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Premium Pricing under a Ruin Probability with Policy Deductible or with Benefit Limit

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Abstract

This study calculates the premium price of an insurance policy, one with policy deductible and another with benefit limit, using exponentially distributed claims such that the ruin probability does not exceed the insurer's desired ruin probability. Incorporating policy deductible and benefit limit lessen the risk of ruin of insurers. A general expression is derived to compute the premium prices of an insurance policy that includes deductible as well as a general expression for an insurance policy that includes benefit limit. Furthermore, the behavior of premium prices was observed in relation to ruin probability, policy deductible, and benefit limit. Based on derived expressions, the premium price increases as the ruin probability decreases. Moreover, the premium price varies inversely with deductible decreases and varies directly with benefit limit.

Keywords: Ruin probability, insurance, premium pricing

2010 MSC: 91B30, 97M30

1 Introduction

It is common in the insurance industry that once an insurance policy commences, the insurer will offer a fixed premium to be paid by the policyholder with the corresponding initial wealth. Insurers will take no other adjustments as long as the policy is in force. If the insurer could no longer pay claims to the insured, we say that the insurer incurs ruin. This can be attributed to the case where lots of loss from the insured have been reported or the insurer offers a very low premium that it cannot sustain the policy for the untimely claim of benefits.

Ruin probability is the probability that the total claim amount of the insured exceeded the sum of initial surplus and the total amount of premiums collected by the insurer. According to [8], ruin probability accounts for insurance risk and not for possible mismanagement. From this assumption, the insurer may be able to calculate the ruin probability which will suggest the actions to be done on the price of the premium. Adjustments on the premium level can be made to attain the desired ruin probability.

Another way to minimize the chances of ruin is by incorporating limits in the policy contract [1]. Examples of these limits are policy deductible and benefit limit. A policy deductible is an amount of money deducted from the amount of claim stated in the policy contract [1], while a benefit limit is an upper boundary on how much an insurance company will provide for any loss.

Constantinescu [3] cited developments in ruin theory such as well-structured models that allow explicit formulas for lower and upper bound of ruin probabilities, and for observation of asymptotical behavior as the initial wealth increases without bound. On the other hand, only a few studies have been made to show how premium prices can be used to calculate ruin probability, and one of those is that of Chan and Zhang [2] which was used in this study. This paper used their result to calculate the premium from the desired ruin probability level.

2 Theoretical Framework

Consider the collective risk process defined by

$$U(\omega, t) = \omega + pt - S(t) \tag{1}$$

where $U(\omega, t)$ is the insurer's surplus at time t, ω is the initial surplus, p is the premium per unit time and $S(t) = \sum_{i=1}^{N(t)} Z_i$ with Z_i as the ith claim and $N(t)$ as the number of claims on the interval $(0, t]$ and follows a Poisson process [6].

In the study, $N(t) = t$ is assumed which is valid when group insurance is considered and Z_i be the total amount of claims for year i. Thus, $S(t)$ can be rewritten as $S(t) = \sum_{i=1}^{t} Z_i$ where each Z_i are independent and identically distributed (i.i.d.) claim size random variables [7]. With these assumptions, it follows that

$$U(\omega, t) = \omega + pt - \sum_{i=1}^{t} Z_i. \tag{2}$$

Using the collective risk process, Gerber and Shiu [4] derived an equation to compute the finite-time ruin probability given by

$$\psi(\omega, t) = P(U(\omega, t) < 0 \,|\, U(\omega, 0) = \omega). \tag{3}$$

Let L be the loss amount random variable with parameter $\frac{1}{\lambda}$. Then the corresponding probability distribution function and density function is given by

$$F_L(l) = P(L \leq l) = 1 - e^{-\frac{l}{\lambda}}$$

and

$$f_L(l) = \frac{1}{\lambda} e^{-\frac{l}{\lambda}}.$$

The exponential distribution is considered because of its memoryless property for convenience in the computation of ruin probability [7].

2.1 Construction of Density Functions for Insurance Policy with Policy Deductible

Policy deductible is defined as an amount subtracted from the amount of loss incurred in a given policy [1]. If the loss is less than the deductible, no claim will be given. To better understand this concept, consider the following.

