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q-Lah numbers, Laguerre configurations and rook placements

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Abstract

We derive combinatorial interpretations for Lah numbers in terms of pairs of permutations and partitions, and pairs of file placements and non-attacking placements of rooks on staircase boards. These interpretations were obtained using bijections from the classical interpretation of the Lah numbers in terms of Laguerre configurations. We then give a combinatorial intepretation for q-Lah numbers studied by Garsia and Remmel in terms of natural statistics on the rook placements. Some properties of these q-Lah numbers are also derived.

Keywords: Lah numbers, Laguerre configurations, rook placement, Laguerre polynomials

2010 MSC: 05A15, 05A10, 60C05

1 Introduction

Suppose n labelled objects are to be distributed into k indistinguishable tubes such that no tube is left empty. The number of ways this can be done is given by the *Lah number* $L(n,k)$, named after the Slovenian mathematician Ivo Lah, who studied them in 1955 [9]. For example, $L(3,2)=6$, as enumerated in Figure 1. Equivalently, the Lah numbers count the number of ways to partition a set with n elements into k linearly ordered subsets called *lists*. These partitions are known as *Laguerre configurations*.

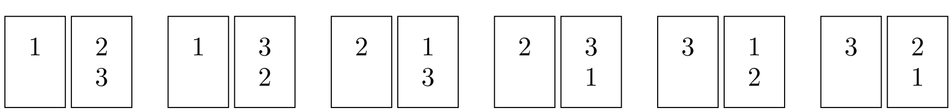


Figure 1: Six ways to distribute numbers $1,2$ and 3 into two tubes

The Lah numbers have a simple explicit formula given by

$$L(n,k)=\frac{n!}{k!}\binom{n-1}{k-1}. \tag{1}$$

Using either the combinatorial definition or formula (1), one can show that the Lah numbers satisfy the recurrence relation

$$L(n,k)=L(n-1,k-1)+(n+k-1)L(n-1,k), \tag{2}$$

with initial conditions $L(n,0) = L(0,n) = \delta_{n,0}$, where $\delta_{n,0}$ is the Kronecker delta.

The Lah numbers occur as connection coefficients between the rising factorial $x^{(n)} = x(x+1)\cdots(x+n-1)$ and the falling factorial $(x)_n = x(x-1)\cdots(x-n+1)$ as follows:

$$x^{(n)} = \sum_{k=1}^{n} L(n,k)(x)_k\,, \tag{3}$$

$$(x)_n = \sum_{k=1}^{n} (-1)^{n-k} L(n,k) x^{(k)}\,. \tag{4}$$

From relations (3) and (4) above, it follows that the matrices

$$[L(i,j)]_{0\le i,j\le n} \qquad \text{and} \qquad [(-1)^{i-j}L(i,j)]_{0\le i,j\le n}$$

are inverses of each other. In addition, the Lah numbers are also the coefficients of the nth *Laguerre polynomial* $L_n^{(-1)}(x)$ given by

$$L_n^{(-1)}(x) = \frac{1}{n!}\sum_{k=0}^{n} L(n,k)(-x)^k\,.$$

These polynomials satisfy the generating function [2, Identity 6]

$$e^{-xt/(1-t)} = \sum_{n=0}^{\infty} L_n^{(-1)}(x)t^n, \quad |t| < 1\,.$$

Laguerre polynomials belong to a class of orthogonal polynomials which are solutions to a certain second-order linear differential equation (see [1, Identity (10), p. 188]). More recently, it has been shown [5] that the Lah numbers occur as coefficients of the higher order derivatives of $\exp(1/x)$.

This paper focuses on a q-analogue of the Lah numbers, or q-Lah numbers. A *q-analogue* is a deformation of a number or sequence into an expression in the indeterminate q, often a polynomial in $\mathbb{N}[q]$, that reduces to the original number or sequence when $q \to 1$. For instance, a q-analogue of a non-negative integer n is given by $[n]_q = 1+q+q^2+q^3+\cdots+q^{n-1}$. This definition can be extended to any real number r by defining $[r]_q = \frac{1-q^r}{1-q}$, provided that $|q| < 1$. While there are multiple ways of defining a q-analogue (for example rq is technically a q-analogue of the real number r, albeit a trivial one), q-analogues that are useful from a combinatorial perspective tend to be those that are also generating functions of statistics on a combinatorial interpretation of the original sequence. This paper focuses on a particular version of q-Lah numbers studied by Garsia and Remmel [6]. This q-analogue satisfies the recurrence relation

$$L_q[n,k] = q^{n-k-2}L_q[n-1,k-1] + [n+k-1]_q L_q[n-1,k]\,,$$

subject to the same initial condition as $L(n,k)$. Note that there exists other q-analogues, such as the one studied by Lindsay, et al [10] which satisfy a different recurrence relation.

