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Complete Tripartite Graphs with Spanning Maximal Planar Subgraphs

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Abstract

We prove that for complete tripartite graphs of order at most 9, the only ones that contain a spanning maximal planar subgraph are $K_{1,1,1}$, $K_{2,2,2}$, $K_{2,3,3}$, and $K_{3,3,3}$. The main result gives a necessary and sufficient condition for the complete tripartite graph $K_{x,y,z}$ to contain a spanning maximal planar subgraph.

Keywords: Complete tripartite graph, planar graph, maximal planar graph

2010 MSC: 05C10, 05C75

1 Introduction

All graphs considered here are undirected, finite and simple, *i.e.*, edges are undirected, the number of vertices is finite, and neither loops nor multiple edges are present. For basic terms not defined here, readers may refer to the books authored by Bondy & Murty [2] and Harary [4].

The drawing of a graph in the plane is not unique because points representing vertices may be chosen to be any set of distinct points, and the edges joining the points may take any shape. The graph K_4 in Figure 1 has one edge-crossing while the same graph, as illustrated in Figure 2, has no edge-crossing.

By a *planar graph* we mean a graph which can be drawn in the plane without edge-crossings. Paths P_n and cycles C_n are planar graphs. Among the complete graphs, only K_1, K_2, K_3, and K_4 are planar graphs.

Not all graphs are planar. One example of a non-planar graph is the complete graph K_5. Any graph containing a non-planar subgraph is itself a non-planar graph. Hence, all complete graphs of order 5 or greater are non-planar graphs.

Another example of a non-planar graph is the complete bipartite graph $K_{3,3}$. All complete bipartite graphs $K_{m,n}$ where $m \geq 3$ and $n \geq 3$ are also non-planar.

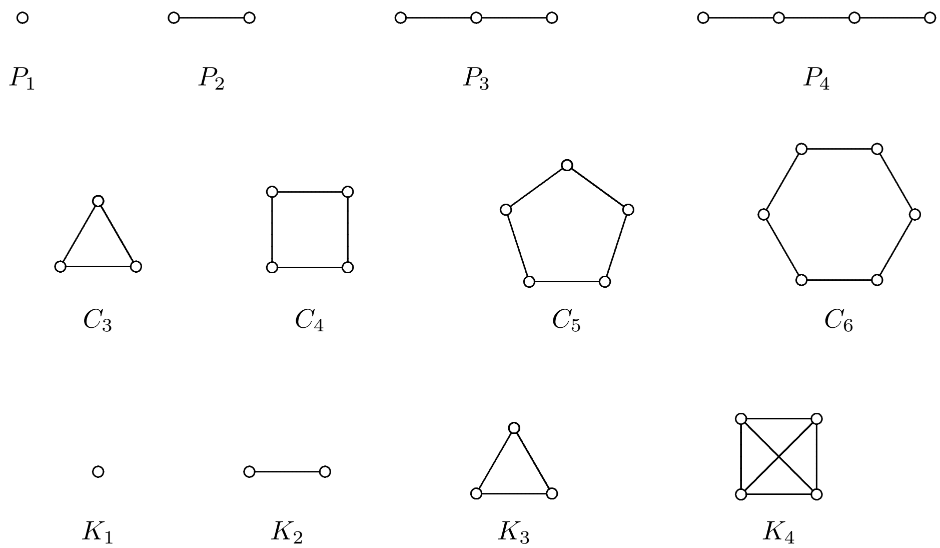


Figure 1: Some planar graphs.

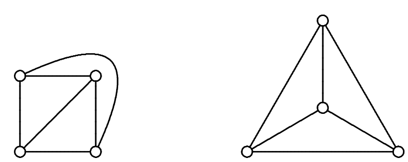


Figure 2: Two drawings of K_4 without edge-crossings.

See the works of Barnette [1], Borodin [3], Kempe [5], and Robertson [7] for more discussions on planar graphs.

