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χ_S-skew symmetric and χ_S-orthogonal matrices

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Abstract

We determine the Jordan Canonical Form of a real matrix similar to its negative via a real skew-involution ($A^2 = -I$), and use this to (i) determine the Jordan Canonical Form of a real nonsingular matrix similar to its inverse via a real skew-involution, and (ii) give necessary and sufficient conditions for a real matrix to be a product of two real skew-involutions.

Keywords: χ_S-skew symmetric matrix, χ_S-orthogonal matrix, involution, skew-involution, primary matrix function
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1 Introduction

Let $M_n(\mathbb{F})$ be the set of all n-by-n matrices with entries in a field $\mathbb{F}$, and denote by $\mathrm{GL}_n(\mathbb{F})$ the set of nonsingular matrices in $M_n(\mathbb{F})$. Let $S \in \mathrm{GL}_n(\mathbb{F})$ and define the function $\chi_S : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ by

$$\chi_S(A) = S^{-1}AS, \text{ for all } A \in M_n(\mathbb{F}).$$

We say that $A \in M_n(\mathbb{F})$ is χ_S-*skew symmetric* if $\chi_S(A) = -A$; and χ_S-*orthogonal* if A is nonsingular and $\chi_S(A) = A^{-1}$. If $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $f(A)$ is a primary matrix function, then $\chi_S(f(A)) = f(\chi_S(A))$. In particular, if A is χ_S-skew symmetric, then $\chi_S\left(e^A\right) = e^{-A} = \left(e^A\right)^{-1}$, since e^A is a primary matrix function. Thus, if A is χ_S-skew symmetric, then e^A is χ_S-orthogonal.

Let $A \in \mathrm{GL}_n(\mathbb{C})$. It is known that there exists $X \in M_n(\mathbb{C})$ such that $e^X = A$ and X is a polynomial in A [6, Chapter 6]. Moreover, if A has no negative eigenvalues, then X is unique such that the spectrum of X lies in $\{z \in \mathbb{C} | -\pi < \mathrm{Im}\ z < \pi\}$ (we refer to this unique X as the *principal logarithm* of A and write $X = \log A$); $\log(A^{-1}) = -\log(A)$; and if $A \in \mathrm{GL}_n(\mathbb{R})$, then $X \in M_n(\mathbb{R})$ [4, Section 1.7 and 11.1]. If S is a real skew-involution, we determine the Jordan Canonical Form of a real χ_S-skew symmetric, and use the preceding to determine the Jordan Canonical Form of a real χ_S-orthogonal.

Suppose $A \in M_n(\mathbb{F})$. Wonenburger proved in 1966 that if the characteristic of $\mathbb{F}$ is not equal to 2, then A is a product of two involutions ($B^2 = I$) if and only if A is χ_S-orthogonal [10]. In the following year, Djoković gave a proof of the preceding result for an arbitrary field [3]. Observe that A is a product of two involutions if and only if A is χ_S-orthogonal for some involution $S \in \mathrm{GL}_n(\mathbb{F})$; and that A is a product of two skew-involutions if and only if A is χ_T-orthogonal for some skew-involution $T \in \mathrm{GL}_n(\mathbb{F})$. Hence A is χ_S-orthogonal for some $S \in \mathrm{GL}_n(\mathbb{F})$ if and only if A is χ_T-orthogonal for some involution $T \in \mathrm{GL}_n(\mathbb{F})$. Suppose x^2+1 has a root r in $\mathbb{F}$. Then A is a skew-involution if and only if rA is an involution. Hence $A = BC$, where B and C are skew-involutions, if and only if $A = (rB)(-rC)$ is a product of two involutions. Thus, if $x^2 + 1$ has a root r in $\mathbb{F}$, then, by the theorem of Wonenburger and Djoković, A is χ_S-orthogonal for some involution $S \in \mathrm{GL}_n(\mathbb{F})$ if and only if A is χ_T-orthogonal for some skew-involution $T \in \mathrm{GL}_n(\mathbb{F})$. We show that this is not the case when x^2+1 has no root in $\mathbb{F}$. In particular, we show that $A \in \mathrm{GL}_{2n}(\mathbb{R})$ is χ_S-orthogonal for some skew-involution $S \in \mathrm{GL}_n(\mathbb{R})$ if and only if A is similar to $Q \oplus Q^{-1}$ for some $Q \in \mathrm{GL}_n(\mathbb{R})$. It was shown in [8] that $A \in M_n(\mathbb{F})$ is a product of an involution and a skew-involution in $\mathrm{GL}_n(\mathbb{F})$ if and only if A is similar to $-A^{-1}$. Square matrices that are sums or products of two matrices satisfying quadratic polynomial equations are also studied in [2].