Example 2.1. *Consider a car insurance policy where the claim amount has a policy deductible of 5,000 PhP. If the total damage of the car in an accident costs less than 5,000 PhP, the insurer would not give any amount to the insured. That means the insured would cover all the expenses for the repair of the car. However, if the total damage costs more than 5,000 PhP, say 50,000 PhP, then the insurer would give an amount of 45,000 PhP (deducting 5,000 PhP from 50,000 PhP) to the insured.*

Definition 2.1 ([5])**.** *Let X be the loss random variable with policy deductible d. If L is a loss random variable, then*

$$X = \begin{cases} 0, & \text{if } L \leq d \\ L - d, & \text{if } L > d. \end{cases} \tag{4}$$

It can be easily shown that the probability distribution function for X is given by

$$F_X(x) = P(X \leq x) = 1 - e^{-\frac{x+d}{\lambda}} \tag{5}$$

and the density function for X is

$$f_X(x) = \frac{1}{\lambda} e^{-\frac{x+d}{\lambda}}. \tag{6}$$

2.2 Construction of Density Functions for Insurance Policy with Benefit Limit

Benefit limit is defined as the amount given to the insured if the loss is greater than the limit [1]. Otherwise, the original claim will be given to the insured. The following is an example to better understand this concept.

Example 2.2. *Consider the same policy in §2.1 but in this case, only benefit limit of 100,000 PhP is incorporated in the policy. Now, if the insured has reported a total damage amounting to 80,000 PhP, he/she would receive the whole amount. However, if the cost of total damage exceeds 100,000 PhP, say 150,000 PhP, the claim the insured would receive is the benefit limit 100,000 PhP.*

Definition 2.2 ([5]). *Let Y be a claim amount random variable with benefit limit denoted as ℓ, then Y is defined by*

$$Y = \begin{cases} \ell, & \text{if } L \geq \ell, \\ L, & \text{if } L < \ell. \end{cases} \tag{7}$$

Consider only the case that $\omega + p \leq \ell$. The insurer is capable enough to cover any losses whenever $\omega + p > \ell$. That is, when $\omega + p > \ell$, the ruin probability is always zero.

The distribution function for Y is given by

$$F_Y(y) = P(Y \leq y) = 1 - e^{-\frac{y}{\lambda}} \tag{8}$$

and the density function for Y is

$$f_Y(y) = \frac{1}{\lambda} e^{-\frac{y}{\lambda}}. \tag{9}$$

3 Derivation of Ruin Probability ψ

This section shows the general expression of ruin probability considering insurance with policy deductible presented in Theorem 3.1 and a general expression for ruin probability considering insurance with a benefit limit presented in Theorem 3.2. The derivations were done in reference to the result of [2].

3.1 Ruin Probability for Insurance Policy with Policy Deductible

The following theorem presents a closed-form expression for the ruin probability ψ with policy deductible d. The expression can be used to solve for the ruin probability while setting a premium price of p. Moreover, this can also be used to solve a premium given a ruin probability. This theorem is proven by induction.

Theorem 3.1. *Let X_i be the ith claim with policy deductible d which is independent, identically and exponentially distributed with parameter $\frac{1}{\lambda}$ with density function*

$$f_X(x) = \frac{1}{\lambda} e^{-\frac{x+d}{\lambda}}.$$

The ruin probability for an insurance policy with policy deductible is

$$\psi(\omega, t) = \sum_{i=1}^{t} \frac{(\omega + p)(\omega + ip)^{i-2}}{(i-1)!\lambda^{i-1}} e^{-\frac{\omega + i(p+d)}{\lambda}}. \tag{10}$$

Proof: We prove the derived general formula by the Principle of Mathematical Induction, patterned after [2].

