Optimal Zero Ring Labeling Scheme for Trees

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Abstract

A new notion of vertex labeling for graphs, called zero ring labeling, is realized by assigning distinct elements of a zero ring to the vertices of the graph such that the sum of the labels of adjacent vertices is not equal to the identity 0 of the zero ring. The zero ring index of a graph $G$ is the smallest positive integer $\xi(G)$ such that there exists a zero ring of order $\xi(G)$ for which $G$ admits a zero ring labeling. Any zero ring labeling $f$ of $G$ is optimal if it uses a zero ring consisting of $\xi(G)$ elements. In this paper, it is shown that any tree of order $n$ has a zero ring index equal to $n$. Additionally, a scheme in obtaining an optimal zero ring labeling for any tree of order $n$ using the zero ring $M_2^0(\mathbb{Z}_n)$ is presented.

Keywords: zero ring labeling, zero ring index, labeling scheme, tree, bipartite

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1 Introduction

A graph labeling is an assignment of labels, traditionally by integers, to the vertices or edges, or both of a graph, subject to certain conditions. Several graph labeling techniques such as graceful, harmonious, magic-type and prime labelings have been studied. A dynamic survey on these labelings is regularly updated by Gallian [2].

A new notion of graph labeling, called zero ring labeling, was introduced by Acharya, Pranjali and Gupta [1]. Their empirical study demonstrated that every graph admits a zero ring labeling with respect to some zero ring.

Definition 1. [3] A ring $R$ in which the product of any two elements is 0, where 0 is the additive identity of $R$, is called a zero ring and is denoted by $R^0$.

In this paper, we consider one of the standard examples of zero rings. Let $R$ be a commutative ring. Denote by $M_2^0(R)$ the set of all $2 \times 2$ matrices of the form

\[
\begin{pmatrix}
a & -a \\
a & -a
\end{pmatrix}, a \in R.
\]
It can be verified that $M^0_2(R)$ is a ring. Moreover, for $a, b \in R, a \neq b$, we have

$$\begin{bmatrix} a & -a \\ a & -a \end{bmatrix} \begin{bmatrix} b & -b \\ b & -b \end{bmatrix} = \begin{bmatrix} ab - ab & -ab + ab \\ ab - ab & -ab + ab \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$ 

This shows that $M^0_2(R)$ is a zero ring.

**Definition 2.** Let $G = (V, E)$ be a graph with vertex set $V =: V(G)$ and edge set $E =: E(G)$, and let $R^0$ be a finite zero ring. An injective function $f : V(G) \rightarrow R^0$ is called a zero ring labeling of $G$ if $f(u) + f(v) \neq 0$ for every edge $uv \in E(G)$.

**Example 3.** Consider the following graph $G$:

![Graph Image](image)

Also, consider the zero ring $M^0_2(\mathbb{Z}_7)$ whose elements are

$$y_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, y_1 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, y_2 = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}, \ldots, y_6 = \begin{bmatrix} 6 & -6 \\ 6 & -6 \end{bmatrix}.$$ 

(a) Define a function $f : V(G) \rightarrow M^0_2(\mathbb{Z}_7)$ such that $f(v_i) = y_i$, for all $0 \leq i \leq 6$. Then, $f$ is an injective function. Moreover, $f(v_1) + f(v_3) = y_4$, $f(v_3) + f(v_0) = y_3$, $f(v_0) + f(v_4) = y_4$, $f(v_2) + f(v_4) = y_6$, $f(v_5) + f(v_3) = y_1$ and $f(v_4) + f(v_6) = y_3$. That is, $f(v_1) + f(v_2) \neq y_6$ for all $v_1v_2 \in E(G)$. Thus, $f$ is a zero ring labeling of $G$ with respect to the zero ring $M^0_2(\mathbb{Z}_7)$.