Katriel [8] showed that the Garsia and Remmel q-Lah numbers occur as coefficients in the higher order q-derivatives of the q-exponential function $\exp_q(1/x)$. There exists a q-analogue of the Laguerre polynomials which has applications in quantum physics, particularly in the study of oscillator algebras [4]. However, to the best of our knowledge the exact q-Lah numbers that appear in them have not been determined yet. Garsia and Remmel [6, p. 58] defined a q-Laguerre polynomial in a different manner than [4] where their q-Lah numbers appear as coefficients.

The paper is organized as follows. In Section 2, we give alternative combinatorial interpretations for the Lah numbers in terms of permutation-partition pairs and rook placements. These interpretations were derived using bijections from the classical interpretation of Lah numbers in terms of Laguerre configurations. In Section 3, we define q-analogues of the Lah numbers using statistics on these combinatorial interpretations and show that they are exactly the Garsia and Remmel q-Lah numbers. Some combinatorial identities and properties are also derived. Finally, in Section 4, we give some recommendations for future study.

2 New Combinatorial Interpretations for $L(n, k)$

A permutation, partition or Laguerre configuration is in *standard form* if each component cycle, block or list, respectively, is arranged by increasing minimal element. These three combinatorial objects are different ways of partitioning a finite set, with permutations and Laguerre configurations having additional structure. We distinguish the three by using parentheses for cycles, curly brackets for blocks and square brackets for lists. For example, the following are permutations, partitions and Laguerre configurations in standard form:

$$(1)(2,6,3)(4,5), \qquad \{1\}\{2,6,3\}\{4,5\}, \qquad [1][2,6,3][4,5]\,.$$

The following lemma shows that every Laguerre configuration can be decomposed into a unique ordered pair consisting of a permutation and a partition. Here, we denote by $\mathcal{C}(n, k)$ the set of all permutations of $\{1, 2, \ldots, n\}$ consisting of k disjoint non-empty cycles, $\mathcal{S}(n, k)$ the set of all partitions of $\{1, 2, \ldots, n\}$ into k disjoint non-empty blocks and $\mathcal{L}(n, k)$ the set of all partitions of $\{1, 2, \ldots, n\}$ into k disjoint non-empty lists, all written in standard form.

Lemma 1. *There exists a bijection between $\mathcal{L}(n, k)$ and $\bigcup_{j=k}^{n} (\mathcal{C}(n, j) \times \mathcal{S}(j, k))$.*

Proof: Let $\lambda = [\lambda_1][\lambda_2]\cdots[\lambda_k] \in \mathcal{L}(n, k)$. Write each list as $[\lambda_i] = [\lambda_{i,1}, \lambda_{i,2}, \ldots, \lambda_{i,m}]$ and construct a permutation σ_i of the elements of λ_i such that $\sigma_i(t) = \lambda_{i,t}$. In other words, the image of an element t under σ_i is the t-th entry of λ_i from the left. Let σ be the product of the σ_i's written as a product of disjoint cycles, i.e., $\sigma = \sigma_1\sigma_2\cdots\sigma_k$. Note that σ at this point is not necessarily in standard form. To obtain a partition π from λ, label each cycle in σ with s if the cycle contains the s-th smallest minimal element among all the cycles that form σ. Then, for each λ_i, form a block which is a subset of $\{1, 2, \ldots, n\}$ whose elements are the labels of the cycles of σ_i. Both σ and π can now be written in standard form.

To recover λ from (σ, π), let π^t be the t-th element from the left of π after suppressing the braces and let π_i be the i-th block of π from the left. Write $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ and label each σ_t with π^t. For every block π_i, let σ^i be the permutation which is the product of cycles of σ labelled by elements of π_i. Then, the list $[\lambda_i]$ is obtained by letting $\lambda_{i,j}$ be the image under σ^i of the j-th largest element of σ^i.

To illustrate the bijection described in Lemma 1, consider the Laguerre configuration

$$\lambda = [\lambda_1][\lambda_2][\lambda_3][\lambda_4][\lambda_5] = [4, 11, 1][2, 9, 8][7, 10, 3, 13][5, 6][12]\,.$$

The first permutation σ_1 satisfies $\sigma_1(1) = 4$, $\sigma_1(4) = 11$, $\sigma_1(11) = 1$ so that $\sigma_1 = (1, 4, 11)$. Doing this for the other lists we obtain the permutation $\sigma = (1, 4, 11)(2)(8, 9)(3, 7, 10)(13)(5)$

(6)(12). We then label each cycle by increasing minimal element. For instance, we label the first cycle 1, the second cycle 2, the fourth cycle 3, and so on. From this labeling of the cycles we get the sequence $1, 2, 6, 3, 8, 4, 5, 7$. To obtain the corresponding partition, we group the elements of this sequence based on the list where the cycles that they label belong. For example, the cycles labelled 3 and 8 belong to the same block in the partition since $(3, 7, 10)$ and (13) were both obtained from the list $[7, 10, 3, 13]$. The resulting partition is then $\{1\}\{2, 6\}\{3, 8\}\{4, 5\}\{7\}$. Rewriting both σ and π in standard form we obtain the pair $((1, 4, 11)(2)(3, 7, 10)(5)(6)(8, 9)(12)(13), \{1\}\{2, 6\}\{3, 8\}\{4, 5\}\{7\})$, which is the image of λ under the bijection.