At time $t = 1$, suppose that the first claim is X_1. The corresponding ruin probability is

$$\begin{aligned} \psi(\omega, 1) &= P(\omega + p - X_1 < 0) \\ &= P(X_1 > \omega + p) \\ &= 1 - P(X_1 \leq \omega + p) \\ &= e^{-\frac{\omega + p + d}{\lambda}} \end{aligned} \tag{11}$$

At time $t = 2$, suppose that the claims are X_1 and X_2 at times $t = 1$ and $t = 2$, respectively. The corresponding ruin probability is

$$\begin{aligned}\psi(\omega,2) &= \psi(\omega,1) + \int_0^{\omega+p} P(\omega+2p-X_1-X_2<0 \,|\, X_1=x_1) f_{X_1}(x_1)dx_1 \\ &= \psi(\omega,1) + \frac{\omega+p}{\lambda} e^{-\frac{\omega+2p+2d}{\lambda}} \qquad (12)\end{aligned}$$

The first term in (12) means that ruin has occurred at time $t = 1$ while the second term means that ruin has not occurred at time $t = 1$ but occurred at time $t = 2$.

In general, we get the expression

$$\psi(\omega,t) = \sum_{i=1}^{t} \frac{(\omega+p)(\omega+ip)^{i-2}}{(i-1)!\lambda^{i-1}} e^{-\frac{\omega+i(p+d)}{\lambda}} \qquad (13)$$

Now, suppose (13) holds. From [2], we know that

$$\psi(\omega,t) = \psi(\omega,1) + \int_0^{\omega+p} \psi(\omega+p-x,t-1) f(x)dx$$

and this implies

$$\psi(\omega,t) = \psi(\omega,1) + \int_0^{\omega+p} \psi(x,t-1) f(\omega+p-x)dx.$$

Thus,

$$\begin{aligned}\psi(\omega,t+1) &= \psi(\omega,1) + \int_0^{\omega+p} \psi(x,t-1) f(\omega+p-x)dx \\ &= e^{-\frac{\omega+p+d}{\lambda}} + \int_0^{\omega+p} \sum_{i=1}^{t} \frac{(x+p)(x+ip)^{i-2}}{(i-1)!\lambda^{i}} e^{-\frac{x+ip+id+\omega+p-x+d}{\lambda}} dx \\ &= e^{-\frac{\omega+p+d}{\lambda}} + \sum_{i=1}^{t} \frac{1}{\lambda^i} e^{-\frac{\omega+(i+1)(p+d)}{\lambda}} \frac{1}{i!}(\omega+p)(\omega+(i+1)p)^{i-1} \\ &= e^{-\frac{\omega+p+d}{\lambda}} + \sum_{i=1}^{t} \frac{(\omega+p)[\omega+(i+1)p]^{i-1}}{\lambda^i i!} e^{-\frac{\omega+(i+1)(p+d)}{\lambda}} \\ &= \sum_{i=1}^{t+1} \frac{(\omega+p)(\omega+ip)^{i-2}}{(i-1)!\lambda^{i-1}} e^{-\frac{\omega+i(p+d)}{\lambda}}\end{aligned}$$

This completes the induction. □

3.2 Ruin Probability for Insurance Policy with Benefit Limit

The following theorem presents a closed-form expression for the ruin probability ψ with benefit limit ℓ.

Theorem 3.2. *Let Y_i be the ith claim which is independent, identically, and exponentially distributed random variable with parameter $\frac{1}{\lambda}$ with density function*

$$f_Y(y) = \frac{1}{\lambda} e^{-\frac{y}{\lambda}}.$$

Then the ruin probability for an insurance policy with benefit limit at any time t is given by

$$\psi(\omega,t)=\begin{cases}(1-e^{-\frac{\ell}{\lambda}})e^{-\frac{\omega+p}{\lambda}}, & t=1,\\ \sum_{i=1}^{t-1}\left(1-e^{-\frac{\ell}{\lambda}}\right)^{i+1}\frac{(\omega+p)(\omega+ip)^{i-2}}{(i-1)!\lambda^{i-1}}e^{-\frac{\omega+ip}{\lambda}} & \\ \qquad +\left(1-e^{-\frac{\ell}{\lambda}}\right)^{t}\frac{(\omega+p)(\omega+tp)^{t-2}}{(t-1)!\lambda^{t-1}}e^{-\frac{\omega+tp}{\lambda}}, & t\geq 2.\end{cases} \tag{14}$$

Proof: We show the cases at times 1 and 2. At time $t=1$, since the limit is the maximum amount that the insurer will provide, we consider its possible relationship to the surplus of the insurer.