When a planar graph is drawn in the plane without edge-crossings, disjoint open regions are created. An open region of the drawing is a maximal connected open subset of the plane bounded by vertices and edges of the graph. The planar graph K_4 for example, when drawn in the plane without edge-crossings will create four regions R_1, R_2, R_3, and R_4 as shown in Figure 3.

In Figure 3, we refer to R_4 as the outer region. Note that the drawing of a planar graph can always be modified so that any of the regions is the outer region. This is seen easily if we draw the planar graph on the surface of a sphere with the vertices distributed as evenly as possible on the surface.

Euler was the first to observe an interesting relationship among the order v, size e, and number of regions r of a connected planar graph.

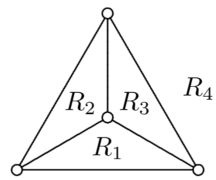


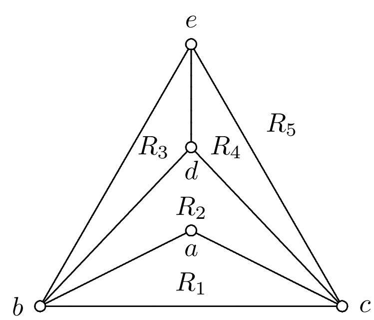
Figure 3: The regions created by a drawing of K_4 without edge-crossings.

Theorem 1 (Euler)**.** *If a connected planar graph has order v, size e, and r regions, then*

$$v - e + r = 2.$$

The formula in Theorem 1 is well-known as Euler's formula. An example to illustrate this is shown in Figure 4.

$v = 5$ vertices
$e = 8$ edges
$r = 5$ regions



Euler's formula:
$v - e + r = 2$
$5 - 8 + 5 = 2$

Figure 4: A planar graph of order $v = 5$, size $e = 9$, with $r = 5$ regions.

Euler published in 1754 a proof of the formula in Theorem 1 by induction. Unfortunately, it was discovered later that the proof was faulty. Legendre published a correct proof of the theorem in 1794.

A planar graph is maximal if the addition of a new edge produces a non-planar graph. A maximal planar graph is necessarily connected.

The complete graph K_4 is a maximal planar graph but the graph in Figure 4 is not a maximal planar graph. However, by adding the edge ad, we obtain a maximal planar graph of size 9.

Theorem 2 and Corollary 2 below are well-known results on planar graphs.

Theorem 2. *The size e of a maximal planar graph of order $v \geq 3$ is $e = 3v - 6$.*

It is an easy exercise to prove Theorem 2 using Euler's formula and the fact that in a maximal planar graph of order $v \geq 3$, every region is bounded by a 3-cycle. In addition, we use the fundamental theorem about graphs which states that the summation of vertex degrees is equal to twice the size.

Corollary 2. *In a planar graph G there exists a vertex v in G such that $deg_G(v) \leq 5$.*

Now, if the degree of every vertex in a planar graph with order v and size e is at least 6, then $6v \leq 2e$ implying that $3v \leq e$, a contradiction. Theorem 2 and Corollary 2 tell us that every planar graph has at least one vertex with degree at most 5.

The complete tripartite graph denoted by $K_{p,q,r}$ is the graph with three disjoint vertex-sets V_1, V_2, V_3 having cardinalities $|V_1| = p$, $|V_2| = q$, and $|V_3| = r$, such that the edges are all the pairs uv where $u \in V_i$, $v \in V_j$ $(i \neq j)$.

Shown in Figure 5 is a drawing of a complete tripartite graph.