To illustrate how λ is obtained from (σ, π), we pick one list $[7, 10, 3, 13]$ and show how it is recovered. The third block of π is given by $\pi_3 = \{3, 8\}$. The third and eight cycles of σ are $(3, 7, 10)$ and (13), respectively, and hence $\sigma^3 = (3, 7, 10)(13)$. The elements of σ^3 arranged in increasing order are $3, 7, 10, 13$ and under σ^3, the image of 3 is 7, the image of 7 is 10, the image of 10 is 3 and the image of 13 is 13. From this sequence of images we form the list $[7, 10, 3, 13]$. Once all the other lists have been obtained, it suffices to arrange them by increasing minimal element to obtain λ.

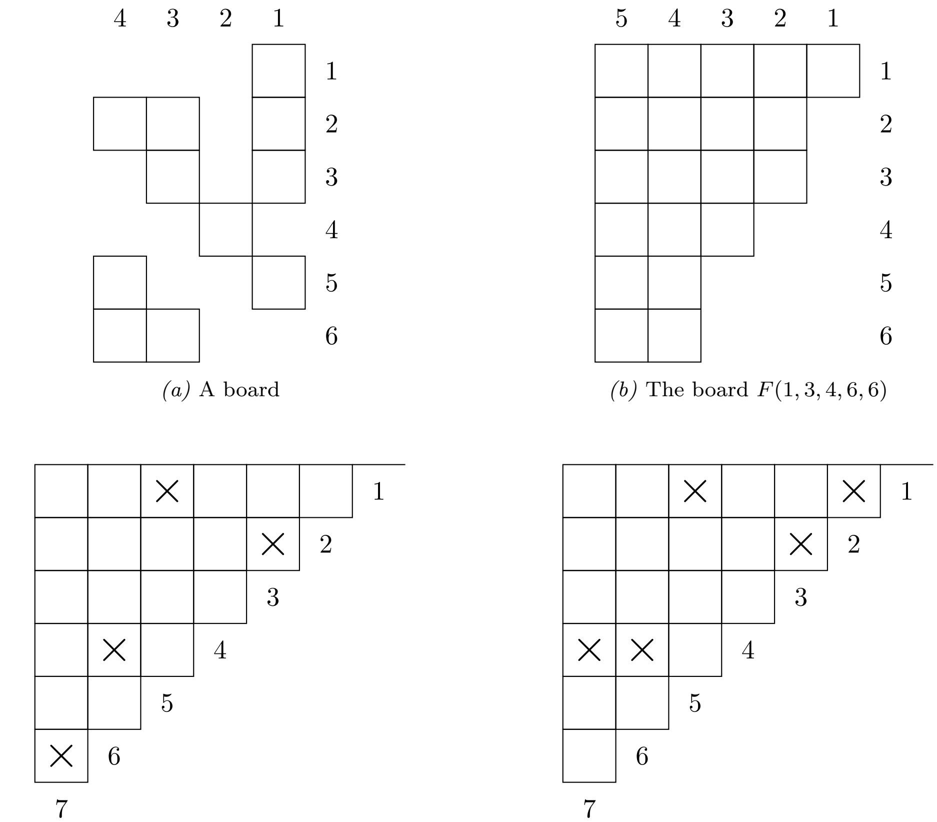
The theorem that follows is an immediate consequence of Lemma 1 and gives our first combinatorial interpretation for the Lah numbers.

Theorem 2. *The Lah number $L(n, k)$ counts the number of pairs (σ, π) where $\sigma \in \mathcal{C}(n, j)$ and $\pi \in \mathcal{S}(j, k)$, for some $k \leq j \leq n$.*

Our next combinatorial interpretation for $L(n, k)$ involves configurations on Young diagrams called *rook placements*. The definitions we use here are based from the monograph by Butler, et al [3], except for some changes in convention and notation.

