S$, and $H_2/K_2 = \mathbb{Z}_2 \rtimes D_2$.

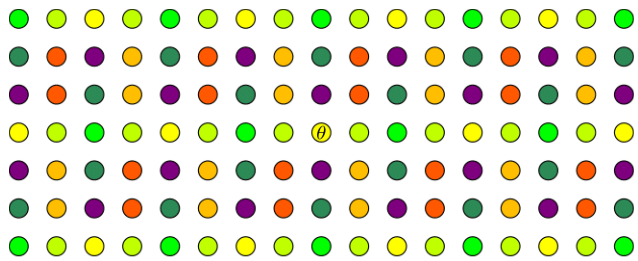
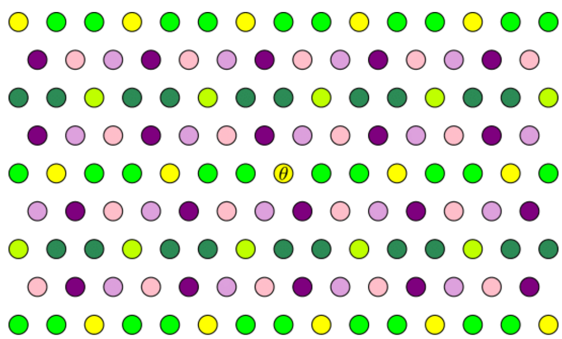


Illustration. The S-orbit colouring of the hexagonal lattice determined by $L = \left[\begin{smallmatrix} 3 & 1 \\ 0 & 4 \end{smallmatrix}\right]$ and $S = \langle u^3, uv^4, \gamma^3 \rangle \cong p2$ gives us $H_2 = \langle u^3, u^2v^2, \gamma^3 \rangle \cong p2$, $K_2 = S$, and $H_2/K_2 \cong \mathbb{Z}_2$.



3.1 Square Lattice

Theorem 3.2 (**Felix & de las Peñas, 2007**). *Consider a sublattice colouring of the square lattice Λ determined by a sublattice $L = \left[\begin{smallmatrix} i & j \\ 0 & k \end{smallmatrix}\right]$. Then the following are true:*

1. *L is invariant under $\langle \alpha \rangle \Leftrightarrow L = k\left[\begin{smallmatrix} l & m \\ 0 & 1 \end{smallmatrix}\right]$ where l divides $(m^2+1) \Leftrightarrow p4 \cong \langle u, v, \alpha \rangle \leq H_1$.*

2. *L is invariant under $\langle \alpha, \beta \rangle \Leftrightarrow L = \left[\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} 2k & k \\ 0 & k \end{smallmatrix}\right] \Leftrightarrow p4m \cong \langle u, v, \alpha, \beta \rangle = H_1$.*

3. *L is invariant under $\langle \beta, \alpha^2\beta \rangle \Leftrightarrow L = \left[\begin{smallmatrix} i & 0 \\ 0 & k \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} 2j & j \\ 0 & k \end{smallmatrix}\right] \Leftrightarrow pmm \cong \langle u, v, \beta, \alpha^2\beta \rangle \leq H_1$.*

4. *L is invariant under $\langle \alpha\beta, \alpha^3\beta \rangle \Leftrightarrow L = k\left[\begin{smallmatrix} l & m \\ 0 & 1 \end{smallmatrix}\right]$ where l divides $(m^2-1) \Leftrightarrow cmm \cong \langle u, v, \alpha\beta, \alpha^3\beta \rangle \leq H_1$.*

For the proof, the reader may see [2].

Theorem 3.3. *Consider an S-orbit colouring of the square lattice Λ determined by a sublattice $L = \left[\begin{smallmatrix} i & j \\ 0 & k \end{smallmatrix}\right]$ and a subgroup S having the following properties:*

1. *The normaliser H_1 in G of the translation subgroup $T[L] = \langle u^i, u^jv^k \rangle$ is of type $p4m$, pmm, or cmm.*

2. *Let $r \in H_1 \backslash K_1$. Form the subgroup $S = \langle u^i, u^jv^k, r \rangle$ of type cm or pm.*

Then the following are true:

1. *If* $r = \beta$, *then* $u \in T(H_2)$.
2. *If* $r = \alpha\beta$, *then* $uv \in T(H_2)$.
3. *If* $r = \alpha^2\beta$, *then* $v \in T(H_2)$.
4. *If* $r = \alpha^3\beta$, *then* $u^{-1}v \in T(H_2)$.

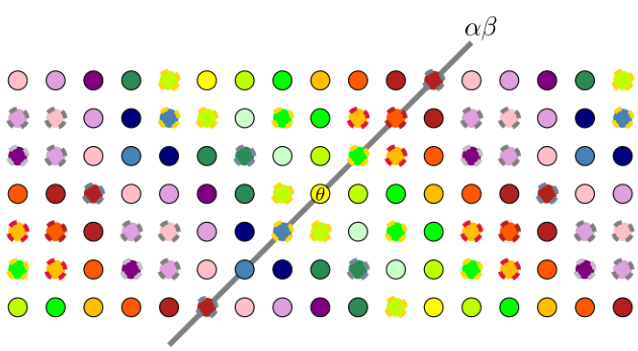
Proof: Suppose Λ is a square lattice; $L = \left[\begin{smallmatrix} i & j \\ 0 & k \end{smallmatrix}\right]$ is a sublattice; and $T[L] = \langle u^i, u^j v^k \rangle$. Without loss of generality, suppose that the mirror reflection β is contained in the normaliser H_1 of $T[L]$ in G but not in K_1. Then from Table 1 of [2], either $L = \left[\begin{smallmatrix} i & 0 \\ 0 & k \end{smallmatrix}\right]$, where $i, k \geq 3$; or $L = \left[\begin{smallmatrix} 2j & j \\ 0 & k \end{smallmatrix}\right]$, where $j, k \geq 2$. Without loss of generality, suppose $L = \left[\begin{smallmatrix} i & 0 \\ 0 & k \end{smallmatrix}\right]$, where $i, k \geq 3$; and form the subgroup $S = \langle u^i, v^k, \beta \rangle$ of type cm or pm. It too follows that $T(S) = \langle u^i, v^k \rangle$.

The complete set of coset representatives of L in Λ is $\left\{ \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix} i-1 \\ 0 \end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix} 0 \\ k-1 \end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix} i-1 \\ k-1 \end{smallmatrix}\right] \right\}$. The normaliser H_1 of $T(S)$ in G contains $pmm \cong \langle u, v, \beta, \alpha^2\beta \rangle$ as a subgroup. The image of the translation generators of S under f is $f(u^i) = f(v^k) = \left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\right)$. Also, since $\beta \notin K_1$, the image of β under f is $f(\beta) = \left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\right)\left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\right) \cdots \left(\left[\begin{smallmatrix} i-1 \\ 0 \end{smallmatrix}\right]\right)\left(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\ \left[\begin{smallmatrix} 0 \\ k-1 \end{smallmatrix}\right]\right) \cdots \left(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]\ \left[\begin{smallmatrix} a \\ k-b \end{smallmatrix}\right]\right) \cdots \left(\left[\begin{smallmatrix} i-1 \\ 1 \end{smallmatrix}\right]\ \left[\begin{smallmatrix} i-1 \\ k-1 \end{smallmatrix}\right]\right)$ and the $f(S)$-orbits are $\left\{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\right\}$, $\left\{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\right\}$, ..., $\left\{\left[\begin{smallmatrix} i-1 \\ 0 \end{smallmatrix}\right]\right\}$, $\left\{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ k-1 \end{smallmatrix}\right]\right\}$, ..., $\left\{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right], \left[\begin{smallmatrix} a \\ k-b \end{smallmatrix}\right]\right\}$, ..., $\left\{\left[\begin{smallmatrix} i-1 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} i-1 \\ k-1 \end{smallmatrix}\right]\right\}$. The modified permutation representation of $f(u)$ with respect to the $f(S)$-orbits is

$$\begin{aligned} f(u) \;=\; & \left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\ \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right]\ \left[\begin{smallmatrix} a+1 \\ 0 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} i-1 \\ 0 \end{smallmatrix}\right]\right)\left(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\ \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} a \\ 1 \end{smallmatrix}\right]\ \left[\begin{smallmatrix} a+1 \\ 1 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} i-1 \\ 1 \end{smallmatrix}\right]\right) \cdots \\ & \left(\left[\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right]\ \left[\begin{smallmatrix} 1 \\ b \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]\ \left[\begin{smallmatrix} a+1 \\ b \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} i-1 \\ b \end{smallmatrix}\right]\right) \cdots \left(\left[\begin{smallmatrix} 0 \\ k-b \end{smallmatrix}\right]\ \left[\begin{smallmatrix} 1 \\ k-b \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} a \\ k-b \end{smallmatrix}\right]\ \left[\begin{smallmatrix} a+1 \\ k-b \end{smallmatrix}\right] \cdots \\ & \left[\begin{smallmatrix} i-1 \\ k-b \end{smallmatrix}\right]\right) \cdots \left(\left[\begin{smallmatrix} 0 \\ k-1 \end{smallmatrix}\right]\ \left[\begin{smallmatrix} 1 \\ k-1 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} a \\ k-1 \end{smallmatrix}\right]\ \left[\begin{smallmatrix} a+1 \\ k-1 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} i-1 \\ k-1 \end{smallmatrix}\right]\right) \\ \to\; & \left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\ \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right]\ \left[\begin{smallmatrix} a+1 \\ 0 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} i-1 \\ 0 \end{smallmatrix}\right]\right)\left(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\ \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} a \\ 1 \end{smallmatrix}\right]\ \left[\begin{smallmatrix} a+1 \\ 1 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} i-1 \\ 1 \end{smallmatrix}\right]\right) \cdots \\ & \left(\left[\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right]\ \left[\begin{smallmatrix} 1 \\ b \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]\ \left[\begin{smallmatrix} a+1 \\ b \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} i-1 \\ b \end{smallmatrix}\right]\right) \cdots \left(\left[\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right]\ \left[\begin{smallmatrix} 1 \\ b \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]\ \left[\begin{smallmatrix} a+1 \\ b \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} i-1 \\ b \end{smallmatrix}\right]\right) \\ & \cdots \left(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\ \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} a \\ 1 \end{smallmatrix}\right]\ \left[\begin{smallmatrix} a+1 \\ 1 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} i-1 \\ 1 \end{smallmatrix}\right]\right). \end{aligned}$$

This shows that u effects permutation of the coloured lattice, which means that $u \in T(H_2)$. The proofs for the cases $r = \alpha\beta$, $\alpha^2\beta$, or $\alpha^3\beta$ follow a similar construction.

Illustration. Consider an S-orbit colouring of the square lattice determined by $L = \left[\begin{smallmatrix} 12 & 9 \\ 0 & 3 \end{smallmatrix}\right]$ and $S = \langle u^{12}, u^9 v^3, \alpha\beta \rangle \cong cm$. The colour groups are $H_2 = \langle u^3, uv, \alpha\beta, \alpha^3\beta \rangle$, $K_2 = S$, and $H_2/K_2 \cong D_6$.