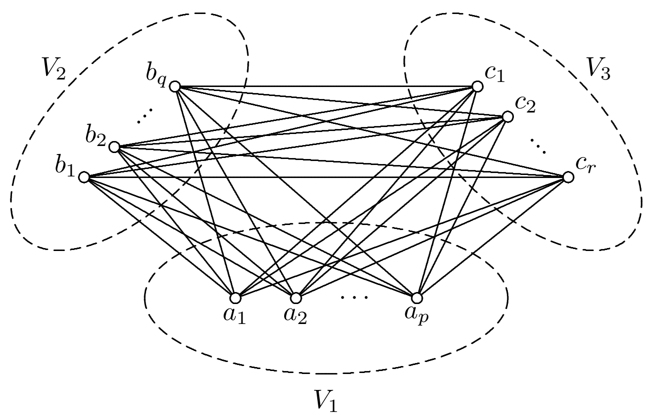


Figure 5: The complete tripartite graph $K_{p,q,r}$.

For convenience, we write $K_{p,q,r} = V_1 + V_2 + V_3$. Note that $K_{p,q,r}$ is isomorphic to $K_{a,b,c}$ provided that abc is a permutation of pqr.

A subgraph H of a graph G is a *spanning subgraph* of G if H contains all the vertices of G. In the next section, we shall determine all complete tripartite graphs that contain a spanning maximal planar subgraph. We state our main result in the following theorem, which we prove in Section 2.

Theorem 3. *Let x, y, z be positive integers, $x \leq y \leq z$, and $n = x + y + z$. The complete tripartite graph $K_{x,y,z}$ of order $n \geq 6$ contains a spanning maximal planar subgraph if and only if either*

(a) n is even and $x + y - z \geq 2$, or

(b) n is odd and $x + y - z \geq 3$.

In Section 2, we show by Proposition 6 that the only complete tripartite graphs of order $n \leq 9$ that contain a spanning maximal planar subgraph are $K_{1,1,1}$, $K_{2,2,2}$, $K_{2,3,3}$, and $K_{3,3,3}$.

It should be mentioned that a result of Nakamoto, et al. [6] will imply the result of this paper. However, the method used in this paper is very elementary and independent of the method used in their paper.

We need to establish a number of propositions that will lead to the proof of our main results.

2 Main Results

Proposition 1. *The complete tripartite graph $K_{1,1,z}$ contains a spanning maximal planar graph if and only if $z = 1$.*

Proof. If $z = 1$, the complete tripartite graph $K_{1,1,1}$ is isomorphic to K_3, which is a maximal planar subgraph. Now, the size of $K_{1,1,z}$ is $2z+1$. On the other hand, the size of a maximal planar graph of order $z+2$ is $3(z+2)-6 = 3z$. If $K_{1,1,z}$ contains a spanning maximal planar graph, we must have $3z \leq 2z+1$. This implies that $z \leq 1$ and therefore, $z = 1$. Conversely, if $z = 1$, then $K_{1,1,1}$ is a maximal planar graph. □

Proposition 2. *Let $x+y+z > 3$. If $K_{x,y,z}$ contains a spanning maximal planar graph G, then $\deg_G(v)$ is even and $\deg_G(v) \geq 4$ for all vertices v in G.*

Proof. Let $x+y+z = n > 3$ and let G be a spanning maximal planar subgraph of $K_{x,y,z}$. Let $abca$ be any 3-cycle in G. There is a drawing of G in the plane without edge-crossings such that $abca$ is the boundary of the outer region (unbounded region). Since the order of G is greater than 3, there are vertices of G inside the 3-cycle $abca$. Furthermore, a, b, and c are always connected to some vertex inside $abca$, and the vertices inside $abca$ to some vertex inside $abca$. Otherwise, edges can still be added without edge-crossings in G, a contradiction that G is maximal. Since all regions are bounded by 3-cycles, the degrees of a, b, c are all greater than 2.