A *board* is any subset of $\mathbb{N} \times \mathbb{N}$, which we visualize as arrays of squares. The rows (resp. columns) of a board B are labelled $1, 2, 3, \ldots$ from top to bottom (respectively, right to left). An example is shown in Figure 2a. Given $b_1, b_2, \ldots, b_n \in \mathbb{N} \cup \{0\}$, we denote by $F(b_1, b_2, \ldots, b_n)$ the board consisting of cells (i, j), where $0 \leq i \leq b_j$ and $1 \leq j \leq n$. Equivalently, $F(b_1, b_2, \ldots, b_n)$ is the top and left-justified board whose i-th column has length b_i. An example is shown in Figure 2b. We denote by St_n the *staircase board* $F(0, 1, \ldots, n-2, n-1)$. The shape of a staircase board allows the simultaneous labeling of the rows and columns by placing the numbers below the border diagonal, as shown in Figures 2c and 2d. A *rook* is a marking of the cells of a board B with "$\times$". A placement of rooks is *non-attacking* if no two rooks occupy the same row or column, an example of which is shown in Figure 2c. A placement of rooks is a *file placement* if no two rooks occupy the same column. Figure 2d shows an example of a file placement. We denote the set of all file placements and non-attacking placements on a board B with k rooks by $\mathcal{F}_k(B)$ and $\mathcal{R}_k(B)$, respectively. The k-th file number and k-th rook number of a board B are defined, respectively, as $f(B, k) = |\mathcal{F}_k(B)|$ and $r(B, k) = |\mathcal{R}_k(B)|$.

We now present the following bijections between permutations (respectively, partitions) and non-attacking (respectively, file) placements of rooks. For each bijection, position (i, j) refers to the cell in row i, column j of a given board.



(a) A board

(b) The board $F(1,3,4,6,6)$

(c) A non-attacking rook placement on St_7

(d) A file placement of rooks on St_7

Figure 2: Boards and rook placements

1. **Bijection $\mathcal{B}_1$ between $\mathcal{C}(n,k)$ and $\mathcal{F}_{n-k}(St_n)$.** This bijection is a slight modification of the map described in [3, p. 8]. Let $\sigma \in \mathcal{C}(n,k)$. If i is a minimal element of a cycle, then column i contains no rook. Now, for $1 \leq i \leq n$, let $\sigma^{(i)}$ be the permutation obtained by removing the elements greater than i. If i is not a minimal element and occurs in the j-th position of $\sigma^{(i)}$ from the left, place a rook in position $(j-1,i)$. For example, let $\sigma = (1,2)(3,6,5)(4)$. Then, we have the following sequence of $\sigma^{(i)}$'s:

$$\begin{aligned}
\sigma^{(1)} &= (1) \\
\sigma^{(2)} &= (1,2) \\
\sigma^{(3)} &= (1,2)(3) \\
\sigma^{(4)} &= (1,2)(3)(4) \\
\sigma^{(5)} &= (1,2)(3,5)(4) \\
\sigma^{(6)} &= (1,2)(3,6,5)(4)\,.
\end{aligned}$$

Since the minimal elements are 1, 3 and 4, there are no rooks in columns 1, 3 and 4. Now, 2 is in the second position of $\sigma^{(2)}$, so we place a rook in position $(1,2)$. The number 5 is in the 4th position of $\sigma^{(5)}$, so we place a rook in position $(3,5)$. Also, 6 is in the 4th position of $\sigma^{(6)}$, so we place a rook in position $(3,6)$. The resulting non-attacking rook placement is shown in Figure 3.

We now describe the reverse map. First, let $\sigma_{(1)} = (1)$. For $i > 1$, the other permutations $\sigma_{(i)}$ are obtained recursively as follows. If column i contains no rook, then adjoin the singleton cycle (i) to $\sigma_{(i-1)}$, i.e., $\sigma_{(i)} = \sigma_{(i-1)}(i)$. Otherwise, if there is a rook in row j, add i after the element in the j-th position of $\sigma_{(i-1)}$. Finally, let $\sigma = \sigma_{(n)}$.

Given the file placement in Figure 3, one observes that the sequence of $\sigma_{(i)}$'s is exactly the sequence of $\sigma^{(i)}$'s.

2. **Bijection $\mathcal{B}_2$ between $\mathcal{S}(n,k)$ and $\mathcal{R}_{n-k}(St_n)$.** This bijection is stated in [3, p. 7], which we describe here for completeness. Let $\pi \in \mathcal{S}(n,k)$. For each non-singleton block in π, arrange the elements in each block by increasing order $i_1 < i_2 < \cdots < i_m$ and for $t = 1, 2, \ldots, m-1$, place a rook in position (i_t, i_{t+1}).

   For example, if $\pi = \{1\}\{3,7\}\{2,4,5\}\{6\}$, place a rook in positions $(3,7)$, $(2,4)$ and $(4,5)$. The resulting file placement is shown in Figure 4.

   A partition is obtained from a rook placement as follows. If column i contains no rook, then i is a minimal element of a block. If there is a rook in position (i,j), then i and j belong to the same block.

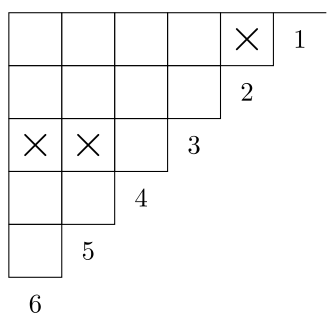


Figure 3: The file placement of rooks corresponding to the permutation $\sigma = (1,2)(3,6,5)(4)$

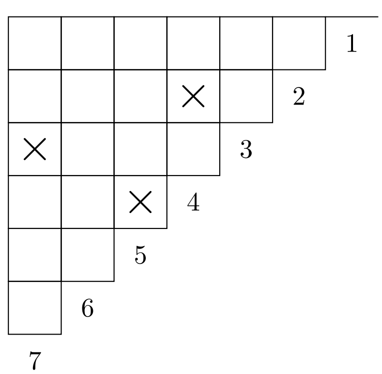


Figure 4: The non-attacking rook placement corresponding to the partition $\pi = \{1\}\{3,7\}\{2,4,5\}\{6\}$.