3.2 Hexagonal Lattice

Theorem 3.4 (**Felix & de las Peñas, 2007**). *Consider a sublattice colouring of the hexagonal lattice* Λ *determined by a sublattice* $L = \left[\begin{smallmatrix} i & j \\ 0 & k \end{smallmatrix}\right]$. *Then the following are true:*

1. *L is invariant under $\langle\gamma\rangle \Leftrightarrow L = k\begin{bmatrix} l & m \\ 0 & 1\end{bmatrix}$ where l divides $(m^2-m+1) \Leftrightarrow p6 \cong \langle u, v, \gamma\rangle \le H_1$.*
2. *L is invariant under $\langle\gamma, \delta\rangle \Leftrightarrow L = \begin{bmatrix} k & 0 \\ 0 & k\end{bmatrix}$ or $\begin{bmatrix} 3k & 2k \\ 0 & k\end{bmatrix} \Leftrightarrow p6m \cong \langle u, v, \gamma, \delta\rangle = H_1$.*
3. *L is invariant under $\langle\delta, \gamma^3\delta\rangle \Leftrightarrow L = k\begin{bmatrix} l & m \\ 0 & 1\end{bmatrix}$ where l divides $(m^2-1) \Leftrightarrow cmm \cong \langle u, v, \delta, \gamma^3\delta\rangle \le H_1$.*
4. *L is invariant under $\langle\gamma\delta, \gamma^4\delta\rangle \Leftrightarrow L = \begin{bmatrix} i & j \\ 0 & k\end{bmatrix}$ where i divides $(2j-k) \Leftrightarrow cmm \cong \gamma^5\langle u, v, \delta, \gamma^3\delta\rangle\gamma^{-5} \le H_1$.*
5. *L is invariant under $\langle\gamma^2\delta, \gamma^5\delta\rangle \Leftrightarrow L = k\begin{bmatrix} l & m \\ 0 & 1\end{bmatrix}$ where l divides $(m^2-2m) \Leftrightarrow cmm \cong \gamma\langle u, v, \delta, \gamma^3\delta\rangle\gamma^{-1} \le H_1$.*

For the proof, the reader may see [2].

Theorem 3.5. *Consider an S-orbit colouring of the hexagonal lattice Λ determined by a sublattice $L = \begin{bmatrix} i & j \\ 0 & k\end{bmatrix}$ and a subgroup S having the following properties:*

a. *The normaliser H_1 in G of the translation subgroup $T[L] = \langle u^i, u^j v^k\rangle$ is of type $p6m$ or cmm.*

b. *Let $r \in H_1 \backslash K_1$. Form the subgroup $S = \langle u^i, u^j v^k, r\rangle$ of type cm or pm.*

Then the following are true:

1. *If $r = \delta$, then $uv \in T(H_2)$.*
2. *If $r = \gamma\delta$, then $uv^2 \in T(H_2)$.*
3. *If $r = \gamma^2\delta$, then $v \in T(H_2)$.*
4. *If $r = \gamma^3\delta$, then $u^{-1}v \in T(H_2)$.*
5. *If $r = \gamma^4\delta$, then $u \in T(H_2)$.*
6. *If $r = \gamma^5\delta$, then $u^2v \in T(H_2)$.*

Proof: Suppose Λ is a hexagonal lattice; $L = \begin{bmatrix} i & j \\ 0 & k\end{bmatrix}$; and $T[L] = \langle u^i, u^j v^k\rangle$. Without loss of generality, suppose that the mirror reflection δ is contained in the normaliser H_1 of $T[L]$ in G but not in K_1. Then from Table 2 of [2], either $L = \begin{bmatrix} k & 0 \\ 0 & k\end{bmatrix}$, where $k \ge 3$; or $L = \begin{bmatrix} 3k & 2k \\ 0 & k\end{bmatrix}$, where $k \ge 2$; or $L = \begin{bmatrix} kl & km \\ 0 & k\end{bmatrix}$, where l divides (m^2-1). Without loss of generality, suppose $L = \begin{bmatrix} kl & km \\ 0 & k\end{bmatrix}$, where l divides (m^2-1); and form the subgroup $S = \langle u^{kl}, u^{km}v^k, \delta\rangle$ of type cm. It also follows that $T(S) = \langle u^{kl}, u^{km}v^k\rangle$.

The complete set of coset representatives of L in Λ is $\left\{\begin{bmatrix} 0 \\ 0\end{bmatrix}, \begin{bmatrix} 1 \\ 0\end{bmatrix}, \ldots, \begin{bmatrix} kl-1 \\ 0\end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ k-1\end{bmatrix}, \ldots, \begin{bmatrix} kl-1 \\ k-1\end{bmatrix}\right\}$. The normaliser H_1 of $T(S)$ in G contains $cmm \cong \langle u, v, \delta, \gamma^3\delta\rangle$ as a subgroup. The images of the translation generators of S under f are $f(u^{kl}) = f(u^{km}v^k) = \left(\begin{bmatrix} 0 \\ 0\end{bmatrix}\right)$. And since $\delta \notin K_1$, the image of δ under f is

$$\begin{aligned} f(\delta) \quad = \quad & \left(\begin{bmatrix} 0 \\ 0\end{bmatrix}\right)\left(\begin{bmatrix} 1 \\ 1\end{bmatrix}\right)\cdots\left(\begin{bmatrix} k-1 \\ k-1\end{bmatrix}\right)\left(\begin{bmatrix} k(l-m+1) \\ 0\end{bmatrix}\right)\cdots\left(\begin{bmatrix} k(l-m+1)+(k-1) \\ k-1\end{bmatrix}\right)\left(\begin{bmatrix} 2k(l-m+1) \\ 0\end{bmatrix}\right)\cdots \\ & \left(\begin{bmatrix} rk(l-m+1) \\ 0\end{bmatrix}\right)\cdots\left(\begin{bmatrix} rk(l-m+1)+(k-1) \\ k-1\end{bmatrix}\right)\cdots\left(\begin{bmatrix} 1 \\ 0\end{bmatrix}\ \begin{bmatrix} 0 \\ 1\end{bmatrix}\right)\left(\begin{bmatrix} 2 \\ 1\end{bmatrix}\ \begin{bmatrix} 1 \\ 2\end{bmatrix}\right)\cdots \\ & \left(\begin{bmatrix} k \\ k-1\end{bmatrix}\ \begin{bmatrix} k(l-m+1)-1 \\ 0\end{bmatrix}\right)\ \left(\begin{bmatrix} k(l-m+1)+1 \\ 0\end{bmatrix}\ \begin{bmatrix} k(l-m+1) \\ 1\end{bmatrix}\right)\cdots \\ & \left(\begin{bmatrix} k(l-m+1)+k \\ k-1\end{bmatrix}\ \begin{bmatrix} 2k(l-m+1)-1 \\ 0\end{bmatrix}\right)\left(\begin{bmatrix} 2k(l-m+1)+1 \\ 0\end{bmatrix}\ \begin{bmatrix} 2k(l-m+1) \\ 1\end{bmatrix}\right)\cdots \\ & \left(\begin{bmatrix} rk(l-m+1)+1 \\ 0\end{bmatrix}\ \begin{bmatrix} rk(l-m+1) \\ 1\end{bmatrix}\right)\cdots\left(\begin{bmatrix} rk(l-m+1)+k \\ k-1\end{bmatrix}\ \begin{bmatrix} kl-1 \\ 0\end{bmatrix}\right)\cdots \\ & \left(\begin{bmatrix} a \\ b\end{bmatrix}\ \begin{bmatrix} b \\ a\end{bmatrix}\right)\left(\begin{bmatrix} a+1 \\ b+1\end{bmatrix}\ \begin{bmatrix} b+1 \\ a+1\end{bmatrix}\right)\cdots\left(\begin{bmatrix} a+c \\ b+c\end{bmatrix}\ \begin{bmatrix} b+c \\ a+c\end{bmatrix}\right)\cdots\left(\begin{bmatrix} a+(k-1) \\ b+(k-1)\end{bmatrix}\ \begin{bmatrix} b+(k-1) \\ a+(k-1)\end{bmatrix}\right)\cdots \\ & \left(\begin{bmatrix} k(l-m) \\ 0\end{bmatrix}\ \begin{bmatrix} k \\ 0\end{bmatrix}\right)\cdots\left(\begin{bmatrix} kl-1 \\ k-1\end{bmatrix}\ \begin{bmatrix} rk(l-m+1)+(k-1)+k \\ k-1\end{bmatrix}\right). \end{aligned}$$

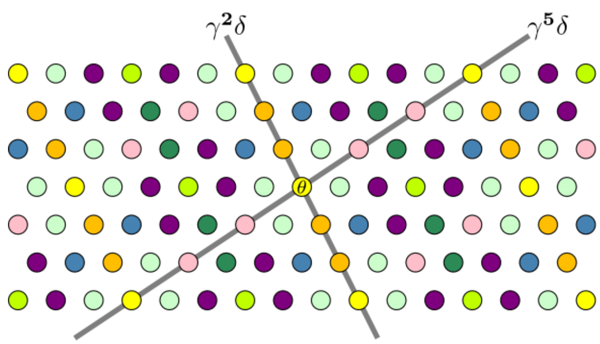
The modified permutation representation of $f(uv)$ with respect to the $f(S)$-orbits is