Let v be any vertex of G and let $\deg(v) = k > 2$. Since each vertex v in G is some vertex in a 3-cycle of G, there is always a cycle in G of order at least 3 with v as its start and terminus. Let $v_1, v_2, \ldots, v_k$ be the neighbors of v. Since vertex v was arbitrarily chosen, without loss of generality, we let $v_1v_2\cdots v_kv_1$ be a k-cycle in G. Let $K_{x,y,z} = V_1+V_2+V_3$ with $|V_1| = x$, $|V_2| = y$, $|V_3| = z$. We may assume that $v \in V_1$. Therefore, the vertices $v_i, i = 1, 2, ..., k$, belong to $V_2 \cup V_3$. Since any vertex v_i in the k-cycle can be selected as the starting vertex, without loss of generality, assume that $v_1 \in V_2$. Then $v_2 \in V_3$, $v_3 \in V_2$, $\ldots$, $v_k \in V_3$. Necessarily, k is even. Since $k > 2$, then $k \geq 4$. □

By Corollary 2, we know that for any planar graph, there is a vertex v with $\deg(v) \leq 5$. If G is a spanning maximal planar subgraph of $K_{x,y,z}$, then by Corollary 2 there exists a vertex v in G with $\deg(v) \leq 5$. By Proposition 2, since $\deg(v)$ is even and $\deg(v) \geq 4$, it follows that $\deg(v) = 4$. Thus, we obtain the the next result.

Proposition 3. *Let $2 \leq x \leq y \leq z$. If $K_{x,y,z}$ contains a spanning maximal planar graph G, then there exists a vertex v in G with $\deg(v) = 4$.*

Proposition 4. *The complete tripartite graph $K_{1,2,z}$ does not contain a spanning maximal planar subgraphs for $z \geq 2$.*

Proof. If $K_{1,2,z} = V_1+V_2+V_3$ with $|V_1| = 1$, $|V_2| = 2$, and $|V_3| = z$ then every vertex in V_3 has degree 3. By Proposition 2, since $1+2+z > 3$, $K_{1,2,z}$ cannot contain a spanning maximal planar graph. □

Proposition 5. *The complete tripartite graph $K_{2,2,z}$, where $2 \leq z$, contains a spanning maximal planar graph if and only if $z = 2$.*

Proof. Let $K_{2,2,z} = V_1+V_2+V_3$ with $|V_1| = 2$, $|V_2| = 2$, and $|V_3| = z$. Let G be a spanning maximal planar subgraph of $K_{2,2,z}$. By Proposition 2, since the degree of each vertex in V_3 is 4, then all edges of the form ab where $a \in V_1 \cup V_2$ and $b \in V_3$ are in G. It follows that the size of G is at least $4z$. But the size of a maximal planar graph of order $4+z$ is $3(4+z)-6 = 3z+6$. Therefore, $4z \leq 3z+6$, which implies that $z \leq 6$. Thus, $(4z+4)-(3z+6) = z-2$ edges must be removed from $K_{2,2,z}$ to obtain G. Since vertices in G have degree at least 4, by Proposition 2, and all vertex in V_3 have degree 4 in $K_{2,2,z}$, only edges of the form uv, where $u \in V_1$, $v \in V_2$ are allowed to be removed. If $z = 6$, then all such edges are removed, and we are left with a complete bipartite graph G, which is a contradiction since spanning planar graphs must have 3-cycles. If $3 \leq z \leq 5$, removing $z-2$ edges leaves us with a vertex in $V_1 \cup V_2$ having odd degree in G, which is still a contradiction by Proposition 2. If $z = 2$, the graph $K_{2,2,2}$ is a maximal planar graph. □

Proposition 6. *The only complete tripartite graphs of order $n \leq 9$ that contain a spanning maximal planar subgraph are $K_{1,1,1}$, $K_{2,2,2}$, $K_{2,3,3}$, and $K_{3,3,3}$.*

Proof. Propositions 1 and 5 show that $K_{1,1,1}$ and $K_{2,2,2}$ contain spanning maximal planar subgraphs. By Proposition 3 the spanning maximal planar subgraphs for $K_{2,3,3}$, and $K_{3,3,3}$ have only even vertices with degree at least 4, as constructed. We show the complete tripartite graphs $K_{1,1,z}$ $(2 \leq z \leq 7)$, $K_{1,2,z}$ $(2 \leq z \leq 6)$, $K_{1,3,z}$ $(3 \leq z \leq 5)$, $K_{1,4,4}$, $K_{2,2,z}$ $(3 \leq z \leq 5)$ do not have spanning maximal planar graphs. We arrange these into the following seven cases.