(b) Define a function $g : V(G) \rightarrow M^0_2(\mathbb{Z}_7)$ such that $g(v_i) = y_{i+2}$, for all $0 \leq i \leq 6$. Note that $g$ is also an injective function. Consider the edge $v_0v_3$ of the graph $G$. Then, $g(v_0) + g(v_3) = y_2 + y_5 = y_0$. Since $g(v_i) + g(v_j) = y_0$ for some $v_iv_j \in E(G)$, we conclude that $g$ is not a zero ring labeling of the graph $G$.

**Remark 4.** If $f$ is a zero ring labeling of a graph $G$, then $f$ is also a zero ring labeling of every subgraph of $G$.

**Definition 5.** Let $G = (V, E)$ be a finite graph of order $n$. The zero ring index of $G$ is the least positive integer $\xi(G)$ such that there exists a zero ring $R^0$ of order $\xi(G)$, for which $G$ admits a zero ring labeling. Any zero ring labeling $f$ of $G$ is optimal if it uses a zero ring consisting of $\xi(G)$ elements.

In [1], lower and upper bounds for the zero ring index of a graph were obtained, that is,

$$n \leq \xi(G) \leq 2^k,$$ 

(1)

where $n$ is the order of the graph $G$ and $k$ is the ceiling of $\log_2 n$. Moreover, the zero ring index of a complete graph was found to be equal to its order $n$ if and only if $n = 2^{k_0}$ for some positive integer $k_0$. The zero ring index of cycle graphs and the Petersen graph was studied in [1] and both were likewise found to have zero ring indices equal to their orders.
2 Zero Ring Index of Trees

In this section, we establish that trees, a special class of bipartite graphs, have zero ring
index equal to the order of the graph.

A connected graph without any cycle is called a tree. In a tree, a vertex of degree 1 is
called a leaf.

One interesting class of graphs akin to trees are bipartite graphs. A graph $G = (V_1, V_2, E)$
is said to be bipartite if its vertex set can be partitioned into nonempty disjoint subsets $V_1$
and $V_2$, called partite sets, in such a way that each edge $e \in E$ connects a vertex in $V_1$ to
one vertex in $V_2$.

Bipartite graphs have several different characterizations. One popular characterization
is given in the following lemma:

**Lemma 6.** [5] A graph $G$ is bipartite if and only if $G$ does not have any odd cycle.

One special kind of bipartite graph is a complete bipartite graph $(V_1, V_2, E)$ in which
every vertex in $V_1$ is adjacent to each vertex in $V_2$. If $|V_1| = m$ and $|V_2| = q$, then the
complete bipartite graph is denoted by $K_{m,q}$.

The following theorem indicates the existence of a zero ring labeling of a complete
bipartite $K_{m,q}$ with respect to a zero ring of order $m + q$.

**Theorem 7.** For positive integers $m$ and $q$ such that $m + q = n$, $\xi(K_{m,q}) = n$.

**Proof:** Without loss of generality, we assume $m \leq q$. Let $V_1$ and $V_2$ denote the partite
sets of the vertices of $K_{m,q}$ such that

$$V_1 = \{v_0, v_1, v_2, ..., v_{m-1}\} \text{ and } V_2 = \{v_m, v_{m+1}, v_{m+2}, ..., v_{n-1}\}.$$

An edge in $K_{m,q}$ is of the form $v_iv_j$ where $0 \leq i \leq m - 1$ and $m \leq j \leq n - 1$. To prove the
desired result, we need to tackle three cases.

**Case 1: m=1.** Define $f : V(K_{m,q}) \to M_2^0(\mathbb{Z}_n)$ such that

$$f(v_i) = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix}, \text{ for all } 0 \leq i \leq n - 1.$$

Then, $f$ is an injective function. Moreover, for all $v_0v_j \in E(K_{1,q})$, $1 \leq j \leq n - 1$,

$$f(v_0) + f(v_j) = \begin{bmatrix} j & -j \\ j & -j \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (mod \ n).$$

Thus, $f$ is a zero ring labeling of $K_{1,q}$ with respect to the ring $M_2^0(\mathbb{Z}_n)$. In view of Definition
1.3, we conclude that $\xi(K_{1,q}) = n$. 