$$\begin{aligned}
f(uv) \quad = \quad & \Big(\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\cdots\begin{bmatrix}k-1\\k-1\end{bmatrix}\begin{bmatrix}k(l-m+1)\\0\end{bmatrix}\cdots\begin{bmatrix}k(l-m+1)+(k-1)\\k-1\end{bmatrix}\begin{bmatrix}2k(l-m+1)\\0\end{bmatrix}\cdots \\
& \begin{bmatrix}rk(l-m+1)\\0\end{bmatrix}\cdots\begin{bmatrix}rk(l-m+1)+(k-1)\\k-1\end{bmatrix}\Big)\Big(\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\cdots\begin{bmatrix}k\\k-1\end{bmatrix}\begin{bmatrix}k(l-m+1)+1\\0\end{bmatrix}\cdots \\
& \begin{bmatrix}k(l-m+1)+k\\k-1\end{bmatrix}\begin{bmatrix}2k(l-m+1)+1\\0\end{bmatrix}\cdots\begin{bmatrix}rk(l-m+1)+1\\0\end{bmatrix}\cdots\begin{bmatrix}rk(l-m+1)+k\\k-1\end{bmatrix}\Big)\cdots \\
& \Big(\begin{bmatrix}a\\b\end{bmatrix}\begin{bmatrix}a+1\\b+1\end{bmatrix}\cdots\begin{bmatrix}a+c\\b+c\end{bmatrix}\cdots\begin{bmatrix}a+(k-1)\\b+(k-1)\end{bmatrix}\Big)\cdots\Big(\begin{bmatrix}k(l-m)\\0\end{bmatrix}\cdots\begin{bmatrix}kl-1\\k-1\end{bmatrix}\Big)\cdots \\
& \Big(\begin{bmatrix}k\\0\end{bmatrix}\cdots\begin{bmatrix}rk(l-m+1)+(k-1)+k\\k-1\end{bmatrix}\Big)\cdots\Big(\begin{bmatrix}b\\a\end{bmatrix}\begin{bmatrix}b+1\\a+1\end{bmatrix}\cdots\begin{bmatrix}b+c\\a+c\end{bmatrix}\cdots\begin{bmatrix}b+(k-1)\\a+(k-1)\end{bmatrix}\Big) \\
& \cdots\Big(\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}\cdots\begin{bmatrix}k(l-m+1)-1\\0\end{bmatrix}\begin{bmatrix}k(l-m+1)\\1\end{bmatrix}\cdots\begin{bmatrix}2k(l-m+1)-1\\0\end{bmatrix}\begin{bmatrix}2k(l-m+1)\\1\end{bmatrix} \\
& \cdots\begin{bmatrix}rk(l-m+1)\\1\end{bmatrix}\cdots\begin{bmatrix}kl-1\\0\end{bmatrix}\Big) \\
\rightarrow \quad & \Big(\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\cdots\begin{bmatrix}k-1\\k-1\end{bmatrix}\begin{bmatrix}k(l-m+1)\\0\end{bmatrix}\cdots\begin{bmatrix}k(l-m+1)+(k-1)\\k-1\end{bmatrix}\begin{bmatrix}2k(l-m+1)\\0\end{bmatrix}\cdots \\
& \begin{bmatrix}rk(l-m+1)\\0\end{bmatrix}\cdots\begin{bmatrix}rk(l-m+1)+(k-1)\\k-1\end{bmatrix}\Big)\Big(\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\cdots\begin{bmatrix}k\\k-1\end{bmatrix}\begin{bmatrix}k(l-m+1)+1\\0\end{bmatrix}\cdots \\
& \begin{bmatrix}k(l-m+1)+k\\k-1\end{bmatrix}\begin{bmatrix}2k(l-m+1)+1\\0\end{bmatrix}\cdots\begin{bmatrix}rk(l-m+1)+1\\0\end{bmatrix}\cdots\begin{bmatrix}rk(l-m+1)+k\\k-1\end{bmatrix}\Big)\cdots \\
& \Big(\begin{bmatrix}a\\b\end{bmatrix}\begin{bmatrix}a+1\\b+1\end{bmatrix}\cdots\begin{bmatrix}a+c\\b+c\end{bmatrix}\cdots\begin{bmatrix}a+(k-1)\\b+(k-1)\end{bmatrix}\Big)\cdots\Big(\begin{bmatrix}k(l-m)\\0\end{bmatrix}\cdots\begin{bmatrix}kl-1\\k-1\end{bmatrix}\Big)\cdots \\
& \Big(\begin{bmatrix}k(l-m)\\0\end{bmatrix}\cdots\begin{bmatrix}kl-1\\k-1\end{bmatrix}\Big)\cdots\Big(\begin{bmatrix}a\\b\end{bmatrix}\begin{bmatrix}a+1\\b+1\end{bmatrix}\cdots\begin{bmatrix}a+c\\b+c\end{bmatrix}\cdots\begin{bmatrix}a+(k-1)\\b+(k-1)\end{bmatrix}\Big)\cdots \\
& \Big(\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\cdots\begin{bmatrix}k\\k-1\end{bmatrix}\begin{bmatrix}k(l-m+1)+1\\0\end{bmatrix}\cdots\begin{bmatrix}k(l-m+1)+k\\k-1\end{bmatrix}\begin{bmatrix}2k(l-m+1)+1\\0\end{bmatrix}\cdots \\
& \begin{bmatrix}rk(l-m+1)+1\\0\end{bmatrix}\cdots\begin{bmatrix}rk(l-m+1)+k\\k-1\end{bmatrix}\Big).
\end{aligned}$$

Therefore uv effects permutation of the coloured lattice, which shows that $uv \in T(H_2)$.

The proofs for the cases $r = \gamma\delta$, $\gamma^2\delta$, $\gamma^3\delta$, $\gamma^4\delta$, or $\gamma^5\delta$ follow a similar construction.

Illustration. In an S-orbit colouring of the hexagonal lattice determined by $L = \begin{bmatrix}6 & 12\\0 & 3\end{bmatrix}$ and $S = \langle u^6, u^{12}v^3, \gamma^2\delta, \gamma^5\delta\rangle \cong cmm$, we simply obtain $H_2 = K_2 = S$, which means $H_2/K_2 = \mathbb{Z}_1$.



4 Algorithm for S-Orbit Colouring

This algorithm is an elaborated version of the algorithm in [4]. The advantage of this algorithm over the original one is that the steps were specifically tailored to use S-orbit colouring on the square and the hexagonal lattices. This algorithm also facilitates in determining the associated colour groups.

1. Choose a sublattice L from Table 1 (square) or 2 (hexagonal) of [2]. Enumerate all its coset representatives in Λ.
2. Choose a subgroup S of H_1 such that $T(S) = T[L]$.

3. Get the permutation representation of each generator of S with respect to the cosets in Step 1.
4. By using the permutation representation of each generator, we obtain the orbit of each coset under S.
5. Use Theorems 3.1, 3.3, or 3.5 to aid in determining the translation subgroup $T(H_2)$.
6. Determine the associated sublattice $L[T(H_2)]$. This sublattice is not necessarily equal to L.
7. Use Theorems 3.2 or 3.4 in determining to which among the sublattices $L[T(H_2)]$ belongs. These theorems could identify the non-translation elements of the colour group H_2.

Example 1. Consider the sublattice $L = \left[\begin{smallmatrix}8&4\\0&4\end{smallmatrix}\right]$ of the square lattice Λ. See Illustration 1 for the complete set of coset representatives of L in Λ. From Table 1 of [2], the normaliser of $T[L] = \langle u^8, u^4v^4\rangle$ in G is $H_1 = \langle u, v, \alpha, \beta\rangle \cong p4m$. Consider $S = \langle u^8, u^4v^4, u\alpha, (uv)^{-1}\alpha^3\beta\rangle \cong p4g$, where $u\alpha$ is a $90°$ counterclockwise rotation and $(uv)^{-1}\alpha^3\beta$ is a mirror reflection. In the following diagram, the location of $u\alpha$ is marked by a black square. The images of the generators of S are under f are

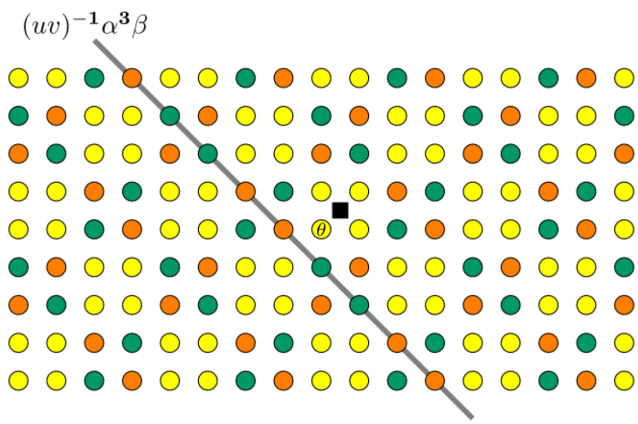
$$\begin{aligned} f(u^8) = f(u^4v^4) &= (\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]);\\ f(u\alpha) &= (\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right])\cdots(\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]\left[\begin{smallmatrix}7\\2\end{smallmatrix}\right]\left[\begin{smallmatrix}3\\3\end{smallmatrix}\right]\left[\begin{smallmatrix}6\\3\end{smallmatrix}\right])\cdots(\left[\begin{smallmatrix}7\\3\end{smallmatrix}\right]\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\left[\begin{smallmatrix}6\\2\end{smallmatrix}\right]\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right]);\\ f((uv)^{-1}\alpha^3\beta) &= (\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}3\\3\end{smallmatrix}\right])(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right])\cdots(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right])\cdots(\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right])\cdots(\left[\begin{smallmatrix}7\\3\end{smallmatrix}\right]\left[\begin{smallmatrix}4\\0\end{smallmatrix}\right]). \end{aligned}$$

The $f(S)$-orbits are

$$\{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}3\\3\end{smallmatrix}\right], \left[\begin{smallmatrix}6\\3\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\2\end{smallmatrix}\right], \left[\begin{smallmatrix}7\\2\end{smallmatrix}\right], \left[\begin{smallmatrix}3\\2\end{smallmatrix}\right], \left[\begin{smallmatrix}7\\3\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right], \left[\begin{smallmatrix}6\\2\end{smallmatrix}\right], \left[\begin{smallmatrix}4\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}4\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}5\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}5\\1\end{smallmatrix}\right]\};$$

$$\{\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right], \left[\begin{smallmatrix}7\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}4\\3\end{smallmatrix}\right], \left[\begin{smallmatrix}3\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\3\end{smallmatrix}\right], \left[\begin{smallmatrix}6\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]\};$$

$$\{\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right], \left[\begin{smallmatrix}6\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}4\\2\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\2\end{smallmatrix}\right], \left[\begin{smallmatrix}7\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}5\\3\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]\}.$$



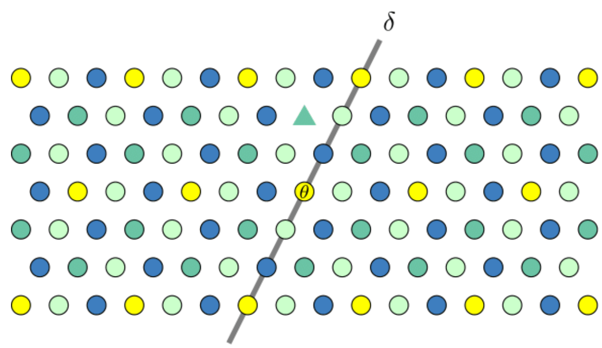
The subgroup of $f(H_1)$ that effects permutation of the $f(S)$-orbits is $\langle f(u^4), f(u^2v^2), f(u\alpha), f(uv\alpha^3\beta)\rangle \cong (\mathbb{Z}_2\times\mathbb{Z}_2)\rtimes D_4$. Thus, the colour symmetry groups are $H_2 = \langle u^4, u^2v^2, u\alpha, uv\alpha^3\beta\rangle \cong p4m$, $K_2 = \langle u^4, v^4, u\alpha, (uv)^{-1}\alpha^3\beta$, and $H_2/K_2 = \langle u^2v^2K_2\rangle \cong \mathbb{Z}_2$.