Case 1. $K_{1,1,z}$ $(2 \leq z \leq 7)$. By Proposition 1, this graph has no spanning maximal planar subgraph.

Case 2. $K_{1,2,z}$ $(2 \leq z \leq 7)$. By Proposition 4, this graph has no spanning maximal planar subgraph.

Case 3. $K_{1,3,z}$ $(3 \leq z \leq 5)$. Assume that G is a spanning maximal planar subgraph of this complete tripartite graph. Let $K_{1,3,z} = V_1 + V_2 + V_3$ with $|V_1| = 1$, $|V_2| = 3$, $|V_3| = z$. The size of this graph is $4z + 3$. On the other hand, the size of G is $3(z+4) - 6 = 3z + 6$. Therefore, exactly $(4z+3) - (3zx+6) = z - 3$ edges must be removed to obtain G from $K_{1,3,z}$. Let us consider the following subcases.

Case 3.1 $z = 5$. If we remove $z - 3 = 2$ edges of the form uv where $u \in V_1$ and $v \in V_2$, then there are exactly two vertices in V_2 having degree 5, which is a contradiction by Proposition 2.

Case 3.2 $z = 4$. If we remove $z - 3 = 1$ edge of the form uv where $u \in V_1$ and $v \in V_2$, then there are exactly two vertices in V_2 having degree 5, which is a contradiction by Proposition 2.

Case 3.3 $z = 3$. In this case, $G = K_{1,3,3}$. But this graph contains the subgraph $K_{3,3}$ which is non-planar. This is a contradiction.

Case 4. $K_{1,4,4}$. The degree of each vertex in $V_2 \cup V_3$ in $K_{1,4,4}$ is 5, hence the degree of those vertices in G is 4 by Proposition 2. The sum of the degrees of the vertices is twice the size, by the Handshaking lemma. The preceding two statements imply that the degree of the sole vertex in V_1 must be 10. This is a contradiction.

Case 5. $K_{2,2,z}$ $(3 \leq z \leq 5)$. By Proposition 5, this graph does not contain a spanning maximal planar subgraph.

Case 6. $K_{2,3,3}$. Let $K_{2,3,3} = V_1 + V_2 + V_3$ with $|V_1| = 2$, $|V_2| = 3$, and $|V_3| = 3$. This graph contains a spanning maximal planar subgraph as illustrated in Figure 6. The vertices labeled 1 belong to V_1; the vertices labeled 2 belong to V_2; the vertices labeled 3 belong to V_3.

Case 7. $K_{3,3,3}$. Let $K_{3,3,3} = V_1 + V_2 + V_3$ with $|V_1| = |V_2| = |V_3| = 3$. This graph contains a spanning maximal planar subgraph as illustrated in Figure 7. The vertices labeled 1 belong to V_1; the vertices labeled 2 belong to V_2; the vertices labeled 3 belong to V_3. □

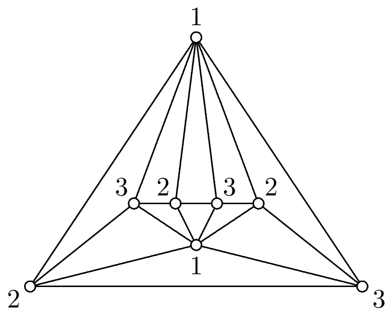


Figure 6: A spanning maximal planar subgraph of $K_{2,3,3}$.