2 Preliminaries

If the polynomial $x^2 + 1$ has no root in $\mathbb{F}$ and $S \in \mathrm{GL}_n(\mathbb{F})$ is a skew-involution, then $n = 2k$ for some positive integer k and S is similar to $H_k := \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix}$, since the minimal polynomial of A is $x^2 + 1$, which is irreducible in $\mathbb{F}[x]$.

Lemma 2.1. *Let $\mathbb{F}$ be a field such that $x^2 + 1$ is irreducible in $\mathbb{F}[x]$. Let $A \in M_n(\mathbb{F})$ be given.*

(a) *If $n = 2k$, then A is similar to a χ_{H_k}-skew symmetric in $M_n(\mathbb{F})$ if and only if A is χ_S-skew symmetric for some skew-involution $S \in \mathrm{GL_n}(\mathbb{F})$.*

(b) *If $C \in M_{2n}(\mathbb{F})$ is similar to the direct sum $A \oplus -A$, then C is χ_S-skew symmetric for some skew-involution $S \in \mathrm{GL_{2n}}(\mathbb{F})$.*

(c) *If $n = 2k$, then A is similar to a χ_{H_k}-orthogonal if and only if A is χ_S-orthogonal for some skew-involution $S \in \mathrm{GL_n}(\mathbb{F})$.*

(d) *If A is nonsingular and $C \in \mathrm{GL_{2n}}(\mathbb{F})$ is similar to $A \oplus A^{-1}$, then C is χ_S-orthogonal for some skew-involution $S \in \mathrm{GL_{2n}}(\mathbb{F})$.*

(e) *If $n = 2k$ and $A \in \mathrm{GL_n}(\mathbb{R})$ is χ_S-orthogonal for some real skew-involution S such that A has no negative eigenvalues and $X = \log A$, then X is χ_S-skew symmetric.*

Proof: Let $A \in M_n(\mathbb{F})$. Suppose that $n = 2k$ and $P \in \mathrm{GL}_n(\mathbb{F})$. Then $(PAP^{-1})H_k = -H_k(PAP^{-1})$ if and only if $A(P^{-1}H_kP) = -\left(P^{-1}H_kP\right)A$. Hence A is similar to a χ_{H_k}-skew symmetric matrix if and only if A is χ_S-skew symmetric, where $S := P^{-1}H_kP$ is a skew-involution. This proves (a).

Let $B := A \oplus -A$. Observe that H_n is a skew-involution and $BH_n = -H_nB$, so that B is χ_{H_n}-skew symmetric. Hence (b) follows from (a).

The proofs for (c) and (d) are analogous to the proofs of (a) and (b), respectively.

Let $n = 2k$ and $A \in \mathrm{GL}_n(\mathbb{R})$ be χ_S-orthogonal for some real skew-involution S such that A has no negative eigenvalues. If $X = \log A$, then

$$e^{S^{-1}XS} = S^{-1}e^X S = S^{-1}AS = A^{-1}.$$

Since A has no negative eigenvalues, we have that A^{-1} has no negative eigenvalues, $\log A^{-1} = S^{-1}XS$, and $\chi_S(X) = -\log A = -X$. This proves (e).