The following corollary follows from Theorem 2 and the bijections $\mathcal{B}_1$ and $\mathcal{B}_2$.

Corollary 3. *The Lah number $L(n,k)$ counts the number of pairs (ϕ,ρ) where $\phi \in \mathcal{F}_{n-j}(St_n)$ and $\rho \in \mathcal{R}_{j-k}(St_j)$, for some $1 \le j \le n$.*

We now briefly discuss some identities which are immediate consequences of Theorem 2. These identities are not new and they can be established algebraically using formulas (2) and (4), or under a general framework, such as those of generalized Stirling numbers [13]. However, Theorem 2 provides an alternative method which is of combinatorial interest.

First, it is known that the cardinalities of $\mathcal{C}(n,k)$ and $\mathcal{S}(n,k)$ are given by the Stirling number of the first kind $c(n,k)$ and Stirling number of the second kind $S(n,k)$, respectively. Hence, Theorem 2 immediately implies that

$$L(n,k) = \sum_{j=k}^{n} c(n,j) S(j,k) \tag{5}$$

from which we get the matrix factorization

$$[L(i,j)]_{0\le i,j\le n} = [c(i,j)]_{0\le i,j\le n}[S(i,j)]_{0\le i,j\le n}\,.$$

3 q-Lah Numbers

In this section, we define a q-analogue of $L(n,k)$ using the rook theoretic intepretations in Section 2. We then show that this q-Lah number is exactly the q-analogue studied by Garsia and Remmel [6]. Recall that certain cells in a board are "cancelled" when a rook is placed in the sense that these cells are no longer allowed to contain additional rooks. In a file placement, a marking cancels the marked cell and all cells above the cell, while in a non-attacking rook placement a marking also cancels all cells to the left of a cell. For a file or non-attacking rook placement ϕ, we define the *weight* $\mathrm{wt}(\phi)$ as the number of uncancelled cells of ϕ. For example, the non-attacking rook placement in Figure 2c has weight 6 since the uncancelled cells are $(1,2)$, $(3,4)$, $(1,4)$, $(4,5)$, $(3,5)$ and $(5,6)$. On the other hand, the file placement in Figure 2d has weight 9. Notice that all cells in column 4 are not cancelled by the rooks in columns 2 and 3 as this is a file placement.

We define the *q-file* and *q-rook numbers of a board B* as follows:

$$f_q[B,k] = \sum_{\phi\in\mathcal{F}_k(B)} q^{\mathrm{wt}(\phi)} \tag{6}$$

$$r_q[B,k] = \sum_{\phi\in\mathcal{R}_k(B)} q^{\mathrm{wt}(\phi)}\,. \tag{7}$$

Since the file and rook numbers of staircase boards are the Stirling numbers, we can define q-analogues of $c(n,k)$ and $S(n,k)$ for $n>k>0$ by

$$c_q[n,k] = \sum_{\phi\in\mathcal{F}_{n-k}(St_n)} q^{\mathrm{wt}(\phi)} \tag{8}$$

$$S_q[n,k] = \sum_{\phi\in\mathcal{R}_{n-k}(St_n)} q^{\mathrm{wt}(\phi)}\,. \tag{9}$$

These numbers satisfy the recurrence formulas

$$c_q[n,k] = q^{n-1} c_q[n-1,k-1] + [n-1]_q\, c_q[n-1,k]\,. \tag{10}$$

$$S_q[n,k] = q^{k-1} S_q[n-1,k-1] + [k]_q \, S_q[n-1,k] \tag{11}$$

with initial conditions $c_q[n,0] = c_q[0,n] = \delta_{n,0}$ and $S_q[n,0] = S_q[0,n] = \delta_{n,0}$. Furthermore, $c_q[n,k] = S_q[n,k] = 0$ if $n < k$. Note that other non-equivalent q-Stirling numbers exist [13]. To show (10), consider a file placement in St_{n-1}. If the file placement already has $n-k$ rooks, we can form a file placement in $\mathcal{F}_{n-k}(St_n)$ by adjoining a column of length $n-1$, which contributes a weight of q^{n-1}. On the other hand, if the file placement has $n-k-1$ markings, we can form a file placement in $\mathcal{F}_{n-k}(St_n)$ by adjoining a column of length $n-1$, on which a marking can be placed on one of the $n-1$ cells. The weights contributed by such markings from bottom to top are $1, q, q^2, \ldots, q^{n-2}$, the sum of which is $[n-1]_q$. The other recurrence formula (11) is proved similarly.