First, suppose $\omega+p>\ell$. We have:

$$\begin{aligned}\psi(\omega,1)&=P(L\leq\ell)(\psi(\omega)\,|\,\omega+p>\ell)+P(L>\ell)(\psi(\omega)\,|\,\omega+p>\ell)\\ &=\left(1-e^{-\frac{\ell}{\lambda}}\right)(0)+e^{-\frac{\ell}{\lambda}}(0)=0.\end{aligned}$$

For any amount of loss L greater than the benefit limit ℓ, the insurer will provide a benefit equal to the benefit limit. Moreover, the insurer will always be able to provide for the claims no matter what the amount is. In other words, the probability of ruin will always be zero since the money at hand will always be sufficient to provide for any claim at any time. Consequently, we will not consider the condition that $\omega+p>\ell$ throughout the process to eliminate the certainty of not being ruined.

Suppose now that $\omega+p\leq\ell$. Then

$$\psi(\omega,1)=P(L\leq\ell)P(\omega+p-Y_1<0|\omega+p>\ell)+P(L>\ell)P(\omega+p-Y_1<0|\omega+p>\ell).$$

This is the same as saying that the loss incurred is equal to the benefit limit where the insured gets a benefit equal to the benefit limit. Hence, $P(L>\ell)$ is already contained in $P(L\leq\ell)$. Thus,

$$\psi(\omega,1)=P(L\leq\ell)P(\omega+p-Y_1<0|\omega+p>\ell), \tag{15}$$

which leads to

$$\begin{aligned}\psi(\omega,1)&=P(L\leq l)P(L>\omega+p)\\ &=\left(1-e^{-\frac{l}{\lambda}}\right)e^{-\frac{\omega+p}{\lambda}}.\end{aligned} \tag{16}$$

For $t=2$, let us assume that the claims are Y_1 and Y_2 at times $t=1$ and $t=2$, respectively. We have

$$\begin{aligned}\psi(\omega,2)&=\left(1-e^{-\frac{\ell}{\lambda}}\right)\left[\psi(\omega,1)+\int_0^{\omega+p}\psi(\omega+p-y,1)f_Y(y)dy\right]\\ &=\left(1-e^{-\frac{\ell}{\lambda}}\right)\left[\left(1-e^{-\frac{l}{\lambda}}\right)e^{-\frac{\omega+p}{\lambda}}+\frac{\omega+p}{\lambda}\left(1-e^{-\frac{\ell}{\lambda}}\right)e^{-\frac{\omega+2p}{\lambda}}\right].\end{aligned} \tag{17}$$

Suppose now that

$$\begin{aligned}\psi(\omega,t)=&\sum_{i=1}^{t-1}\left(1-e^{-\frac{\ell}{\lambda}}\right)^{i+1}\frac{(\omega+p)(\omega+ip)^{i-2}}{(i-1)!\lambda^{i-1}}e^{-\frac{\omega+ip}{\lambda}}\\ &+\left(1-e^{-\frac{\ell}{\lambda}}\right)^{t-1}\frac{(\omega+p)(\omega+tp)^{t-2}}{(t-1)!\lambda^{t-1}}e^{-\frac{\omega+tp}{\lambda}}.\end{aligned} \tag{18}$$