If $A \in M_n(\mathbb{F})$, let $\sigma(A)$ denote the spectrum or the set of all eigenvalues of A.

Lemma 2.2. *Let $\mathbb{F}$ be a field such that x^2+1 is irreducible in $\mathbb{F}[x]$. Suppose $A := \oplus_{i=1}^k A_i \in M_{2n}(\mathbb{F})$, where each $A_i \in M_{2n_i}(\mathbb{F})$.*

(a) *If A_i is similar to a $\chi_{H_{n_i}}$-skew symmetric in $M_{2n_i}(\mathbb{F})$ for each i, then A is similar to a χ_{H_n}-skew symmetric in $M_{2n}(\mathbb{F})$.*

(b) *If $\sigma(A_i) \cap \sigma(-A_j) = \emptyset$ for all $i \neq j$ and A is χ_S-skew symmetric for some skew-involution $S \in \mathrm{GL}_{2n}(\mathbb{F})$, then each A_i is χ_{S_i}-skew symmetric for some skew-involution $S_i \in \mathrm{GL}_{2n_i}(\mathbb{F})$.*

(c) *If A_i is similar to a $\chi_{H_{n_i}}$-orthogonal in $\mathrm{GL}_{2n_i}(\mathbb{F})$ for each i, then A is similar to a χ_{H_n}-orthogonal in $\mathrm{GL}_{2n}(\mathbb{F})$.*

(d) *If $\sigma(A_i) \cap \sigma(A_j^{-1}) = \emptyset$ for all $i \neq j$ and A is χ_S-orthogonal for some skew-involution $S \in \mathrm{GL}_{2n}(\mathbb{F})$, then each A_i is χ_{S_i}-orthogonal for some skew-involution $S_i \in \mathrm{GL}_{2n_i}(\mathbb{F})$.*

Proof: Suppose $P_i \in \mathrm{GL}_{n_i}(\mathbb{F})$ such that $P_iA_iP_i^{-1}$ is $\chi_{H_{n_i}}$-skew symmetric for $i = 1, \ldots, k$. Then $P := \oplus_{i=1}^k P_i \in \mathrm{GL}_n(\mathbb{F})$, $K := \oplus_{i=1}^k H_{n_i}$ is a skew-involution, and PAP^{-1} is χ_K-skew symmetric. By Lemma 2.1(a), A is similar to a χ_{H_n}-skew symmetric in $M_{2n}(\mathbb{F})$. This proves (a).

To prove (b), suppose $AS = -SA$, for some skew-involution $S \in \mathrm{GL}_{2n}(\mathbb{F})$, and partition $S = [S_{ij}]$ conformal to A. Then $A_iS_{ij} = -S_{ij}A_j$ for all i, j. Since $\sigma(A_i) \cap \sigma(-A_j) = \emptyset$ for all $i \neq j$, we have $S_{ij} = 0$ for $i \neq j$, by Sylvester's theorem for linear matrix equations ([5], Theorem 2.4.4.1). Hence $S = \oplus_{i=1}^k S_{ii}$, each S_{ii} is a skew-involution, and each A_i is χ_{S_i}-skew symmetric.

The proof of (c) is analogous to that of (a) and follows from Lemma 2.1(c); while the proof of (d) is analogous to that of (b).

3 Jordan Canonical Form of a χ_H-skew symmetric

Let $S \in M_{2n}(\mathbb{R})$ be a skew-involution. We now take a look at the Jordan Canonical Form of a matrix which is similar to a χ_S-skew symmetric matrix. By Lemma 2.1(a) and (c), it suffices to consider $S = H_n$. For brevity, we simply write H when its size is clear from the context.