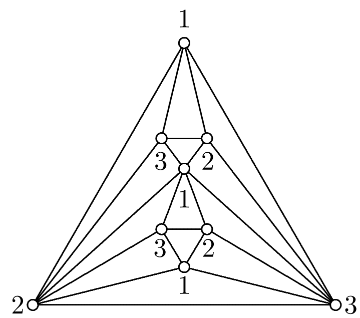


Figure 7: A spanning maximal planar subgraph of $K_{3,3,3}$.

Next, we show how to construct from a given complete tripartite graph of order $n \geq 6$ which contains a spanning maximal planar subgraph a complete tripartite graph of order $n+2$ which also contains a spanning maximal planar subgraph.

Proposition 7. *Let x, y, z be positive integers, $x+y+z \geq 6$. If $K_{x,y,z}$ contains a spanning maximal planar subgraph, then each of $K_{x+1,y+1,z}$, $K_{x+1,y,z+1}$, $K_{x,y+1,z+1}$ contains a spanning maximal planar subgraph.*

Proof. Let G be a spanning maximal planar subgraph of $K_{x,y,z}$ and let $K_{x,y,z} = V_1+V_2+V_3$ with $|V_1| = x$, $|V_2| = y$, $|V_3| = z$. Observe that the size of G is $3(x+y+z)-6$. Let uv be an edge of G having the form $u \in V_1$ and $v \in V_2$. The edge uv is the common boundary of two regions bounded by 3-cycles $uvau$ and $uvbu$. Necessarily, $a \in V_3$ and $b \in V_3$. Let us modify the sets V_1 and V_2 as follows. Let $V_1' = V_1 \cup \{a_0\}$ and let $V_2' = V_2 \cup \{b_0\}$ where a_0 and b_0 are new vertices. Modify G by removing the edge uv and adding the new edges ub_0, b_0a_0, a_0v, aa_0, ab_0, ba_0, and bb_0. We have increased the order of G by 2 and the size by 6. The new graph G' is still planar and has size $3(x+y+x+2)-6$ By Thereom 2, G is maximal. This maximal planar graph is a spanning subgraph of $K_{x+1,y+1,z}$.

Since $K_{x+1,y,z+1}$ and $K_{x,y+1,z+1}$ are isomorphic to $K_{x+1,y+1,z}$, each contains a spanning maximal planar subgraph. □

Two corollaries easily follow from Proposition 7.

Corollary 7.1. *Let $K_{x,y,z}$ be a complete tripartite graph that contains a spanning maximal planar subgraph. Then for each non-negative integer k, each of $K_{x+k,y+k,z}$, $K_{x+k,y,z+k}$, $K_{x,y+k,z+k}$ contains a spanning maximal planar subgraph.*

Corollary 7.2. *Let $K_{x,y,z}$ be a complete tripartite graph that contains a spanning maximal planar subgraph. Then for all non-negative integers a, b, c, the complete tripartite graph $K_{x+a+b,y+a+c,z+b+c}$ contains a spanning maximal planar subgraph.*

Proof. Let $K_{x,y,z}$ be a complete tripartite graph that contains a spanning maximal planar graph. By Corollary 7.1, $K_{x+a,y+a,z}$ contains a spanning maximal planar subgraph for all non-negative integers a. By the same Corollary applied to this new graph, the complete tripartite graph $K_{x+a+b,y+a,z+b}$ contains a spanning maximal planar subgraph for all non-negative integers b. The same Corollary applied to this graph implies that $K_{x+a+b,y+a+c,z+b+c}$ contains a spanning maximal planar subgraph for all non-negative integers c. □

We show that the construction in Corollary 7.2 generates all complete tripartite graphs of order greater than 6 with spanning maximal planar subgraphs when applied to $K_{2,2,2}$ and $K_{3,3,3}$.