From [2], we know that

$$\psi(\omega,t)=\psi(\omega,1)+\int_0^{\omega+p}\psi(\omega+p-y,t-1)f(y)dy,$$

and this implies that

$$\psi(\omega,t)=\psi(\omega,1)+\int_0^{\omega+p}\psi(y,t-1)f(\omega+p-y)dy.$$

Thus,

$$\begin{aligned}\psi(\omega,t+1)&=\psi(\omega,1)+\int_0^{\omega+p}\psi(y,t-1)f(\omega+p-y)dy\\&=\big(1-e^{-\frac{\ell}{\lambda}}\big)e^{-\frac{\omega+p}{\lambda}}\\&\quad+\int_0^{\omega+p}\left(\sum_{i=1}^{t-1}\big(1-e^{-\frac{\ell}{\lambda}}\big)^{i+1}\frac{(y+p)(y+ip)^{i-2}}{(i-1)!\lambda^{i-1}}e^{-\frac{y+ip+\omega+p-y}{\lambda}}\right)dy\\&\quad+\int_0^{\omega+p}\left(\big(1-e^{-\frac{l}{\lambda}}\big)^{t-1}\frac{(\omega+p)(\omega+tp)^{t-2}}{(t-1)!\lambda^{t-1}}e^{-\frac{y+tp+\omega+p-x}{\lambda}}\right)dy\\&=\sum_{i=1}^{t-1}\big(1-e^{-\frac{\ell}{\lambda}}\big)^{i+1}\frac{(\omega+p)(\omega+(i+1)p)^{i-1}}{(i-1)!\lambda^{i-1}}e^{-\frac{\omega+(i+1)p}{\lambda}}\\&\quad+\big(1-e^{-\frac{\ell}{\lambda}}\big)^{t}\frac{(\omega+p)(\omega+(t+1)p)^{t-1}}{(t-1)!\lambda^{t-1}}e^{-\frac{\omega+(t+1)p}{\lambda}}.\end{aligned}$$

This completes the induction and the proof of the theorem. □

For Theorems 3.1 and 3.2, the ruin probability ψ is dependent on t. This is the time for which the insurer receives a premium and gives claims to the insured. In this study, we are not concerned with t. Hence, in the next section, we fixed $t=10$. Moreover, for premium calculations, we set $\alpha=\psi(\omega,t)$ to be the desired ruin probability of the insurer.

4 Premium Calculation and Analysis

This section shows the calculation of the premium using the derived closed-form expressions in Theorem 3.1 and Theorem 3.2. Values for d, ω, λ, α and l in the following numerical simulations were taken for illustration purposes.

4.1 Insurance Policy with Policy Deductible

Using the closed-form expression in Theorem 3.1 and the Newton-Raphson method, we calculated the premiums by setting desired ruin probabilities along with different values of the deductible. We used the Newton-Raphson method since the function in (10) is differentiable in p and the method is is easy to implement. Figure 1 shows the behavior of premium prices with respect to ruin probabilities and different values of deductibles.

Our general observation from the graph is that the premium varies inversely with the ruin probability. This tells us that setting an insurer's desired ruin probability to a much lower value would require a great increase in the premium. The effect on premium price may be lessened by adding deductibles on the policy. As can be seen in Figure 1, the ruin