Suppose that $A \in M_{2n}(\mathbb{R})$ is similar to a real χ_{H_n}-skew symmetric matrix. Then there exists $P \in \mathrm{GL}_{2n}(\mathbb{R})$ such that PAP^{-1} is real χ_{H_n}-skew symmetric. If we partition $PAP^{-1} = [Q_{ij}]$ conformal to H_n, then $Q_{22} = -Q_{11}$ and $Q_{12} = Q_{21}$. Thus A is similar to a real χ_{H_n}-skew symmetric matrix if and only if A is similar to $\begin{bmatrix} Y & Z \\ Z & -Y \end{bmatrix}$ for some $Y, Z \in M_n(\mathbb{R})$.

Consider the matrix $V = \frac{1}{\sqrt{2}} \begin{bmatrix} -iI_n & -iI_n \\ I_n & -I_n \end{bmatrix}$. Note that $V^{-1} = V^*$ and that

$$V^{-1} \begin{bmatrix} Y & Z \\ Z & -Y \end{bmatrix} V = \begin{bmatrix} 0_n & Y - iZ \\ Y + iZ & 0_n \end{bmatrix} \equiv \begin{bmatrix} 0_n & \overline{B} \\ B & 0_n \end{bmatrix}.$$

That is, A is similar to $\begin{bmatrix} 0_n & \overline{B} \\ B & 0_n \end{bmatrix}$ for some $B \in M_n(\mathbb{C})$. This implies that A^2 is similar to $\overline{B}B \oplus B\overline{B}$. Let $p_A(x)$ denote the characteristic polynomial of A. It was proven in [9, Theorem 3] that if A, B, C, $D \in M_n(\mathbb{F})$ and $E = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2n}(\mathbb{F})$ such that $CD = DC$, then $\det E = \det(AD - BC)$. Since xI_n commutes with $-B$ and $-\overline{B}$, we have $p_A(x) = \det(xI_{2n} - A) = \det(x^2 I_n - B\overline{B}) = p_{B\overline{B}}(x^2)$, which is a polynomial with real coefficients. Thus, if λ is an eigenvalue of $B\overline{B}$, then $\sqrt{\lambda}$ and $-\sqrt{\lambda}$ are eigenvalues of A. Moreover, $B\overline{B}$ is similar to the square of a real matrix [6, Corollary 4.6.16]. In particular, if λ is a nonreal eigenvalue of $B\overline{B}$ and $J_k(\lambda)$ is in the Jordan Canonical Form of $B\overline{B}$, then $J_k(\overline{\lambda})$ is also in the Jordan Canonical Form of $B\overline{B}$; and if λ is a negative eigenvalue of $B\overline{B}$, then $J_k(\lambda) \oplus J_k(\lambda)$ is in the Jordan Canonical Form of $B\overline{B}$.

Suppose that λ is an eigenvalue of $B\overline{B}$ and let $\mu^2 = \lambda$. If λ is nonreal and $J_k(\lambda)$ is in the Jordan Canonical Form of $B\overline{B}$, then $J_k(\mu)$ or $J_k(-\mu)$ is in the Jordan Canonical Form of A, where $\mu = a + bi$ for some nonzero $a, b \in \mathbb{R}$. Without loss of generality, assume $J_k(\mu)$ is in the Jordan Canonical Form of A. Since A is similar to a real χ_{H_n}-skew symmetric matrix, then $J_k(\overline{\mu}) \oplus J_k(-\mu) \oplus J_k(-\overline{\mu})$ is also in the Jordan Canonical Form of A.

If λ is negative, then μ is pure imaginary. If $J_k(\lambda)$ is in the Jordan Canonical Form of $B\overline{B}$, then either $J_k(\mu)$ or $J_k(-\mu)$ is in the Jordan Canonical Form of A. Since A is similar to a real matrix and $\overline{\mu} = -\mu$, then both blocks $J_k(\mu)$ and $J_k(-\mu)$ are in the Jordan Canonical Form of A. Because the number of Jordan blocks $J_k(\lambda)$ in the Jordan Canonical Form of $\overline{B}B \oplus B\overline{B}$ is a multiple of four, and A is a real matrix which is similar to a χ_H-skew symmetric square root of $\overline{B}B \oplus B\overline{B}$, then $J_k(\mu) \oplus J_k(-\mu) \oplus J_k(\mu) \oplus J_k(-\mu)$ is in the Jordan Canonical Form of A.