For $n > k > 0$, we define the *q-Lah numbers* by

$$L_q[n,k] = \sum_{j=k}^{n} \left(\sum_{\substack{\phi \in \mathcal{F}_{n-j}(St_n) \\ \psi \in \mathcal{R}_{j-k}(St_j)}} q^{\mathrm{wt}(\phi)+\mathrm{wt}(\psi)} \right) . \tag{12}$$

Otherwise, we define $L_q[n,0] = L_q[0,n] = \delta_{n,0}$ and $L_q[n,k] = 0$ if $n < k$.

The next results give some properties of $L_q[n,k]$.

Theorem 4. *The number $L_q[n,k]$ satisfies*

$$L_q[n,k] = \sum_{j=k}^{n} c_q[n,j] S_q[j,k] \,, \tag{13}$$

Equivalently, we have the matrix factorization

$$\left[L_q[i,j]\right]_{0\le i,j\le n} = \left[c_q[i,j]\right]_{0\le i,j\le n} \left[S_q[i,j]\right]_{0\le i,j\le n} .$$

Proof: This property follows from (8), (9) and (12).

Theorem 5. *For $n > k > 0$, the number $L_q[n,k]$ satisfies the recurrence relation*

$$L_q[n,k] = q^{n+k-2} L_q[n-1,k-1] + [n+k-1]_q L_q[n-1,k] \,. \tag{14}$$

Proof: From (10) and (13),

$$\begin{aligned} L_q[n,k] &= \sum_{j=k}^{n} \left(q^{n-1} c_q[n-1,j-1] + [n-1]_q c_q[n-1,j] \right) S_q[j,k] \\ &= \sum_{j=k}^{n} q^{n-1} c_q[n-1,j-1] S_q[j,k] + \sum_{j=k}^{n} [n-1]_q c_q[n-1,j] S_q[j,k] \,. \end{aligned} \tag{15}$$

Using (11), the first sum in (15) becomes

$$q^{n-1} \sum_{j=k}^{n} c_q[n-1,j-1] (q^{k-1} S_q[j-1,k-1] + [k]_q S_q[j-1,k])$$

$$
\begin{aligned}
&= q^{n-k-2}\sum_{j=k}^{n} c_q[n-1,j-1]S_q[j-1,k-1] + q^{n-1}[k]_q\sum_{j=k}^{n} c_q[n-1,j-1]S_q[j-1,k]\\
&= q^{n-k-2}\sum_{j=k-1}^{n-1} c_q[n-1,j]S_q[j,k-1] + q^{n-1}[k]_q\sum_{j=k-1}^{n-1} c_q[n-1,j]S_q[j,k]\\
&= q^{n-k-2}L_q[n-1,k-1] + q^{n-1}[k]_qL_q[n-1,k] - q^{n-1}[k]_qc_q[n-1,k-1]S_q[k-1,k]\,.\\
&= q^{n-k-2}L_q[n-1,k-1] + q^{n-1}[k]_qL_q[n-1,k]\,. \qquad (16)
\end{aligned}
$$

On the other hand, the second sum in (15) becomes

$$
[n-1]_q\sum_{j=k}^{n-1} c_q[n-1,j]S_q[j,k] + [n-1]_qc_q[n-1,n]S_q[n,k] = [n-1]_qL_q[n-1,k]\,. \qquad (17)
$$

Identity (14) then follows by combining (16) and (17) and using the fact that $q^{n-1}[k]_q + [n-1]_q = [n+k-1]_q$.

The q-Lah numbers $L_{n,k}(q)$ studied by Garsia and Remmel satisfy same initial conditions as $L_q[n,k]$ and the recurrence relation (see [6, Identity (2.15)])

$$
L_{n+1,k}(q) = q^{n+k-1}L_{n,k-1}(q) + [n+k]L_{n,k}(q)
$$

Comparing this identity with (14) shows that $L_q[n,k] = L_{n,k}(q)$.

The next theorem gives a generating function for $L_q[n,k]$ and a q-analogue of (3) which belongs to a class of formulas in rook theory known as *product formulas* [14]. The identity is a special case of a more general result by Garsia and Remmel [7, Identity (1.3)] which was derived using a different combinatorial approach.