Case 2: \(1 < m \leq q\) and \(m\) is even. Define \(f : V(K_{m,q}) \rightarrow M_2^n(Z_n)\) such that

\[
f(v_i) = \begin{cases} 
\begin{bmatrix} i + 1 & -(i + 1) \\
 i + 1 & -(i + 1) 
\end{bmatrix}, & 0 \leq i < \frac{m}{2} \\
\begin{bmatrix} n - i - 1 + \frac{m}{2} & -(n - i - 1 + \frac{m}{2}) \\
 n - i - 1 + \frac{m}{2} & -(n - i - 1 + \frac{m}{2}) 
\end{bmatrix}, & \frac{m}{2} \leq i < m \\
\begin{bmatrix} 0 & 0 \\
 0 & 0 
\end{bmatrix}, & i = m \\
\begin{bmatrix} i - \frac{m}{2} & -(i - \frac{m}{2}) \\
 i - \frac{m}{2} & -(i - \frac{m}{2}) 
\end{bmatrix}, & m < i \leq n - 1 
\end{cases}.
\]

Note that \(f\) is an injective function. Let \(v_iv_j \in E(K_{m,q})\). We have \(0 \leq i \leq m - 1\) and \(m \leq j \leq n - 1\).

If \(0 \leq i \leq \frac{m}{2} - 1\) and \(j = m\), then

\[
f(v_i) + f(v_j) = \begin{bmatrix} i + 1 & -(i + 1) \\
 i + 1 & -(i + 1) 
\end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\
 0 & 0 
\end{bmatrix},
\]

since \(1 \leq i + 1 \leq \frac{m}{2}\).

If \(\frac{m}{2} \leq i \leq m - 1\) and \(j = m\), then

\[
f(v_i) + f(v_j) = \begin{bmatrix} n - i - 1 + \frac{m}{2} & -(n - i - 1 + \frac{m}{2}) \\
 n - i - 1 + \frac{m}{2} & -(n - i - 1 + \frac{m}{2}) 
\end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\
 0 & 0 
\end{bmatrix},
\]

since \(n - \frac{m}{2} \leq n - i - 1 + \frac{m}{2} \leq n - 1\).

If \(0 \leq i \leq \frac{m}{2} - 1\) and \(m + 1 \leq j \leq n - 1\), then

\[
f(v_i) + f(v_j) = \begin{bmatrix} i + j + 1 - \frac{m}{2} & -(i + j + 1 - \frac{m}{2}) \\
 i + j + 1 - \frac{m}{2} & -(i + j + 1 - \frac{m}{2}) 
\end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\
 0 & 0 
\end{bmatrix},
\]

since \(\frac{m}{2} + 2 \leq i + j + 1 - \frac{m}{2} \leq n - 1\).

Finally, if \(\frac{m}{2} \leq i \leq m - 1\) and \(m + 1 \leq j \leq n - 1\), then

\[
f(v_i) + f(v_j) = \begin{bmatrix} n - 1 - i + j & -(n - 1 - i + j) \\
 n - 1 - i + j & -(n - 1 - i + j) 
\end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\
 0 & 0 
\end{bmatrix},
\]

since \(1 \leq n - 1 - i + j \leq n - 2 - \frac{m}{2}\) (mod \(n\)).
Case 3: \(1 < m \leq q\) and \(m\) is odd. Define \(f: V(K_{m,q}) \to M_2^0(\mathbb{Z}_n)\) such that