Suppose that λ is positive and $J_k(\lambda)$ is in the Jordan Canonical Form of $B\overline{B}$. If $J_k(\mu)$ is in the Jordan Canonical Form of A, then $J_k(-\mu)$ is also in the Jordan Canonical Form of A since A is similar to a χ_{H_n}-skew symmetric matrix.

Observe that A^{2k} is similar to $(\overline{B}B)^k \oplus (B\overline{B})^k$, and A^{2k-1} is similar to

$$\begin{bmatrix} 0 & (\overline{B}B)^{k-1}\overline{B} \\ (B\overline{B})^{k-1}B & 0 \end{bmatrix}$$

for any positive integer k. Hence rank $\left(A^k\right)$ is even, for each positive integer k. This implies that the number of Jordan blocks $J_k(0)$ in the Jordan Canonical Form of A is even for any positive integer k.

To summarize, if $A \in M_{2n}(\mathbb{R})$ is similar to a real χ_{H_n}-skew symmetric matrix, then its Jordan Canonical Form consists of matrices of the following types: (i) $J_k(\mu) \oplus J_k(\overline{\mu}) \oplus J_k(-\mu) \oplus J_k(-\overline{\mu})$, for some nonreal μ, and positive integer k; or (ii) $J_k(\mu) \oplus J_k(-\mu)$, for some real μ, and positive integer k.

To show the converse, it is enough to show that a Jordan matrix of the form given in (i) or (ii) is similar to a χ_H-skew symmetric matrix, by Lemma 2.2(a). For Jordan matrices of type (i), $J_k(\mu) \oplus J_k(\overline{\mu})$ is similar to a real matrix C and $J_k(-\mu) \oplus J_k(-\overline{\mu})$ is similar to the real matrix $-C$. Hence a Jordan matrix of type (i) is similar to a real matrix $C \oplus -C$. For Jordan matrices of type (ii), since μ is real, we have $J_k(\mu) \oplus J_k(-\mu)$ is similar to the real matrix $J_k(\mu) \oplus -J_k(\mu)$, which is similar to a real χ_{H_k}-skew symmetric matrix by Lemma 2.1(b) and (a).

Theorem 3.1. *A 2k-by-2k real matrix is similar to a real χ_{H_k}-skew symmetric matrix if and only if its Jordan Canonical Form is a direct sum of matrices of types:*

(i) $J_m(\lambda) \oplus J_m(-\lambda) \oplus J_m(\overline{\lambda}) \oplus J_m(-\overline{\lambda})$, *for some nonreal λ, and positive integer m,*

(ii) $J_m(\lambda) \oplus J_m(-\lambda)$, *for some real* λ, *and positive integer* m.

4 Jordan Canonical Form of a χ_H-orthogonal

Suppose that $A \in \mathrm{GL}_{2n}(\mathbb{R})$ is similar to a real χ_{H_n}-orthogonal matrix. Since A is similar to its inverse, for each Jordan block $J_m(\lambda)$ in the Jordan Canonical Form of A, there is a corresponding $J_m(\lambda^{-1})$ in the Jordan Canonical Form of A. Since A is real, the Jordan blocks come in conjugate pairs. If λ is a nonreal scalar with $|\lambda| \neq 1$, then the four blocks $J_m(\lambda), J_m(\overline{\lambda}), J_m(\lambda^{-1})$ and $J_m(\overline{\lambda}^{-1})$ are pairwise different. If λ is real and $|\lambda| \neq 1$, then there are two distinct blocks $J_m(\lambda)$ and $J_m\left(\lambda^{-1}\right)$. If λ is nonreal and $|\lambda| = 1$, there are again two distinct blocks: $J_m(\lambda)$ and $J_m(\overline{\lambda})$. If $\lambda = \pm 1$, these four blocks are the same. We wish to determine if eigenvalues with modulus 1 exhibit a pairing. By Lemma 2.2(d), it suffices to consider the following Jordan matrices: $J_m(1)$, $J_m(-1)$, and $J_m\left(e^{i\theta}\right) \oplus J_m\left(e^{-i\theta}\right)$ for $\theta \in (0, \pi)$.