Example 2. Consider the sublattice $L = \left[\begin{smallmatrix}9&6\\0&3\end{smallmatrix}\right]$ of the hexagonal lattice Λ. See Illustration 2 for the complete set of coset representatives of L in Λ. From Table 2 of [2], the normaliser of $T[L] = \langle u^9, u^6v^3\rangle$ in G is $H_1 = \langle u, v, \gamma, \delta\rangle \cong p6m$. Let $S = \langle u^9, u^6v^3, u^3v^3\gamma^2, \delta\rangle \cong p31m$, where δ is a mirror reflection and $u^3v^3\gamma^2$ is a $120°$ counterclockwise rotation whose location is represented by a triangle in the diagram. The images of the generators of S under f are

$$\begin{aligned}
f(u^9) = f(u^6v^3) &= \left(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\right);\\
f(u^3v^3\gamma^2) &= \left(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}6\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right]\right)\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}6\\1\end{smallmatrix}\right]\left[\begin{smallmatrix}8\\2\end{smallmatrix}\right]\right)\cdots\left(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]\left[\begin{smallmatrix}4\\0\end{smallmatrix}\right]\right)\cdots\left(\left[\begin{smallmatrix}7\\2\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\left[\begin{smallmatrix}4\\2\end{smallmatrix}\right]\right);\\
f(\delta) &= \left(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\right)\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right)\cdots\left(\left[\begin{smallmatrix}8\\2\end{smallmatrix}\right]\right).
\end{aligned}$$

The complete list of $f(S)$-orbits is

$$\begin{aligned}
&\left\{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}6\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right]\right\}; &&\left\{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}6\\1\end{smallmatrix}\right],\left[\begin{smallmatrix}8\\2\end{smallmatrix}\right],\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right],\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right],\left[\begin{smallmatrix}4\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}7\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right],\left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]\right\};\\
&\left\{\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right],\left[\begin{smallmatrix}5\\1\end{smallmatrix}\right],\left[\begin{smallmatrix}8\\1\end{smallmatrix}\right],\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right],\left[\begin{smallmatrix}4\\2\end{smallmatrix}\right],\left[\begin{smallmatrix}7\\2\end{smallmatrix}\right]\right\}; &&\left\{\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}6\\2\end{smallmatrix}\right],\left[\begin{smallmatrix}7\\1\end{smallmatrix}\right],\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right],\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right],\left[\begin{smallmatrix}5\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}8\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right],\left[\begin{smallmatrix}4\\1\end{smallmatrix}\right]\right\}.
\end{aligned}$$



The subgroup of $f(H_1)$ that effects permutation of the $f(S)$-orbits is $\langle f(u^3), f(v^3), f(\gamma), f(\delta)\rangle \cong \mathbb{Z}_3 \rtimes D_6$. Thus $H_2 = \langle u^3, v^3, \gamma, \delta\rangle \cong p6m$, $K_2 = S$, and $H_2/K_2 = \langle u^3K_2, \gamma K_2\rangle \cong D_3$.

5 Conclusions and Recommendations

In this paper, we have determined groups arising for each S-orbit colouring of the square and the hexagonal lattices Λ induced by a given sublattice L and a subgroup S. These colours groups are the colour symmetry group H_2 whose elements are symmetries of the lattice which induce a permutation of the colours; and the colour-fixing group K_2 whose elements are symmetries of the lattice which fix all the colours in the lattice colouring. Theorems 3.1, 3.3, and 3.5 were formulated to determine translation generators of H_2 while Theorems 3.2 and 3.4 aided in identifying non-translation generators of H_2. We summarise in the end tables containing different subgroups S together with their associated colour groups. These tables are generalisations of Tables 1 and 2 in [2]. For sublattices L of small index in Λ, S-orbit colourings of the lattice Λ can be easily done to show that $K_2 \neq S$. But in most cases, $K_2 = S$ especially if S is not of type pg, pmg, pgg, and $p4g$.

For future studies, we want to verify whether or not $K_2 = S$. We believe that this is true if S is not of type pg, pmg, pgg, or $p4g$. But for subgroups of type pg, pmg, pgg, or $p4g$, they yield mixed results. We recommend finding methods of constructing subgroups of these types. These would involve the characterisation of their generators which are compatible to a given set of translation generators. Using these types of subgroup to induce colouring of Λ, they could produce analogous results to Theorems 3.1, 3.3, and 3.5.

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Appendix 1: Square Lattice

L	S	Cases		H	K
$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$	$\langle u^k, v^k, \alpha^2\rangle \cong p2$	k odd		$\langle u^k, v^k, \alpha, \beta\rangle \cong p4m$	S
		k even		$\langle u^{k/2}, v^{k/2}, \alpha, \beta\rangle \cong p4m$	S
$k \geq 3$	$\langle u^k, v^k, \beta\rangle \cong pm$	k odd		$\langle u, v^k, \beta, \alpha^2\beta\rangle \cong pmm$	S
		k even		$\langle u, v^{k/2}, \beta, \alpha^2\beta\rangle \cong pmm$	S
	$\langle u^k, v^k, \alpha\beta\rangle \cong cm$	k odd		$\langle (u^{-1}v)^k, uv, \alpha\beta, \alpha^3\beta\rangle \cong pmm$	S
		k even		$\langle (u^{-1}v)^{k/2}, uv, \alpha\beta, \alpha^3\beta\rangle \cong pmm$	S
	$\langle u^k, v^k, (uv)^{k/2}\beta\rangle \cong pg$	$k \geq 4$, even		$\langle u, v^{k/2}, \beta, \alpha^2\beta\rangle \cong pmm$	S
	$\langle u^k, v^k, \beta, \alpha^2\beta\rangle \cong pmm$	k odd		$\langle u^k, v^k, \alpha, \beta\rangle \cong p4m$	S
		k even		$\langle u^{k/2}, v^{k/2}, \alpha, \beta\rangle \cong p4m$	S
	$\langle u^k, v^k, \alpha\beta, \alpha^3\beta\rangle \cong cmm$	k odd		$\langle u^k, v^k, \alpha, \beta\rangle \cong p4m$	S
		k even		$\langle (u^{-1}v)^{k/2}, (uv)^{k/2}, \alpha, \beta\rangle \cong p4m$	S
	$\langle u^k, v^k, \alpha^2\beta, (uv)^{k/2}\beta\rangle \cong pmg$	$k \geq 4$, even		$\langle u^{k/2}, v^{k/2}, \beta, \alpha^2\beta\rangle \cong pmm$	S
	$\langle u^k, v^k, u^{k/2}v\beta, uv^{k/2}\alpha^2\beta\rangle \cong pgg$	$k \geq 4$, even		$\langle u^{k/2}, v^{k/2}, u\alpha, v\beta\rangle \cong p4m$	S
	$\langle u^k, v^k, \alpha\rangle \cong p4$	$k = 3$		$\langle u^3, v^3, \alpha, \beta\rangle \cong p4m$	(1)
		$k \geq 5$, odd		$\langle u^k, v^k, \alpha, \beta\rangle \cong p4m$	S
		$k = 4$		$\langle u^4, u^2v^2, \alpha, \beta\rangle \cong p4m$	(2)
		$k \geq 6$, even		$\langle u^k, u^{k/2}v^{k/2}, \alpha, \beta\rangle \cong p4m$	S
	$\langle u^k, v^k, \alpha, \beta\rangle \cong p4m$	k odd		S	S
		k even		$\langle u^k, u^{k/2}v^{k/2}, \alpha, \beta\rangle \cong p4m$	S
	$\langle u^k, v^k, u^{(k/2)-1}\alpha, (uv)^{-1}\alpha^3\beta\rangle \cong p4g$	$k \geq 4$, even		$\langle u^k, u^{k/2}v^{k/2}, u^{(k/2)-1}\alpha, \alpha\beta\rangle \cong p4m$	S
$k\begin{bmatrix} l & m \\ 0 & 1 \end{bmatrix}$	$\langle u^{kl}, u^{km}v^k, \alpha^2\rangle \cong p2$	$k\left[\begin{smallmatrix}\text{odd} & \text{odd}\\ 0 & 1\end{smallmatrix}\right]$ or $k\left[\begin{smallmatrix}\text{odd} & \text{even}\\ 0 & 1\end{smallmatrix}\right]$	k odd	$\langle u^{kl}, u^{km}v^k, \alpha\rangle \cong p4$	S
l divides m^2+1			k even	$\langle u^{kl/2}, u^{km/2}v^{k/2}, \alpha\rangle \cong p4$	S
		$k\left[\begin{smallmatrix}\text{even} & \text{odd}\\ 0 & 1\end{smallmatrix}\right]$	k odd	$\langle u^{kl/2}, u^{km}v^k, \alpha\rangle \cong p4$	S
			k even	$\langle u^{kl/2}, u^{km/2}v^{k/2}, \alpha\rangle \cong p4$	S
	$\langle u^{kl}, u^{km}v^k, \alpha\rangle \cong p4$	$k\left[\begin{smallmatrix}\text{odd} & \text{odd}\\ 0 & 1\end{smallmatrix}\right]$ or $k\left[\begin{smallmatrix}\text{odd} & \text{even}\\ 0 & 1\end{smallmatrix}\right]$		S	S
		$k\left[\begin{smallmatrix}\text{even} & \text{odd}\\ 0 & 1\end{smallmatrix}\right]$		$\langle u^{kl/2}, u^{km}v^k, \alpha\rangle \cong p4$	S

Notes:

(1) $\langle u^3, v^3, \alpha, \beta\rangle \cong p4m$

(2) $\langle u^4, v^4, \alpha, \beta\rangle \cong p4m$

L	S		H	K
$\begin{bmatrix}2k & k\\ 0 & k\end{bmatrix}$	$\langle u^{2k}, u^k v^k, \alpha^2\rangle \cong p2$	k odd	$\langle u^k, v^k, \alpha, \beta\rangle \cong p4m$	S
		k even	$\langle u^k, u^{k/2}v^{k/2}, \alpha, \beta\rangle \cong p4m$	S
$k \geq 3$	$\langle u^{2k}, u^k v^k, \beta\rangle \cong cm$		$\langle u, v^k, \beta, \alpha^2\beta\rangle \cong pmm$	S
	$\langle u^{2k}, u^k v^k, \alpha\beta\rangle \cong pm$	k odd	$\langle (u^{-1}v)^k, uv, \alpha\beta, \alpha^3\beta\rangle \cong pmm$	S
		k even	$\langle (u^{-1}v)k/2, uv, \alpha\beta, \alpha^3\beta\rangle \cong pmm$	S
	$\langle u^{2k}, u^k v^k, v^k\alpha\beta\rangle \cong pg$	$k = 2$	$\langle u^2, uv, \alpha\beta, \alpha^3\beta\rangle \cong pmm$	(1)
		k odd	$\langle (u^{-1}v)^k, uv, \alpha\beta, \alpha^3\beta\rangle \cong pmm$	S
		$k \geq 4$, even	$\langle (u^{-1}v)k/2, uv, \alpha\beta, \alpha^3\beta\rangle \cong pmm$	S
	$\langle u^{2k}, u^k v^k, \beta, \alpha^2\beta\rangle \cong cmm$		$\langle u^k, v^k, \alpha, \beta\rangle \cong p4m$	S
	$\langle u^{2k}, u^k v^k, \alpha\beta, \alpha^3\beta\rangle \cong pmm$	k odd	$\langle u^k, v^k, \alpha, \beta\rangle \cong p4m$	S
		k even	$\langle (uv)^{k/2}, (u^{-1}v)^{k/2}v^{k/2}, \alpha, \beta\rangle \cong p4m$	S
	$\langle u^{2k}, u^k v^k, \alpha^3\beta, v^k\alpha\beta\rangle \cong pmg$	k odd	$\langle u^k, v^k, \alpha\beta, \alpha^3\beta\rangle \cong pmm$	S
		k even	$\langle (uv)^{k/2}, (u^{-1}v)^{k/2}, \alpha\beta, \alpha^3\beta\rangle \cong pmm$	S
	$\langle u^{2k}, u^k v^k, uv\alpha^3\beta, v^k\alpha\beta\rangle \cong pmg$	$k = 2$	$\langle u^2, uv, \alpha\beta, \alpha^3\beta\rangle \cong pmm$	(2)
		k odd	$\langle u^k, v^k, \alpha\beta, uv\alpha^3\beta\rangle \cong pmm$	S
		$k \geq 4$, even	$\langle (uv)^{k/2}, (u^{-1}v)^{k/2}, \alpha\beta, uv\alpha^3\beta\rangle \cong pmm$	S
	$\langle u^{2k}, u^k v^k, v^k\alpha\beta, v^k\alpha^3\beta\rangle \cong pgg$	$k = 2$	$\langle u^2, uv, \alpha, \beta\rangle \cong p4m$	(3)
		k odd	$\langle u^k, v^k, \alpha, \beta\rangle \cong p4m$	S
		$k \geq 4$, even	$\langle (uv)^{k/2}, (u^{-1}v)^{k/2}, \alpha, \beta\rangle \cong p4m$	S
	$\langle u^{2k}, u^k v^k, \alpha\rangle \cong p4$	$k = 2, 3$	$\langle u^k, v^k, \alpha, \beta\rangle \cong p4m$	(4)
		$k \geq 4$	$\langle u^k, v^k, \alpha, \beta\rangle \cong p4m$	S
	$\langle u^{2k}, u^k v^k, \alpha, \beta\rangle \cong p4m$		$\langle u^k, v^k, \alpha, \beta\rangle \cong p4m$	S
	$\langle u^{2k}, u^k v^k, u^k\alpha, \alpha^2\beta\rangle \cong p4g$	$k = 2$	$\langle u^2, uv, \alpha, \beta\rangle \cong p4m$	(3)
		$k \geq 3$	$\langle u^k, v^k, \alpha, \beta\rangle \cong p4m$	S
	$\langle u^{2k}, u^k v^k, u^{(k/2)-1}\alpha, (uv)^{-1}\alpha^3\beta\rangle \cong p4g$	k even	$\langle u^k, u^{k/2}v^{k/2}, u^{(k/2)-1}\alpha, \alpha\beta\rangle \cong p4m$	(5)
	$\langle u^{2k}, u^k v^k, u\alpha, (uv)^{-(k/2)+1}\alpha^3\beta\rangle \cong p4g$	k even	$\langle u^k, u^{k/2}v^{k/2}, u\alpha, \alpha\beta\rangle \cong p4m$	(6)
$\begin{bmatrix}2 & 1\\ 0 & k\end{bmatrix}$	$\langle u^2, uv^k, \alpha^2\rangle \cong p2$		$\langle u, v^k, \beta, \alpha^2\beta\rangle \cong pmm$	(7)
$k \geq 2$	$\langle u^2, uv^k, \beta\rangle \cong cm$		$\langle u, v^k, \beta, \alpha^2\beta\rangle \cong pmm$	(7)
$\begin{bmatrix}2j & j\\ 0 & 1\end{bmatrix}$	$\langle u^{2j}, u^j v, \alpha^2\rangle \cong p2$		$\langle u^j, v, \beta, \alpha^2\beta\rangle \cong pmm$	(8)
$j \geq 2$	$\langle u^{2j}, u^j v, \alpha^2\beta\rangle \cong cm$		$\langle u^j, v, \beta, \alpha^2\beta\rangle \cong pmm$	(8)
$\begin{bmatrix}2j & j\\ 0 & k\end{bmatrix}$	$\langle u^{2j}, u^j v^k, \alpha^2\rangle \cong p2$	j, k even	$\langle u^j, u^{j/2}v^{k/2}, \beta, \alpha^2\beta\rangle \cong pmm$	S
		otherwise	$\langle u^j, v^k, \beta, \alpha^2\beta\rangle \cong pmm$	S
$j, k \geq 2$	$\langle u^{2j}, u^j v^k, \beta\rangle \cong cm$		$\langle u, v^k, \beta, \alpha^2\beta\rangle \cong pmm$	S
$j \neq k$	$\langle u^{2j}, u^j v^k, \alpha^2\beta\rangle \cong cm$		$\langle u^j, v, \beta, \alpha^2\beta\rangle \cong pmm$	S
	$\langle u^{2k}, u^j v^k, \beta, \alpha^2\beta\rangle \cong pmm$		$\langle u^j, v^k, \beta, \alpha^2\beta\rangle \cong pmm$	S

Notes:

(1) $\langle u^4, u^2v^2, \alpha^3\beta, v^2\alpha\beta\rangle \cong pmg$

(2) $\langle u^4, uv, u^{-1}v\alpha\beta, \alpha^3\beta\rangle \cong pmm$

(3) $\langle u^2, v^2, \alpha, \beta\rangle \cong p4m$

(4) $\langle u^{2k}, u^k v^k, \alpha, \beta\rangle \cong p4m$

(5) $\langle u^k, v^k, u^{(k/2)-1}\alpha, (uv)^{-1}\alpha^3\beta\rangle \cong p4g$

(6) $\langle u^k, v^k, u\alpha, (uv)^{-(k/2)+1}\alpha^3\beta\rangle \cong p4g$

(7) $\langle u^2, uv^k, \beta, \alpha^2\beta\rangle \cong pmm$

(8) $\langle u^{2j}, u^j v, \beta, \alpha^2\beta\rangle \cong pmm$

L	Cases	S		H	K
$\begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}$		$\langle u^i, v, \alpha^2 \rangle \cong p2$	i odd	$\langle u^i, v, \beta, \alpha^2\beta \rangle \cong pmm$	H
			i even	$\langle u^{i/2}, v, \beta, \alpha^2\beta \rangle \cong pmm$	(1)
$i \geq 3$		$\langle u^i, v, \alpha^2\beta \rangle \cong pm$	i odd	$\langle u^i, v, \beta, \alpha^2\beta \rangle \cong pmm$	H
			i even	$\langle u^{i/2}, v, \beta, \alpha^2\beta \rangle \cong pmm$	(1)
$\begin{bmatrix} i & 0 \\ 0 & 2 \end{bmatrix}$		$\langle u^i, v^2, \alpha^2 \rangle \cong p2$	i odd	$\langle u^i, v, \beta, \alpha^2\beta \rangle \cong pmm$	(2)
			i even	$\langle u^{i/2}, v, \beta, \alpha^2\beta \rangle \cong pmm$	(2)
$i \geq 3$		$\langle u^i, v^2, \alpha^2\beta \rangle \cong pm$	i odd	$\langle u^i, v, \beta, \alpha^2\beta \rangle \cong pmm$	(2)
			i even	$\langle u^{i/2}, v, \beta, \alpha^2\beta \rangle \cong pmm$	(2)
		$\langle u^i, v^2, v\alpha^2\beta \rangle \cong pg$	i odd	$\langle u^i, v, \beta, u\alpha^2\beta \rangle \cong pmm$	(3)
			i even	$\langle u^{i/2}, v, \beta, \alpha^2\beta \rangle \cong pmm$	(3)
$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$		$\langle u, v^k, \alpha^2 \rangle \cong p2$	k odd	$\langle u, v^k, \beta, \alpha^2\beta \rangle \cong pmm$	(4)
			k even	$\langle u, v^{k/2}, \beta, \alpha^2\beta \rangle \cong pmm$	(4)
$k \geq 3$		$\langle u, v^k, \beta \rangle \cong pm$	k odd	$\langle u, v^k, \beta, \alpha^2\beta \rangle \cong pmm$	(4)
			k even	$\langle u, v^{k/2}, \beta, \alpha^2\beta \rangle \cong pmm$	(4)
$\begin{bmatrix} 2 & 0 \\ 0 & k \end{bmatrix}$		$\langle u^2, v^k, \alpha^2 \rangle \cong p2$	k odd	$\langle u, v^k, \beta, \alpha^2\beta \rangle \cong pmm$	(5)
			k even	$\langle u, v^{k/2}, \beta, \alpha^2\beta \rangle \cong pmm$	(5)
$k \geq 3$		$\langle u^2, v^k, \beta \rangle \cong pm$	k odd	$\langle u, v^k, \beta, \alpha^2\beta \rangle \cong pmm$	(5)
			k even	$\langle u, v^{k/2}, \beta, \alpha^2\beta \rangle \cong pmm$	(5)
		$\langle u^2, v^k, u\beta \rangle \cong pg$	k odd	$\langle u, v^k, \alpha^2\beta, \beta \rangle \cong pmm$	(6)
			k even	$\langle u, v^{k/2}, \alpha^2\beta, \beta \rangle \cong pmm$	(6)
$\begin{bmatrix} i & 0 \\ 0 & k \end{bmatrix}$		$\langle u^i, v^k, \alpha^2 \rangle \cong p2$		$\langle u^i, v^{k/2}, \beta, \alpha^2\beta \rangle \cong pmm$	S
		$\langle u^i, v^k, \beta \rangle \cong pm$		$\langle u, v^{k/2}, \beta, \alpha^2\beta \rangle \cong pmm$	S
$i, k \geq 3$, $i \neq k$	$\begin{bmatrix} \text{odd} & 0 \\ 0 & \text{even} \end{bmatrix}$	$\langle u^i, v^k, \alpha^2\beta \rangle \cong pm$		$\langle u^i, v, \beta, \alpha^2\beta \rangle \cong pmm$	S
		$\langle u^i, v^k, \beta, \alpha^2\beta \rangle \cong pmm$		$\langle u^i, v^{k/2}, \beta, \alpha^2\beta \rangle \cong pmm$	S
		$\langle u^i, v^k, v^{k/2}\beta, v^{k/2}\alpha^2\beta \rangle \cong pmg$		$\langle u^i, v^{k/2}, \beta, \alpha^2\beta \rangle \cong pmm$	S
		$\langle u^i, v^k, \alpha^2 \rangle \cong p2$		$\langle u^{i/2}, v^k, \beta, \alpha^2\beta \rangle \cong pmm$	S
		$\langle u^i, v^k, \beta \rangle \cong pm$		$\langle u, v^k, \beta, \alpha^2\beta \rangle \cong pmm$	S
		$\langle u^i, v^k, \alpha^2\beta \rangle \cong pm$		$\langle u^{i/2}, v, \beta, \alpha^2\beta \rangle \cong pmm$	S
	$\begin{bmatrix} \text{even} & 0 \\ 0 & \text{odd} \end{bmatrix}$	$\langle u^i, v^k, u^{i/2}\beta \rangle \cong pg$		$\langle u, v^k, \beta, \alpha^2\beta \rangle \cong pmm$	S
		$\langle u^i, v^k, \beta, \alpha^2\beta \rangle \cong pmm$		$\langle u^{i/2}, v^k, \beta, \alpha^2\beta \rangle \cong pmm$	S
		$\langle u^i, v^k, u^{i/2}\alpha^2\beta, u^{i/2}\beta \rangle \cong pmg$		$\langle u^{i/2}, v^k, \beta, \alpha^2\beta \rangle \cong pmm$	S
		$\langle u^i, v^k, \alpha^2 \rangle \cong p2$	i, k odd	$\langle u^i, v^k, \beta, \alpha^2\beta \rangle \cong pmm$	S
			i, k even	$\langle u^{i/2}, v^{k/2}, \beta, \alpha^2\beta \rangle \cong pmm$	S
		$\langle u^i, v^k, \beta \rangle \cong pm$	i, k odd	$\langle u, v^k, \beta, \alpha^2\beta \rangle \cong pmm$	S
			i, k even	$\langle u, v^{k/2}, \beta, \alpha^2\beta \rangle \cong pmm$	S
		$\langle u^i, v^k, \alpha^2\beta \rangle \cong pm$	i, k odd	$\langle u^i, v, \beta, \alpha^2\beta \rangle \cong pmm$	S
			i, k even	$\langle u^{i/2}, v, \beta, \alpha^2\beta \rangle \cong pmm$	S
		$\langle u^i, v^k, \beta, \alpha^2\beta \rangle \cong pmm$	i, k odd	S	S
			i, k even	$\langle u^{i/2}, v^{k/2}, \beta, \alpha^2\beta \rangle \cong pmm$	S
		$\langle u^i, v^k, u^{i/2}\beta \rangle \cong pg$	i, k even	$\langle u, v^{k/2}, \beta, \alpha^2\beta \rangle \cong pmm$	S
		$\langle u^i, v^k, v^{k/2}\alpha^2\beta \rangle \cong pg$	i, k even	$\langle u^{i/2}, v, \beta, \alpha^2\beta \rangle \cong pmm$	S
		$\langle u^i, v^k, \beta, u^{i/2}v^{k/2}\alpha^2\beta \rangle \cong pmg$	i, k even	$\langle u^{i/2}, v^{k/2}, \beta, \alpha^2\beta \rangle \cong pmm$	S
		$\langle u^i, v^k, u^{i/2}v\beta, uv^{k/2}\alpha^2\beta \rangle \cong pgg$	i, k even	$\langle u^{i/2}, v^{k/2}, v\beta, u\alpha^2\beta \rangle \cong pmm$	S