\[
f(v_i) = \begin{cases} 
  i + 1 & -(i + 1) \\
  i + 1 & -(i + 1) \\
  n - i - 1 + \frac{m-1}{2} & -(n - i - 1 + \frac{m-1}{2}) \\
  n - i - 1 + \frac{m-1}{2} & -(n - i - 1 + \frac{m-1}{2}) \\
  0 & 0 \\
  0 & 0 \\
  i - \frac{m-1}{2} & -(i - \frac{m-1}{2}) \\
  i - \frac{m-1}{2} & -(i - \frac{m-1}{2}) \\
\end{cases}, \quad 0 \leq i < \frac{m-1}{2} \\
\begin{cases} 
  0 & 0 \\
  i - \frac{m-1}{2} & -(i - \frac{m-1}{2}) \\
  i - \frac{m-1}{2} & -(i - \frac{m-1}{2}) \\
\end{cases}, \quad i = m - 1 \\
\begin{cases} 
  0 & 0 \\
  i - \frac{m-1}{2} & -(i - \frac{m-1}{2}) \\
  i - \frac{m-1}{2} & -(i - \frac{m-1}{2}) \\
\end{cases}, \quad m \leq i \leq n - 1
\]

Clearly, \(f\) is injective. It remains to show that if \(v_i v_j \in E(K_{m,q})\),

\[
f(v_i) + f(v_j) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \pmod{n}.
\]

If \(0 \leq i \leq \frac{m-1}{2} - 1\) and \(m \leq j \leq n - 1\), then

\[
f(v_i) + f(v_j) = \begin{bmatrix} i + j + 1 - \frac{m-1}{2} & -(i + j + 1 - \frac{m-1}{2}) \\
  n - 1 - i + j & -(n - 1 - i + j) \\
\end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\]

since \(\frac{m+1}{2} \leq i + j + 1 - \frac{m-1}{2} \leq n - 1\).

If \(\frac{m-1}{2} \leq i \leq m - 2\) and \(m \leq j \leq n - 1\), then

\[
f(v_i) + f(v_j) = \begin{bmatrix} n - i - 1 + \frac{m-1}{2} & -(n - i - 1 + \frac{m-1}{2}) \\
  n - 1 - i + j & -(n - 1 - i + j) \\
\end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\]

since \(1 \leq n - 1 - i + j \leq n - 2 - \frac{m-1}{2} \pmod{n}\).

Lastly, if \(i = m - 1\) and \(m \leq j \leq n - 1\), then \(j \geq m > \frac{m-1}{2}\) and

\[
f(v_i) + f(v_j) = \begin{bmatrix} j - \frac{m-1}{2} & -(j - \frac{m-1}{2}) \\
  j - \frac{m-1}{2} & -(j - \frac{m-1}{2}) \\
\end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

We have thus shown that \(f\) is a zero ring labeling of \(K_{m,q}\) and \(\xi(K_{m,q}) = m + q = n\).

**Theorem 8.** If \(G\) is a bipartite graph of order \(n\), then \(\xi(G) = n\).

**Proof:** Let \(G = (V_1, V_2, E)\) be a bipartite graph of order \(n\) such that \(|V_1| = m\) and \(|V_2| = q\). Consider the complete bipartite \(K_{m,q}\) with vertex set \(V_1 \cup V_2\). Since \(E(G) \subseteq E(K_{m,q})\), we see that \(G\) is a spanning subgraph of \(K_{m,q}\). It follows that a zero ring labeling of \(K_{m,q}\) with respect to the zero ring \(M_2^0(\mathbb{Z}_n)\), where \(n = m + q\), is also a zero ring labeling of \(G\). Hence, \(\xi(G) = n\).

**Theorem 9.** For every tree \(T\) of order \(n\), \(\xi(T) = n\).

**Proof:** Let \(T\) be a tree of order \(n\). Since \(T\) does not contain any cycle, \(T\) is bipartite, by Lemma 8. It then follows from Theorem 8 that \(\xi(T) = n\).

The previous theorem guarantees that any tree of order \(n\) admits an optimal zero ring labeling with respect to a zero ring of order \(n\). Interestingly, such optimal zero ring labeling can be constructed using the zero ring \(M_2^0(\mathbb{Z}_n)\).
3 Optimal Zero Ring Labeling of Trees

This section presents a scheme for obtaining an optimal zero ring labeling for trees using the zero ring \( M_2^0(\mathbb{Z}_n) \).