Let $S \in \mathrm{GL}_n(\mathbb{R})$ be a skew-involution. Suppose that $A \in \mathrm{GL}_n(\mathbb{R})$ is χ_S-orthogonal with Jordan Canonical Form consisting entirely of Jordan blocks corresponding to the eigenvalue 1. Then there exists a unique logarithm X of A which is a real matrix whose Jordan Canonical Form consists entirely of Jordan blocks corresponding to the eigenvalue 0. By Lemma 2.1(e), X is χ_S-skew symmetric. Since the Jordan Canonical Form of X consists entirely of blocks corresponding to the eigenvalue 0 and X is similar to a χ_H-skew symmetric matrix, every block $J_m(0)$ in the Jordan Canonical Form of X has a corresponding equal pair by Theorem 3.1. That is, X is similar to $\oplus_i \left(J_{m_i}(0) \oplus J_{m_i}(0)\right)$. Thus, the Jordan Canonical Form of $A = e^X$ is $\oplus_i \left(J_{m_i}(1) \oplus J_{m_i}(1)\right)$, that is, every Jordan block $J_m(1)$ in the Jordan Canonical Form of A has a corresponding equal pair.

Consider $A = \oplus_{i=1}^{\mu} J_{m_i}(1)$ and $B = \oplus_{i=1}^{\mu} J_{m_i}(-1)$. Then B is similar to a real χ_H-orthogonal matrix if and only if $-A$, and consequently A, is similar to a real χ_H-orthogonal matrix. Since A is similar to a real χ_H-orthogonal matrix if and only if each Jordan block of size m_i has a corresponding equal pair, we have that B is similar to a real χ_H-orthogonal matrix if and only if every Jordan block $J_{m_i}(-1)$ in the Jordan Canonical Form of B has a corresponding equal pair.

For $\theta \in (0, \pi)$ and positive integer k, the direct sum $J_k(e^{i\theta}) \oplus J_k(e^{-i\theta})$ is similar to the real matrix

$$A_k = \begin{bmatrix} C_\theta & I_2 & 0 & \cdots & 0 \\ 0 & C_\theta & I_2 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & I_2 \\ 0 & 0 & \cdots & 0 & C_\theta \end{bmatrix}, \tag{1}$$

where A_k has k copies of $C_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ along its diagonal, and $k-1$ copies of I_2 along its superdiagonal [6, Theorem 3.4.1.5]. Let $\theta \in (0, \pi)$ be fixed and let $k_1, k_2, \ldots, k_\mu$ be positive integers. Suppose $A = \oplus_{j=1}^{\mu} A_{k_j}$ is similar to a real χ_H-orthogonal matrix. Since A is a real matrix whose eigenvalues are $e^{i\theta}$ and $e^{-i\theta}$, its principal logarithm X exists, which is a real primary matrix function of A with Jordan Canonical Form $\oplus_{j=1}^{\mu}\left(J_{k_j}(i\theta) \oplus J_{k_j}(-i\theta)\right)$. Since A has no negative eigenvalues, we have, by Lemma 2.1 (c), (e), and (a), that X is similar to a real χ_H-skew symmetric matrix. Since the eigenvalues of X are pure imaginary, by Theorem 3.1, if $J_m(i\theta) \oplus J_m(-i\theta)$ is in the Jordan Canonical Form of X, then $J_m(i\theta) \oplus$

$J_m(-i\theta) \oplus J_m(i\theta) \oplus J_m(-i\theta)$ is in the Jordan Canonical Form of X. Since $A = e^X$, then $J_m\left(e^{i\theta}\right) \oplus J_m\left(e^{-i\theta}\right) \oplus J_m\left(e^{i\theta}\right) \oplus J_m\left(e^{-i\theta}\right)$ is in the Jordan Canonical Form of A.