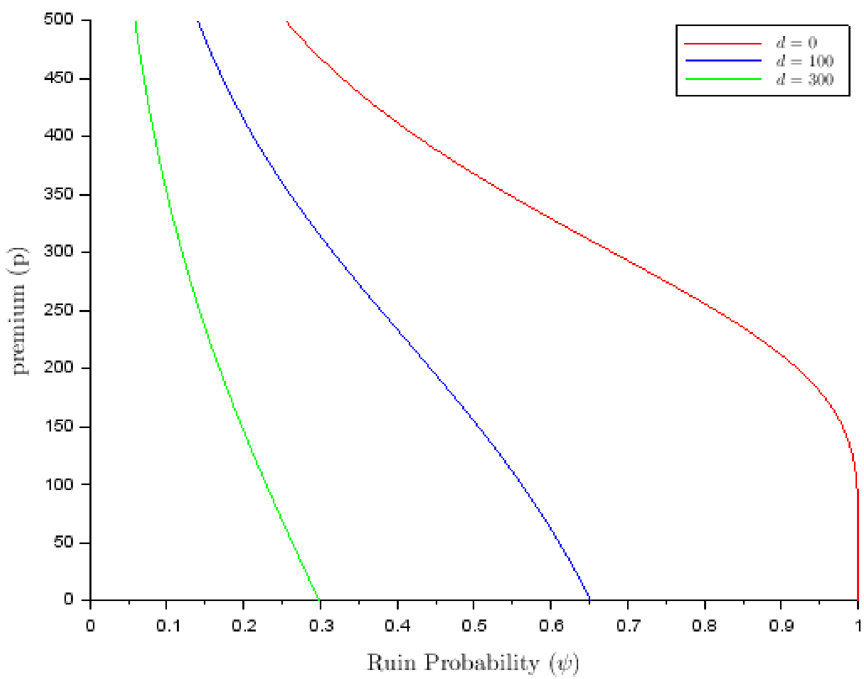

Figure 1: Premiums on different ruin probabilities when $\omega = 100$, $\lambda = 200$ and for various deductible amounts d.

probability decreases when the policy deductible increases. In other words, increasing the value of deductible would lessen the chance of an insurer to become insolvent.

Figure 2 shows that the premium price decreases as the deductible increases. Thus, reducing the claims amount received by an insured means also reducing the amount he or she would pay.

4.2 Insurance Policy with Benefit Limit

Using the closed-form expression in Theorem 3.2 and the secant method, we calculated the premiums by setting desired ruin probabilities along with different values of benefit limit. We used the secant method because obtaining the derivative of the function in (14) with respect to p is a bit complicated. Figure 3 shows the behavior of premium prices with respect to ruin probabilities and different values of benefit limit.

As does Figure 1, Figure 3 also shows that as ruin probability increases, the premium price decreases. Moreover, it can be seen that a higher value of the benefit limit gives a higher value of ruin probability. Thus, if the benefit limit gets higher, it is likely that the insurer will provide for higher claim amounts. This may result in a higher value of the ruin probability.

Figure 4 shows that increasing the benefit limit is also increasing the premium price. However, increasing further the benefit limit to a higher value seems to result to a smaller increase in the premium price.

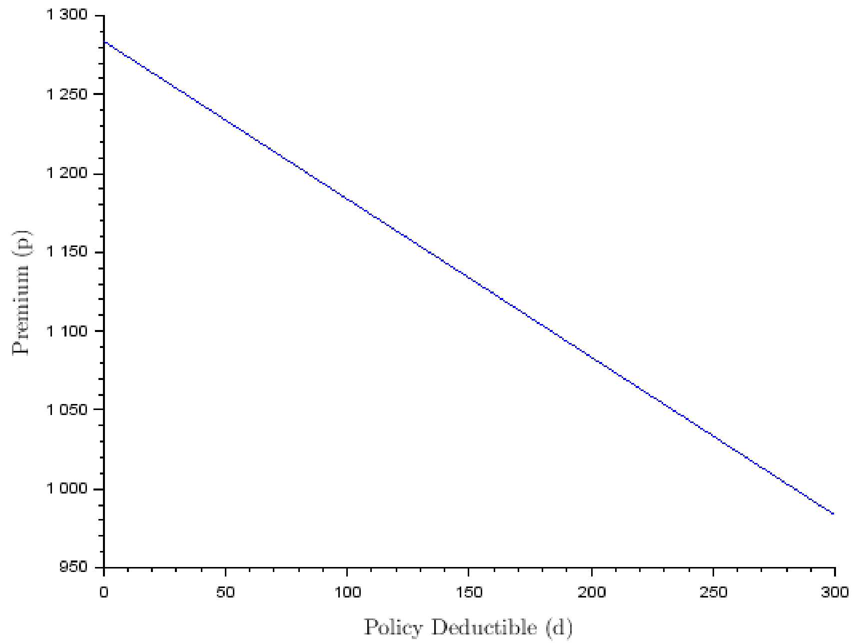

Figure 2: Premium price as policy deductible is varied, when $\omega = 100$, $\lambda = 200$, and $\alpha = 0.001$.

5 Conclusion

One of the main concerns of the insurers is solvency, their ability to meet the financial obligations to the insured over a long period of time. Calculating ruin probability is one way to determine the solvency of a company. Premium can be calculated if the desired ruin probability is known. Other modifications to the insurance policy can be used to lower the probability of ruin, and to meet the financial obligations of the insurer. Theorems 3.1 and 3.2 given in this paper can be used to determine levels of premium, initial surplus, deductible, and benefit limit to attain the desired ruin probability.