Note: For (1)-(6), see notes in the next table.

L	Cases	Subcases	S		H	K
$\begin{bmatrix} i & 1 \\ 0 & 1 \end{bmatrix}$ $i \geq 3$, odd			$\langle u^i, uv, \alpha^2 \rangle \cong p2$		$\langle u^i, uv, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	H
			$\langle u^i, uv, \alpha\beta \rangle \cong cm$		$\langle u^i, uv, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	H
$\begin{bmatrix} i & 1 \\ 0 & 1 \end{bmatrix}$ $i \geq 4$, even			$\langle u^i, uv, \alpha^2 \rangle \cong p2$		$\langle u^{i/2}, uv, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	(7)
			$\langle u^i, uv, \alpha\beta \rangle \cong pm$		$\langle u^{i/2}, uv, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	(7)
			$\langle u^i, uv, v\alpha\beta \rangle \cong pg$		$\langle u^{i/2}, uv, \alpha^3\beta, v\alpha\beta \rangle \cong pmg$	(8)
$\begin{bmatrix} i & i-1 \\ 0 & 1 \end{bmatrix}$ $i \geq 3$, odd			$\langle u^i, u^{i-1}v, \alpha^2 \rangle \cong p2$		$\langle u^i, u^{i-1}v, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	H
			$\langle u^i, u^{i-1}v, \alpha^3\beta \rangle \cong cm$		$\langle u^i, u^{i-1}v, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	H
$\begin{bmatrix} i & i-1 \\ 0 & 1 \end{bmatrix}$ $i \geq 4$, even			$\langle u^i, u^{i-1}v, \alpha^2 \rangle \cong p2$		$\langle u^{i/2}, u^{i-1}v, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	(9)
			$\langle u^i, u^{i-1}v, \alpha^3\beta \rangle \cong pm$		$\langle u^{i/2}, u^{i-1}v, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	(9)
			$\langle u^i, u^{i-1}v, v\alpha^3\beta \rangle \cong pg$		$\langle u^{i/2}, u^{i-1}v, \alpha\beta, v\alpha^3\beta \rangle \cong pmg$	(10)
$k\begin{bmatrix} l & m \\ 0 & 1 \end{bmatrix}$ $l \mid (m^2-1)$	k even		$\langle u^{kl}, u^{km}v^k, \alpha^2 \rangle \cong p2$		$\langle u^{kl/2}, u^{km/2}v^{k/2}, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \alpha\beta \rangle \cong cm$		$\langle u^{k\sigma}, uv, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \alpha^3\beta \rangle \cong cm$		$\langle u^{k\rho}, u^{-1}v, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \alpha\beta, \alpha^3\beta \rangle \cong cmm$		$\langle u^{k\mu}, u^{km/2}v^{k/2}, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	S
	k odd	$k\left[\begin{smallmatrix} \text{odd} & \text{odd} \\ 0 & 1 \end{smallmatrix}\right]$ or $k\left[\begin{smallmatrix} \text{odd} & \text{even} \\ 0 & 1 \end{smallmatrix}\right]$	$\langle u^{kl}, u^{km}v^k, \alpha^2 \rangle \cong p2$		$\langle u^{kl}, u^{km}v^k, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \alpha\beta \rangle \cong cm$		$\langle u^{k\sigma}, uv, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \alpha^3\beta \rangle \cong cm$		$\langle u^{k\rho}, u^{-1}v, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \alpha\beta, \alpha^3\beta \rangle \cong cmm$		S	S
		$k\left[\begin{smallmatrix} \text{even} & \text{odd} \\ 0 & 1 \end{smallmatrix}\right]$	$\langle u^{kl}, u^{km}v^k, \alpha^2 \rangle \cong p2$		$\langle u^{kl/2}, u^{km}v^k, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \alpha\beta \rangle \cong cm$	σ even	$\langle u^{k\sigma/2}, uv, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	S
				otherwise	$\langle u^{k\sigma}, uv, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \alpha^3\beta \rangle \cong cm$	ρ even	$\langle u^{k\rho/2}, u^{kl-1}v, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	S
				otherwise	$\langle u^{k\rho}, u^{kl-1}v, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	ρ or σ is even	$\langle u^{k\mu}, u^{km}v^k, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	S
				ρ, σ are even	$\langle u^{k\mu/2}, u^{km}v^k, \alpha\beta, \alpha^3\beta \rangle \cong cmm$	S
				otherwise	S	S

Notes: $\rho = \dfrac{l}{gcd(l, l-m+1)}$, $\sigma = \dfrac{l}{gcd(l, l-m-1)}$, $\mu = lcm(\rho, \sigma)$.