Theorem 6. *For $n > 0$, the number $L_q[n,k]$ satisfies the identity*

$$
\prod_{t=0}^{n-1}[x+t]_q = \sum_{k=1}^{n} L_q[n,k]\prod_{t=0}^{k-1}[x-t]_q\,. \qquad (18)
$$

Proof: We establish the identity by induction. Using (14),

$$
\begin{aligned}
&\sum_{k=1}^{n+1} L_q[n+1,k]\prod_{t=0}^{k-1}[x-t]_q \qquad (19)\\
&= \sum_{k=1}^{n+1}\left(q^{n+k-1}L_q[n,k-1] + [n+k]_qL_q[n,k]\right)\prod_{t=0}^{k-1}[x-t]_q\\
&= \sum_{k=1}^{n+1}\left(q^{n+k-1}L_q[n,k-1]\prod_{t=0}^{k-1}[x-t]_q\right) + \sum_{k=1}^{n+1}\left([n+k]_qL_q[n,k]\prod_{t=0}^{k-1}[x-t]_q\right)\\
&= \sum_{k=1}^{n+1}\left(q^{n+k-1}L_q[n,k-1]\prod_{t=0}^{k-1}[x-t]_q\right) + \sum_{k=1}^{n}\left([n+k]_qL_q[n,k]\prod_{t=0}^{k-1}[x-t]_q\right) \qquad (20)\\
&= \sum_{k=0}^{n}\left(q^{n+k}L_q[n,k]\prod_{t=0}^{k}[x-t]_q\right) + \sum_{k=1}^{n}\left([n+k]_qL_q[n,k]\prod_{t=0}^{k-1}[x-t]_q\right) \qquad (21)
\end{aligned}
$$

$$= \sum_{k=1}^{n} \left(q^{n+k} L_q[n,k] \prod_{t=0}^{k} [x-t]_q \right) + \sum_{k=1}^{n} \left([n+k]_q L_q[n,k] \prod_{t=0}^{k-1} [x-t]_q \right) \tag{22}$$

$$= \sum_{k=1}^{n} \left(q^{n+k}[x-k]_q + [n+k]_q \right) L_q[n,k] \prod_{t=0}^{k-1} [x-t]_q$$

$$= [x+n]_q \sum_{k=1}^{n} L_q[n,k] \prod_{t=0}^{k-1} [x-t]_q \tag{23}$$

$$= [x+n]_q \prod_{t=0}^{n-1} [x+t]_q$$

$$= \prod_{t=0}^{n} [x+t]_q \,,$$

where, in the computations above, we removed the term $L_q[n, n+1] = 0$ in (20), adjusted the index of the first summation in (21), removed the term containing $L_q[n, 0] = 0$ in (22), and used the fact that $q^{n+k}[x-k]_q + [n+k]_q = [x+n]_q$ in (23).

The corollary that follows allows us to recover the classical rook placement interpretation of the Lah numbers by setting $q = 1$. Let $\mathcal{L}_n$ denote the *Laguerre board* which is the Ferrers board with n columns of height $n-1$. For a bijection between the Laguerre configurations and rook placements on Laguerre boards, see [3, p. 15]. This bijection yields

$$L(n,k) = |\mathcal{R}_{n-k}(\mathcal{L}_n)| \,.$$

To the best of our knowledge a weighting scheme that produces $L_q[n,k]$ from $\mathcal{R}_{n-k}(\mathcal{L}_n)$ has not appeared in the literature yet.

Corollary 7. *The number* $L_q[n,k]$ *satisfies*

$$L_q[n,k] = \sum_{\phi \in \mathcal{R}_{n-k}(\mathcal{L}_n)} q^{\mathrm{wt}(\phi)} \,.$$

Proof: Let

$$l_q[n,k] = \sum_{\phi \in \mathcal{R}_{n-k}(\mathcal{L}_n)} q^{\mathrm{wt}(\phi)} \,.$$

We want to show that

$$\prod_{t=0}^{n-1} [x+t]_q = \sum_{k=1}^{n} l_q[n,k] \prod_{t=0}^{k-1} [x-t]_q \,, \tag{24}$$

which by Theorem 6 implies that $l_q[n,k] = L_q[n,k]$, as desired.

To prove (24), we use the fact that any polynomial identity that holds for any positive integer x also holds when x is any real number (see [11, Remark 2.7]). In addition, we also use a common method in proving product formulas which involves attaching a rectangular board (for example, see [3, p. 4] and [17, Theorem 3.8]).

Let $\mathcal{L}_n^x$ be the board obtained by adjoining a rectangular board containing n columns of height x above $\mathcal{L}_n$, as shown in Figure 5. We establish (24) by showing that its LHS and RHS are different ways of computing $r_q[\mathcal{L}_n^x, n]$.

For the LHS, place a rook on the rightmost column. This can be done in $x+n-1$ ways, the total weight of all such placements being $[x+n-1]_q$. Because of the placement of this