Proposition 8. *Let x, y, z be positive integers, and $x+y+z = n \geq 6$ be even. Then $K_{x,y,z}$ contains a spanning maximal planar subgraph if and only if there exist non-negative integers a, b, c such that $x = 2+a+b$, $y = 2+a+c$, $z = 2+b+c$.*

Proof. Let x, y, z be positive integers, and $x+y+z = n \geq 6$ be even. First, assume that $K_{x,y,z}$ contains a spanning maximal planar subgraph. If $n = 6$, we can take $a = b = c = 0$. Assume that for the even integer $n > 6$, if the complete tripartite graph $K_{x,y,z}$ of order n has a spanning maximal planar subgraph then there exist non-negative integers a, b, c such that $x = 2+a+b$, $y = 2+a+c$, $z = 2+b+c$.

Let $K_{x',y',z'}$ be a complete tripartite graph of order $x'+y'+z' = n+2$ which contains a spanning maximal planar subgraph G. Without loss of generality, we may assume that $x' \leq y' \leq z'$. Let $K_{x',y',z'} = V_1 + V_2 + V_3$ and let G be a spanning maximal planar subgraph of this tripartite graph. By Propositions 5 and 6, we must have both x' and y' greater than 2. Let v be a vertex in G whose degree is 4 and assume without loss of generality that $v \in V_1$. Let v_1, v_2, v_3, v_4 be the neighbors of v and without loss of generality that $v_1v_2v_3v_4v_1$ is the cycle they form. Furthermore, we may assume that $v_1, v_3 \in V_2$ and $v_2, v_4 \in V_3$. Identify the vertices v_1, v_3 as one vertex in V_2 and delete v from V_1. The resulting graph G' is a spanning subgraph of $K_{x'-1,y'-1,z'}$. By hypothesis of induction, there exist non-negative integers a, b, c such that $x'-1 = 2+a+b$, $y'-1 = 2+a+c$, $z' = 2+b+c$. Therefore, $x' = 2+(a+1)+b$, $y' = 2+(a+1)+c$, $z' = 2+b+c$.

Conversely, Assume that there exist non-negative integers a, b, c such that $x = 2+a+b$, $y = 2+a+c$, $z = 2+b+c$. Since $K_{2,2,2}$ is a maximal planar graph, by Corollary 7.2, then $K_{x,y,z}$ has a spanning maximal planar subgraph. □

The following is similarly proven.

Proposition 9. *Let x, y, z be positive integers, and $x+y+z = n \geq 9$ be odd. Then $K_{x,y,z}$ contains a spanning maximal planar subgraph if and only if there exist non-negative integers a, b, c such that $x = 3+a+b$, $y = 3+a+c$, $z = 3+b+c$.*

We now prove Theorem 3.

Proof. We first consider the case when $x+y-z$ is even. In this case, $x+y+z$ is even and according to Proposition 8, the complete tripartite graph $K_{x,y,z}$ contains a maximal planar subgraph if and only if there exist non-negative integers a, b, c such that $x = 2+a+b$, $y = 2+a+c$, $z = 2+b+c$. Therefore, $x+y-z = (2+a+b+2+a+c)-(2+b+c) = 2+2a \geq 2$.

Conversely, assume that $x+y-z\geq 2$. Since $x\leq y\leq z$, it follows that $x-y+z\geq 2$ and $-x+y+z\geq 2$. Take the following three integers:

$$a=\frac{x+y-z-2}{2},\qquad b=\frac{x-y+z-2}{2},\qquad c=\frac{-x+y+z}{2}.$$

Clearly, a,b,c are non-negative integers. It is straightforward to verify that

$$\begin{aligned} x&=2+a+b,\\ y&=2+a+c,\\ z&=2+b+c, \end{aligned}$$

and the fact that $K_{2,2,2}$ is a maximal planar graph.

By Proposition 8, the complete tripartite graph $K_{x,y,z}$ contains a spanning maximal planar graph.

The case when $x+y-z$ is odd is similarly proven using Proposition 9.

The proof is now complete. □

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