(1) $\langle u^i, v, \beta, \alpha^2\beta \rangle \cong pmm$

(2) $\langle u^i, v^2, \beta, \alpha^2\beta \rangle \cong pmm$

(3) $\langle u^i, v^2, \beta, v\alpha^2\beta \rangle \cong pmg$

(4) $\langle u, v^k, \beta, \alpha^2\beta \rangle \cong pmm$

(5) $\langle u^2, v^k, \beta, \alpha^2\beta \rangle \cong pmm$

(6) $\langle u^2, v^k, \alpha^2\beta, u\beta \rangle \cong pmg$

(7) $\langle u^i, uv, \alpha\beta, \alpha^3\beta \rangle \cong cmm$

(8) $\langle u^i, uv, \alpha^3\beta, v\alpha\beta \rangle \cong pmg$

(9) $\langle u^i, u^{i-1}v, \alpha\beta, \alpha^3\beta \rangle \cong cmm$

(10) $\langle u^i, u^{i-1}v, \alpha\beta, v\alpha^3\beta \rangle \cong pmg$

Appendix 2: Hexagonal Lattice

L	S		H	K
$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$	$\langle u^k, v^k, \gamma^3 \rangle \cong p2$	k odd	$\langle u^k, v^k, \gamma, \delta \rangle \cong p6m$	S
		k even	$\langle u^{k/2}, v^{k/2}, \gamma, \delta \rangle \cong p6m$	S
$k \geq 3$	$\langle u^k, v^k, \delta \rangle \cong cm$	k odd	$\langle (u^{-1}v)^k, uv, \delta, \gamma^3\delta \rangle \cong cmm$	S
		k even	$\langle (u^{-1}v)^{k/2}, uv, \delta, \gamma^3\delta \rangle \cong cmm$	S
	$\langle u^k, v^k, \delta, \gamma^3\delta \rangle \cong cmm$	k odd	S	S
		k even	$\langle (uv)^{k/2}, (u^{-1}v)^{k/2}, \delta, \gamma^3\delta \rangle \cong cmm$	S
	$\langle u^k, v^k, \gamma^2 \rangle \cong p3$	$k = 3$	$\langle u^3, v^3, \gamma, \delta \rangle \cong p6m$	(1)
		$k \geq 4$	$\langle u^k, v^k, \gamma, \delta \rangle \cong p6m$	S
	$\langle u^k, v^k, \gamma^2, \delta \rangle \cong p31m$		$\langle u^k, v^k, \gamma, \delta \rangle \cong p6m$	S
	$\langle u^k, v^k, \gamma^2, \gamma\delta \rangle \cong p3m1$	$k = 3$	$\langle u^3, v^3, \gamma, \delta \rangle \cong p6m$	(2)
		$k \geq 4$	$\langle u^k, v^k, \gamma, \delta \rangle \cong p6m$	S
	$\langle u^k, v^k, \gamma \rangle \cong p6$	$k \leq 5$	$\langle u^k, v^k, \gamma, \delta \rangle \cong p6m$	(3)
		$k > 5$	$\langle u^k, v^k, \gamma, \delta \rangle \cong p6m$	S
	$\langle u^k, v^k, \gamma, \delta \rangle \cong p6m$		S	S
$\begin{bmatrix} 3k & 2k \\ 0 & k \end{bmatrix}$	$\langle u^{3k}, u^{2k}v^k, \gamma^3 \rangle \cong p2$	k odd	$\langle u^{3k}, u^{2k}v^k, \gamma, \delta \rangle \cong p6m$	S
		k even	$\langle u^{3k/2}, u^k v^{k/2}, \gamma, \delta \rangle \cong p6m$	S
$k \geq 2$	$\langle u^{3k}, u^{2k}v^k, \delta \rangle \cong cm$	k odd	$\langle (u^{-1}v)^k, uv, \delta, \gamma^3\delta \rangle \cong cmm$	S
		k even	$\langle (u^{-1}v)^{k/2}, uv, \delta, \gamma^3\delta \rangle \cong cmm$	S
	$\langle u^{3k}, u^{2k}v^k, \delta, \gamma^3\delta \rangle \cong cmm$	k odd	S	S
		k even	$\langle (uv)^{k/2}, (u^{-1}v)^{k/2}, \delta, \gamma^3\delta \rangle \cong cmm$	S
	$\langle u^{3k}, u^{2k}v^k, \gamma^2 \rangle \cong p3$	$k = 2$	$\langle u^2, v^2, \gamma, \delta \rangle \cong p6m$	(4)
		$k \geq 3$	$\langle u^k, v^k, \gamma, \delta \rangle \cong p6m$	S
	$\langle u^{3k}, u^{2k}v^k, \gamma^2, \delta \rangle \cong p3m1$		$\langle u^k, v^k, \gamma, \delta \rangle \cong p6m$	S
	$\langle u^{3k}, u^{2k}v^k, u^k v^k \gamma^2, \delta \rangle \cong p31m$	$k = 2$	$\langle u^2, v^2, \gamma, \delta \rangle \cong p6m$	(5)
		$k \geq 3$	$\langle u^k, v^k, \gamma, \delta \rangle \cong p6m$	S
	$\langle u^{3k}, u^{2k}v^k, \gamma \rangle \cong p6$	$k = 2, 3$	$\langle u^{3k}, u^{2k}v^k, \gamma, \delta \rangle \cong p6m$	(6)
		$k \geq 4$	$\langle u^{3k}, u^{2k}v^k, \gamma, \delta \rangle \cong p6m$	S
	$\langle u^{3k}, u^{2k}v^k, \gamma, \delta \rangle \cong p6m$		S	S
$k\begin{bmatrix} l & m \\ 0 & 1 \end{bmatrix}$	$\langle u^{kl}, u^{km}v^k, \gamma^3 \rangle \cong p2$	k odd	$\langle u^{kl}, u^{km}v^k, \gamma \rangle \cong p6$	S
		k even	$\langle u^{kl/2}, u^{km/2}v^{k/2}, \gamma \rangle \cong p6$	S
l divides $m^2 - m + 1$	$\langle u^{kl}, u^{km}v^k, \gamma^2 \rangle \cong p3$		$\langle u^{kl}, u^{km}v^k, \gamma \rangle \cong p6$	S
	$\langle u^{kl}, u^{km}v^k, \gamma \rangle \cong p6$		S	S

Notes:

(1) $\langle u^3, v^3, \gamma^2, \delta \rangle \cong p31m$

(2) $\langle u^3, v^3, \gamma, \delta \rangle \cong p6m$

(3) $\langle u^k, v^k, \gamma, \delta \rangle \cong p6m$

(4) $\langle u^6, u^4v^2, \gamma^2, \delta \rangle \cong p3m1$

(5) $\langle u^2, v^2, \gamma, \delta \rangle \cong p6m$

(6) $\langle u^{3k}, u^{2k}v^k, \gamma, \delta \rangle \cong p6m$

L	Cases	Subcases	S		H	K
$\begin{bmatrix} i & 1 \\ 0 & 1 \end{bmatrix}$ $i \geq 3$, odd			$\langle u^i, uv, \gamma^3 \rangle \cong p2$		$\langle u^i, uv, \delta, \gamma^3\delta \rangle \cong cmm$	H
			$\langle u^i, uv, \delta \rangle \cong cm$		$\langle u^i, uv, \delta, \gamma^3\delta \rangle \cong cmm$	H
$\begin{bmatrix} i & 1 \\ 0 & 1 \end{bmatrix}$ $i \geq 4$, even			$\langle u^i, uv, \gamma^3 \rangle \cong p2$		$\langle u^{i/2}, uv, \delta, \gamma^3\delta \rangle \cong cmm$	(1)
			$\langle u^i, uv, \delta \rangle \cong pm$		$\langle u^{i/2}, uv, \delta, \gamma^3\delta \rangle \cong cmm$	(1)
			$\langle u^i, uv, v\delta \rangle \cong pg$		$\langle u^{i/2}, uv, \gamma^3\delta, v\delta \rangle \cong pmg$	(2)
$\begin{bmatrix} i & i-1 \\ 0 & 1 \end{bmatrix}$ $i \geq 5$, odd			$\langle u^i, u^{i-1}v, \gamma^3 \rangle \cong p2$		$\langle u^i, u^{i-1}v, \delta, \gamma^3\delta \rangle \cong cmm$	H
			$\langle u^i, u^{i-1}v, \gamma^3\delta \rangle \cong cm$		$\langle u^i, u^{i-1}v, \delta, \gamma^3\delta \rangle \cong cmm$	H
$\begin{bmatrix} i & i-1 \\ 0 & 1 \end{bmatrix}$ $i \geq 4$, even			$\langle u^i, u^{i-1}v, \gamma^3 \rangle \cong p2$		$\langle u^{i/2}, u^{i-1}v, \delta, \gamma^3\delta \rangle \cong cmm$	(3)
			$\langle u^i, u^{i-1}v, \gamma^3\delta \rangle \cong pm$		$\langle u^{i/2}, u^{i-1}v, \delta, \gamma^3\delta \rangle \cong cmm$	(3)
			$\langle u^i, u^{i-1}v, v\gamma^3\delta \rangle \cong pg$		$\langle u^{i/2}, u^{i-1}v, \delta, v\gamma^3\delta \rangle \cong pmg$	(4)
$k\begin{bmatrix} l & m \\ 0 & 1 \end{bmatrix}$ $l \mid (m^2-1)$	k even		$\langle u^{kl}, u^{km}v^k, \gamma^3 \rangle \cong p2$		$\langle u^{kl/2}, u^{km/2}v^{k/2}, \delta, \gamma^3\delta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \delta \rangle \cong cm$		$\langle u^{k\sigma}, uv, \delta, \gamma^3\delta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \gamma^3\delta \rangle \cong cm$		$\langle u^{k\rho}, u^{kl-1}v, \delta, \gamma^3\delta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \delta, \gamma^3\delta \rangle \cong cmm$		$\langle u^{k\mu}, u^{km/2}v^{k/2}, \delta, \gamma^3\delta \rangle \cong cmm$	S
	k odd	$k\left[\begin{smallmatrix} \text{odd} & \text{odd} \\ 0 & 1 \end{smallmatrix}\right]$ or $k\left[\begin{smallmatrix} \text{odd} & \text{even} \\ 0 & 1 \end{smallmatrix}\right]$	$\langle u^{kl}, u^{km}v^k, \gamma^3 \rangle \cong p2$		$\langle u^{kl}, u^{km}v^k, \delta, \gamma^3\delta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \delta \rangle \cong cm$		$\langle u^{k\sigma}, uv, \delta, \gamma^3\delta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \gamma^3\delta \rangle \cong cm$		$\langle u^{k\rho}, u^{kl-1}v, \delta, \gamma^3\delta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \delta, \gamma^3\delta \rangle \cong cmm$		S	S
		$k\left[\begin{smallmatrix} \text{even} & \text{odd} \\ 0 & 1 \end{smallmatrix}\right]$	$\langle u^{kl}, u^{km}v^k, \gamma^3 \rangle \cong p2$		$\langle u^{kl/2}, u^{km}v^k, \delta, \gamma^3\delta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \delta \rangle \cong cm$	σ even	$\langle u^{k\sigma}, uv, \delta, \gamma^3\delta \rangle \cong cmm$	S
				otherwise	$\langle u^{k\sigma}, uv, \delta, \gamma^3\delta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \gamma^3\delta \rangle \cong cm$	ρ even	$\langle u^{k\rho}, u^{kl-1}v, \delta, \gamma^3\delta \rangle \cong cmm$	S
				otherwise	$\langle u^{k\rho}, u^{kl-1}v, \delta, \gamma^3\delta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \delta, \gamma^3\delta \rangle \cong cmm$	ρ or σ is even	$\langle u^{k\mu}, u^{km}v^k, \delta, \gamma^3\delta \rangle \cong cmm$	S
				ρ, σ even	$\langle u^{k\mu/2}, u^{km}v^k, \delta, \gamma^3\delta \rangle \cong cmm$	S
				otherwise	S	S

Notes: $\rho = \dfrac{l}{gcd(l, l-m+1)}, \ \sigma = \dfrac{l}{gcd(l, l-m-1)}, \ \mu = lcm(\rho, \sigma).$