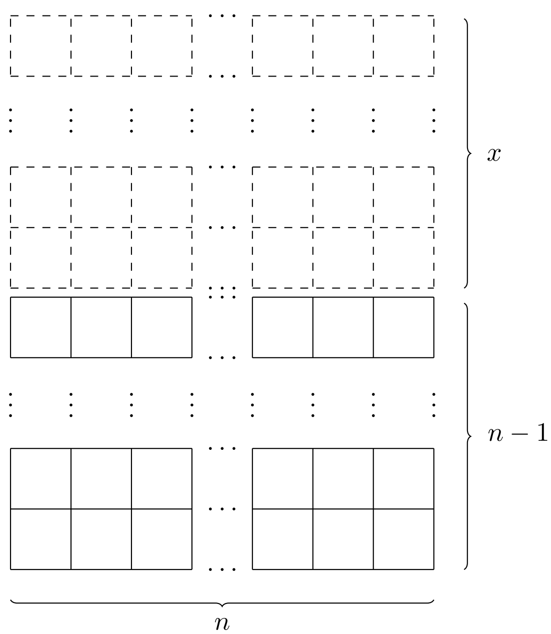


Figure 5: The board $\mathcal{L}_n^x$

rook, the next column has $x+n-2$ uncancelled cells and the total weight of the placement of the next rook is $[x+n-2]_q$. Continuing until all columns have been exhausted we get that the total weight of all rook placements obtained in this manner is indeed the LHS of (24).

For the RHS, fix $1 \le k \le n$ and place $n-k$ rooks on $\mathcal{L}_n$. The total weight of this placement is $l_q[n,k]$, by definition. Next, place k rooks on the uncancelled columns of the x by $n-1$ board we adjoined into $\mathcal{L}_n$, beginning from the rightmost uncancelled columns. Arguing as in the LHS, the total weight of all such placements is $\prod_{t=0}^{k-1}[x-t]_q$.

4 Some Remarks and Recommendations

1. We mentioned that there are other q-Lah numbers nonequivalent to $L_q[n,k]$, one of which are the q-Lah numbers of Lindsay, et al [10], which satisfy the recurrence relation

$$L_q^*[n,k] = L_q^*[n-1,k-1] + ([n-1]_q + [k]_q)\, L_q^*[n-1,k]\,. \tag{25}$$

By a suitable modification of the weight wt, these q-Lah numbers may also be given their own rook theoretic intepretations. Specifically, given a rook placement ϕ we let $\mathrm{wt}^*(\phi)$ be the number of uncancelled cells in columns containing rooks (so that columns not containing rook do not contribute any weight) and define

$$L_q^*[n,k] = \sum_{j=k}^{n} \left(\sum_{\substack{\phi\in\mathcal{F}_{n-j}(St_n)\\ \psi\in\mathcal{R}_{j-k}(St_j)}} q^{\mathrm{wt}^*(\phi)+\mathrm{wt}^*(\psi)} \right). \tag{26}$$

The proof that numbers defined by (26) satisfy the recurrence (25) is similar to that of Theorem 5.

2. The *associated Laguerre polynomial* [15, p. 109] with parameter α is given by

$$L_n^{(\alpha)}(x) = \sum_{k=1}^{n} \binom{n+\alpha}{n-k} \frac{n!}{k!} (-x)^k \,.$$

These polynomials satisfy the generating function

$$\sum_{k=0}^{n} \frac{L_k^{(\alpha)}(x)}{k!} t^k = (1-t)^{-\alpha-1} e^{(xt)/(t-1)} \,.$$

We denote the coefficients, called the *associated Lah numbers*, by $L^{(\alpha)}(n,k)$, i.e.,

$$L^{(\alpha)}(n,k) = \binom{n+\alpha}{n-k} \frac{n!}{k!} \,.$$

These coefficients can be viewed as a generalization of the Lah numbers since

$$L_{-1}(n,k) = \binom{n-1}{n-k} \frac{n!}{k!} = \binom{n-1}{k-1} \frac{n!}{k!} = L(n,k) \,.$$

It follows that the (classical) Laguerre polynomial occurs as the special case when $\alpha = -1$. The methods in our paper may be used to obtain rook theoretic interpretations for $L^{(\alpha)}(n,k)$ and its q-analogue.

3. Since $L(n,k)$ and $L_q[n,k]$ are coefficients in the higher-order derivatives of $\exp(1/x)$ [5] and higher-order q-derivatives of $\exp_q(1/x)$ [8], respectively, it is natural to ask if the numbers $L^{(\alpha)}(n,k)$ also serve as coefficients in the higher-order derivatives of some exponential function.

4. A matrix is *totally non-negative* if all of its minors are non-negative. On the other hand, a matrix is *q-totally non-negative* if all of its minors are polynomials in q with non-negative coefficients. Recently, the total non-negativity of matrices of Lah numbers were established by Martinjak and Škrekovski [12] using an interpretation for Lah number in terms of lattice paths. It is possible that the q-total non-negativity of the matrices of $L_q[n,k]$ and $L_q^*[n,k]$ may be established using appropriate weights on the existing lattice path interpretation. However, the use of the combinatorial interpretations given in this paper as a possible alternative proof may also be explored.

5. A q-Laguerre polynomial was studied by Cao [4], where the q-Lah numbers that appear as coefficients in this polynomial have not been identified yet. If it is a q-analogue other than $L_q[n,k]$ and $L_q^*[n,k]$, it will be interesting to study its combinatorial properties.

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