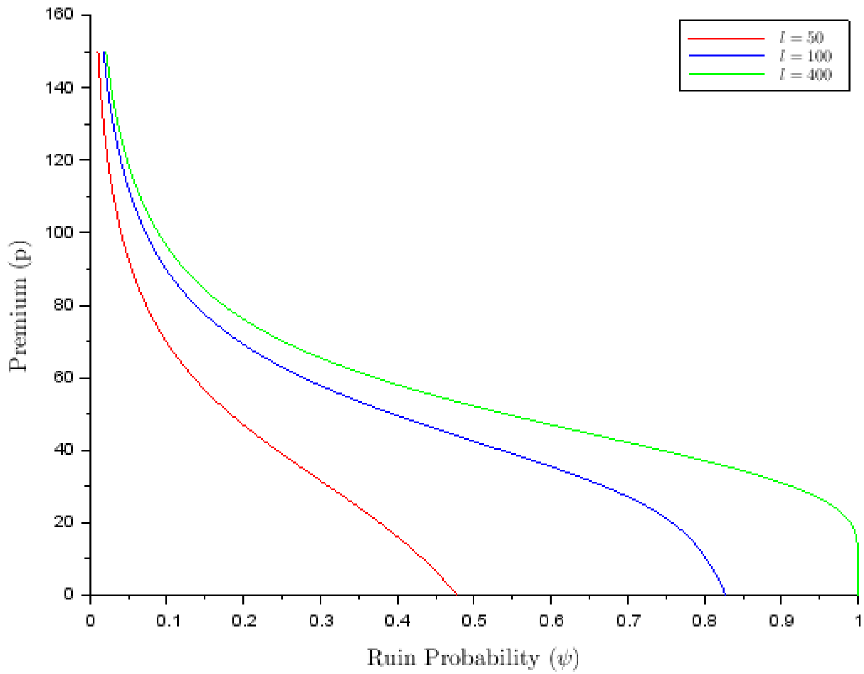

Figure 3: Premiums on different ruin probabilities when $\omega = 10$, $\lambda = 40$, and for various benefit limit values ℓ.

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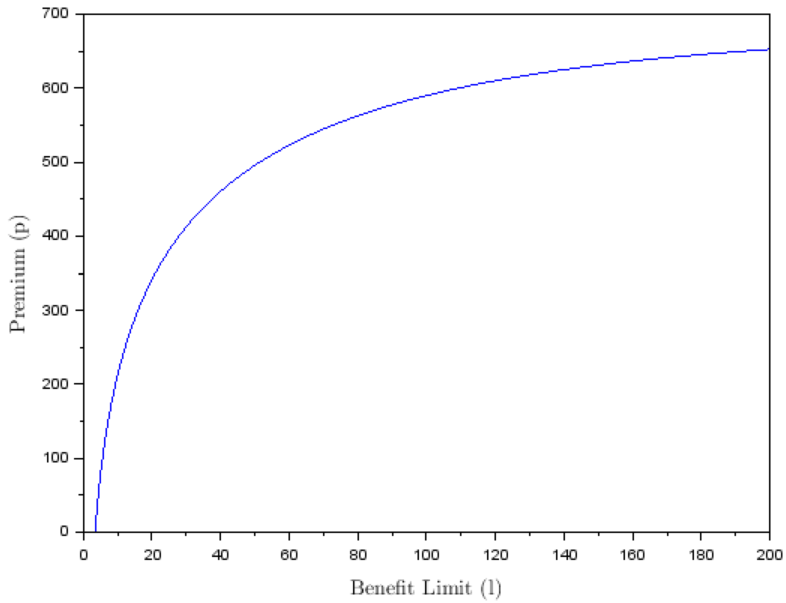

Figure 4: Premium prices with different values of benefit limit where $\omega = 10$, $\lambda = 100$, and $\alpha = 0.001$

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