(1) $\langle u^i, uv, \delta, \gamma^3\delta \rangle \cong cmm$

(2) $\langle u^i, uv, \gamma^3\delta, v\delta \rangle \cong pmg$

(3) $\langle u^i, u^{i-1}v, \delta, \gamma^3\delta \rangle \cong cmm$

(4) $\langle u^i, u^{i-1}v, \delta, v\gamma^3\delta \rangle \cong pmg$

L	Cases	Subcases	S		H	K
$\begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}$ $i \geq 3$, odd			$\langle u^i, v, \gamma^3 \rangle \cong p2$		$\langle u^i, v, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	H
			$\langle u^i, v, \gamma^2\delta \rangle \cong cm$		$\langle u^i, v, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	H
$\begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}$ $i \geq 4$, even			$\langle u^i, v, \gamma^3 \rangle \cong p2$		$\langle u^{i/2}, v, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	(1)
			$\langle u^i, v, \gamma^2\delta \rangle \cong pm$		$\langle u^{i/2}, v, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	(1)
			$\langle u^i, v, u\gamma^2\delta \rangle \cong pg$		$\langle u^{i/2}, v, \gamma^5\delta, u\gamma^2\delta \rangle \cong pmg$	(2)
$\begin{bmatrix} i & 2 \\ 0 & 1 \end{bmatrix}$ $i \geq 5$, odd			$\langle u^i, u^2v, \gamma^3 \rangle \cong p2$		$\langle u^i, u^2v, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	H
			$\langle u^i, u^2v, \gamma^2\delta \rangle \cong cm$		$\langle u^i, u^2v, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	H
$\begin{bmatrix} i & 2 \\ 0 & 1 \end{bmatrix}$ $i \geq 4$, even			$\langle u^i, u^2v, \gamma^3 \rangle \cong p2$		$\langle u^{i/2}, u^2v, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	(3)
			$\langle u^i, u^2v, \gamma^5\delta \rangle \cong pm$		$\langle u^{i/2}, u^2v, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	(3)
			$\langle u^i, u^2v, u\gamma^5\delta \rangle \cong pg$		$\langle u^{i/2}, u^2v, \gamma^2\delta, u\gamma^5\delta \rangle \cong pmg$	(4)
$k\begin{bmatrix} l & m \\ 0 & 1 \end{bmatrix}$ $l \mid (m^2-2m)$	k even		$\langle u^{kl}, u^{km}v^k, \gamma^3 \rangle \cong p2$		$\langle u^{kl/2}, u^{km/2}v^{k/2}, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \gamma^2\delta \rangle \cong cm$		$\langle u^{k\rho}, uv, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \gamma^5\delta \rangle \cong cm$		$\langle u^{k\sigma}, u^{kl-1}v, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$		$\langle u^{k\mu}, u^{km/2}v^{k/2}, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	S
	k odd	$k\left[\begin{smallmatrix} \text{odd} & \text{odd} \\ 0 & 1 \end{smallmatrix}\right]$ or $k\left[\begin{smallmatrix} \text{odd} & \text{even} \\ 0 & 1 \end{smallmatrix}\right]$	$\langle u^{kl}, u^{km}v^k, \gamma^3 \rangle \cong p2$		$\langle u^{kl}, u^{km}v^k, \gamma^2\delta, \gamma^5\delta \rangle \cong p2$	S
			$\langle u^{kl}, u^{km}v^k, \gamma^2\delta \rangle \cong cm$		$\langle u^{k\rho}, v, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \gamma^5\delta \rangle \cong cm$		$\langle u^{k\sigma}, u^2v, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$		S	S
		$k\left[\begin{smallmatrix} \text{even} & \text{even} \\ 0 & 1 \end{smallmatrix}\right]$	$\langle u^{kl}, u^{km}v^k, \gamma^3 \rangle \cong p2$		$\langle u^{kl/2}, u^{km}v^k, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \gamma^2\delta \rangle \cong cm$	ρ even	$\langle u^{k\rho}, v, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	S
				otherwise	$\langle u^{k\rho}, v, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \gamma^5\delta \rangle \cong cm$	σ even	$\langle u^{k\sigma}, u^2v, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	S
				otherwise	$\langle u^{ks}, u^2v, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	S
			$\langle u^{kl}, u^{km}v^k, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	ρ or σ is even	$\langle u^{k\mu}, u^{km}v^k, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	S
				ρ, σ even	$\langle u^{k\mu/2}, u^{km}v^k, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$	S
				otherwise	S	S

Notes: $\rho = \dfrac{l}{gcd(l, l-m+1)}$, $\sigma = \dfrac{l}{gcd(l, l-m-1)}$, $\mu = lcm(\rho, \sigma)$.

(1) $\langle u^i, v, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$

(2) $\langle u^i, v, \gamma^5\delta, u\gamma^2\delta \rangle \cong pmg$

(3) $\langle u^i, u^2v, \gamma^2\delta, \gamma^5\delta \rangle \cong cmm$

(4) $\langle u^i, u^2v, \gamma^2\delta, u\gamma^5\delta \rangle \cong pmg$

(5) $\langle u, v^i, \gamma\delta, \gamma^4\delta \rangle \cong cmm$

(6) $\langle u, v^i, \gamma\delta, v\gamma^4\delta \rangle \cong pmg$

(7) $\langle u^{i/2}, uv^2, \gamma\delta, \gamma^4\delta \rangle \cong cmm$

(8) $\langle u^{i/2}, uv^2, \gamma\delta, v\gamma^4\delta \rangle \cong pmg$

L	Cases	Subcases	S		H	K
$\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}$ $i \geq 3$, odd			$\langle u, v^i, \gamma^3\rangle \cong p2$		$\langle u, v^i, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	H
			$\langle u, v^i, \delta\rangle \cong cm$		$\langle u, v^i, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	H
$\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}$ $i \geq 4$, even			$\langle u, v^i, \gamma^3\rangle \cong p2$		$\langle u, v^{i/2}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	(5)
			$\langle u, v^i, \gamma^4\delta\rangle \cong pm$		$\langle u, v^{i/2}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	(5)
			$\langle u, v^i, v\gamma^4\delta\rangle \cong pg$		$\langle u, v^{i/2}, \gamma\delta, v\gamma^4\delta\rangle \cong pmg$	(6)
$\begin{bmatrix}i & (i+1)/2\\ 0 & 1\end{bmatrix}$ $i \geq 5$, odd			$\langle u^i, u^{(i+1)/2}v, \gamma^3\rangle \cong p2$		$\langle u^i, u^{(i+1)/2}v, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	H
			$\langle u^i, u^{(i+1)/2}v, \gamma\delta\rangle \cong cm$		$\langle u^i, u^{(i+1)/2}v, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	H
$\begin{bmatrix}i/2 & 1\\ 0 & 2\end{bmatrix}$ $i \geq 4$, even			$\langle u^{i/2}, uv^2, \gamma^3\rangle \cong p2$	$4 \mid i$	$\langle u^{i/4}, uv^2, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	(7)
				$4 \nmid i$	$\langle u^{(i+2)/4}v, uv^2, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	(7)
			$\langle u^{i/2}, uv^2, \gamma\delta\rangle \cong pm$	$4 \mid i$	$\langle u^{i/4}, uv^2, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	(7)
				$4 \nmid i$	$\langle u^{(i+2)/4}v, uv^2, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	(7)
			$\langle u^{i/2}, uv^2, v\gamma\delta\rangle \cong pg$	$4 \mid i$	$\langle u^{i/4}, uv^2, \gamma\delta, v\gamma^4\delta\rangle \cong pmg$	(8)
				$4 \nmid i$	$\langle u^{(i+2)/4}v, uv^2, \gamma\delta, v\gamma^4\delta\rangle \cong pmg$	(8)
$\begin{bmatrix}i & j\\ 0 & k\end{bmatrix}$ $i \mid (2j-k)$	$\begin{bmatrix}\text{odd} & \text{odd}\\ 0 & \text{odd}\end{bmatrix}$ or $\begin{bmatrix}\text{odd} & \text{even}\\ 0 & \text{odd}\end{bmatrix}$		$\langle u^i, u^jv^k, \gamma^3\rangle \cong p2$		$\langle u^i, u^jv^k, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
			$\langle u^i, u^jv^k, \gamma\delta\rangle \cong cm$		$\langle u^jv^k, uv^2, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
			$\langle u^i, u^jv^k, \gamma^4\delta\rangle \cong cm$		$\langle u, v^k, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
			$\langle u^i, u^jv^k, \gamma\delta, \gamma^4\delta\rangle \cong cmm$		S	S
	$\begin{bmatrix}\text{odd} & \text{odd}\\ 0 & \text{even}\end{bmatrix}$		$\langle u^i, u^jv^k, \gamma^3\rangle \cong p2$		$\langle u^i, u^{(i+j)/2}v^{k/2}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
			$\langle u^i, u^jv^k, \gamma\delta\rangle \cong cm$		$\langle u^{(i+j)/2}v^{k/2}, uv^2, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
			$\langle u^i, u^jv^k, \gamma^4\delta\rangle \cong cm$	$4 \mid k$	$\langle u, v^{k/2}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
				$4 \nmid k$	$\langle u, v^{k/2}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
			$\langle u^i, u^jv^k, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	$4 \mid k$	$\langle u^{(i+j)/2}v^{k/2}, (uv^2)^{k/4}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
				$4 \nmid k$	$\langle u^i, u^{(i+j)/2}v^{k/2}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
	$\begin{bmatrix}\text{odd} & \text{even}\\ 0 & \text{even}\end{bmatrix}$		$\langle u^i, u^jv^k, \gamma^3\rangle \cong p2$		$\langle u^i, u^{j/2}v^{k/2}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
			$\langle u^i, u^jv^k, \gamma\delta\rangle \cong cm$		$\langle u^{j/2}v^{k/2}, uv^2, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
			$\langle u^i, u^jv^k, \gamma^4\delta\rangle \cong cm$	$4 \mid k$	$\langle u, v^{k/2}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
				$4 \nmid k$	$\langle u, v^{k/2}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
			$\langle u^i, u^jv^k, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	$4 \mid k$	$\langle u^{j/2}v^{k/2}, (uv^2)^{k/4}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
				$4 \nmid k$	$\langle u^i, u^{j/2}v^{k/2}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
	$\begin{bmatrix}\text{even} & \text{odd}\\ 0 & \text{even}\end{bmatrix}$		$\langle u^i, u^jv^k, \gamma^3\rangle \cong p2$		$\langle u^{i/2}, u^jv^k, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
		$2i \mid (2j-k)$	$\langle u^i, u^jv^k, \gamma\delta\rangle \cong cm$		$\langle u^{i/2}, uv^2, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
			$\langle u^i, u^jv^k, \gamma^4\delta\rangle \cong cm$	$4 \mid k$	$\langle u, v^{k/2}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
				$4 \nmid k$	$\langle u, v^{k/2}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
			$\langle u^i, u^jv^k, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	$4 \mid k$	$\langle u^{i/2}, (uv^2)^{k/4}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
				$4 \nmid k$	$\langle u^{i/2}, u^jv^k, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
		$2i \nmid (2j-k)$	$\langle u^i, u^jv^k, \gamma\delta\rangle \cong cm$		$\langle u^{i/2}, uv^2, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
			$\langle u^i, u^jv^k, \gamma^4\delta\rangle \cong cm$		$\langle u, v^k, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
			$\langle u^i, u^jv^k, \gamma\delta, \gamma^4\delta\rangle \cong cmm$		$\langle u^{i/2}, (uv^2)^{k/2}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
	$\begin{bmatrix}\text{even} & \text{even}\\ 0 & \text{even}\end{bmatrix}$		$\langle u^i, u^jv^k, \gamma^3\rangle \cong p2$		$\langle u^{i/2}, u^{j/2}v^{k/2}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
		$2i \mid (2j-k)$	$\langle u^i, u^jv^k, \gamma\delta\rangle \cong cm$		$\langle u^{i/2}, uv^2, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
			$\langle u^i, u^jv^k, \gamma^4\delta\rangle \cong cm$	$4 \mid k$	$\langle u, v^{k/2}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
				$4 \nmid k$	$\langle u, v^{k/2}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
			$\langle u^i, u^jv^k, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	$4 \mid k$	$\langle u^{i/2}, (uv^2)^{k/4}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
				$4 \nmid k$	$\langle u^{i/2}, u^{j/2}v^{k/2}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
		$2i \nmid (2j-k)$	$\langle u^i, u^jv^k, \gamma\delta\rangle \cong cm$		$\langle u^{i/2}, uv^2, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
			$\langle u^i, u^jv^k, \gamma^4\delta\rangle \cong cm$		$\langle u, v^k, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S
			$\langle u^i, u^jv^k, \gamma\delta, \gamma^4\delta\rangle \cong cmm$		$\langle u^{i/2}, (uv^2)^{k/2}, \gamma\delta, \gamma^4\delta\rangle \cong cmm$	S

Note: For (5)-(8), see notes in the previous table.