Thus, if a $2n$-by-$2n$ real matrix A is similar to a real χ_{H_n}-orthogonal matrix, then the Jordan Canonical Form of A is a direct sum of matrices of the following types: (a) $J_m(\lambda) \oplus J_m(\overline{\lambda}) \oplus J_m(\lambda^{-1}) \oplus J_m(\overline{\lambda}^{-1})$, for some nonreal λ and positive integer m; or (b) $J_m(\lambda) \oplus J_m(\lambda^{-1})$, for some nonzero real λ and positive integer m.

To establish the sufficiency of these conditions, it is enough to show that each Jordan matrix listed above is similar to a real χ_H-orthogonal matrix, by Lemma 2.2 (c). If λ is a nonreal scalar, then $J_m(\lambda) \oplus J_m(\bar{\lambda})$ is similar to a real matrix B. Moreover, $J_m\left(\lambda^{-1}\right) \oplus J_m(\overline{\lambda}^{-1})$ is similar to B^{-1}. Therefore $R := J_m(\lambda) \oplus J_m(\overline{\lambda}) \oplus J_m(\lambda^{-1}) \oplus J_m(\overline{\lambda}^{-1})$ is similar to a real matrix $B \oplus B^{-1}$. By Lemma 2.1 (c) and (d), R is similar to a real $\chi_{H_{2m}}$-orthogonal matrix. If λ is a nonzero real scalar, then $J_m(\lambda) \oplus J_m(\lambda^{-1})$ is similar to the real matrix $J_m(\lambda) \oplus J_m(\lambda)^{-1}$, which is similar to a real χ_{H_m}-orthogonal matrix by Lemma 2.1 (c) and (d).

We summarize in the following theorem the possible Jordan Canonical Form of a real χ_H-orthogonal matrix.

Theorem 4.1. *A $2k$-by-$2k$ real matrix is similar to a real χ_{H_k}-orthogonal matrix if and only if its Jordan Canonical Form is a direct sum of matrices of types:*

(a) $J_m(\lambda) \oplus J_m(\lambda^{-1}) \oplus J_m(\overline{\lambda}) \oplus J_m(\overline{\lambda}^{-1})$, *for some nonreal λ, and positive integer m,*

(b) $J_m(\lambda) \oplus J_m(\lambda^{-1})$, *for some nonzero real λ, and positive integer m.*

Let $\epsilon = \pm 1$. In Theorems 1 and 2 of [7], $J_k(\epsilon)$ is similar to a complex orthogonal for odd k, while $J_k(0)$ is similar to a complex skew symmetric for odd k. Theorems 3.1 and 4.1 show that these blocks need to occur in pairs in order to be similar to real χ_H-orthogonal and real χ_H-skew symmetric matrices, respectively. We now give necessary and sufficient conditions for a real nonsingular matrix to be a product of two skew-involutions.

Corollary 4.1. *Let $n = 2k$ be even and $A \in \mathrm{GL_n}(\mathbb{R})$ be given. Then A is χ_S-orthogonal for some skew-involution $S \in \mathrm{GL_n}(\mathbb{R})$ if and only if A is similar to $Q \oplus Q^{-1}$ for some $Q \in \mathrm{GL_k}(\mathbb{R})$.*

Proof: Let $n = 2k$ be even and $A \in \mathrm{GL}_n(\mathbb{R})$ be given. The forward implication follows from Theorem 4.1, while the reverse implication follows from Lemma 2.1(d).

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