The basis for the labeling is the zero ring labeling of the complete bipartite \( K_{m,q} \) discussed in the proof of Theorem 7. We denote the matrix \( \begin{bmatrix} i & -i \\ i & -i \end{bmatrix} \in M_2^0(\mathbb{Z}_n) \) by \( A_i \).

Consider a tree \( T \) of order \( n \). The labeling scheme starts with partitioning the \( n \) vertices of \( T \) to form the partite sets \( V_1 \) and \( V_2 \). Without loss of generality, we choose \( V_1 \) such that \( |V_1| \leq |V_2| \).

Case 1. If \( |V_1| = 1 \), then the lone vertex in \( V_1 \) is labeled \( A_0 \) and we assign the labels \( A_i \), where \( i = 1, 2, ..., n - 1 \), to the vertices in \( V_2 \).

Case 2. If \( 1 < |V_1| \leq |V_2| \) and \( |V_1| \) is even, then the vertices in \( V_1 \) are labeled by \( A_i \), where \( i = 1, 2, ..., \frac{m}{2}, n - 1, n - 2, ..., n - \frac{m}{2} \). On the other hand, the vertices in \( V_2 \) are labeled by \( A_i \), where \( i = 0, \frac{m}{2} + 1, \frac{m}{2} + 2, ..., n - 1 - \frac{m}{2} \).

Case 3. If \( 1 < |V_1| \leq |V_2| \) and \( |V_1| \) is odd, then the vertices in \( V_1 \) are labeled by \( A_i \), where \( i = 0, 1, 2, ..., \frac{m-1}{2}, n - 1, n - 2, ..., n - \frac{m-1}{2} \), while the vertices in \( V_2 \) are labeled by \( A_i \), where \( i = \frac{m+1}{2}, \frac{m+1}{2} + 1, ..., n - 1 - \frac{m-1}{2} \).

For special kinds of trees, the method of partitioning the vertices into partite sets \( V_1 \) and \( V_2 \) is discussed in the following subsections. In the illustrations, vertices belonging to the partite set \( V_1 \) are colored white while the vertices in \( V_2 \) are colored black.

3.1 Star Graph

The star graph \( S_n \) of order \( n \) is a special kind of tree with one central vertex having degree \( n - 1 \) and the other \( n - 1 \) vertices, called leaves, having degree 1. The star graph \( S_n \) is therefore isomorphic to the complete bipartite \( K_{1,n-1} \).

Let \( v_0 \) denote the central vertex of \( S_n \) and let \( v_i \), \( 1 \leq i \leq n - 1 \), be its adjacent vertices. Then, \( V_1 = \{v_0\} \) and \( V_2 = \{v_1, v_2, ..., v_{n-1}\} \).

For the labeling of vertices of \( S_n \), case 1 applies.

Figure 3.1 shows a zero ring labeling of a star graph of order 6.
3.2 Caterpillar and Double Star Graphs

A caterpillar graph is a tree derived from a path by hanging any number of leaves from the vertices of the path.

Let \([w_0, w_1, w_2, ..., w_r]\) denote the path in the caterpillar and let \(n_i, i = 0, 1, 2, ..., r\), denote the number of leaves hanging from \(w_i\). The leaves hanging from \(w_i\) are denoted by \(w_1^i, w_2^i, ..., w_{ni}^i\).

If \(r = 1\), then the caterpillar is called a double star graph.

To determine the partite sets \(V_1\) and \(V_2\) of the vertex set of the caterpillar graph, we classify first the vertices into two sets \(A\) and \(B\). To do this, we place \(w_k\), where \(k\) is even, in set \(A\) and its adjacent vertices \(w_1^k, w_2^k, ..., w_{ni}^k\) in set \(B\). Further, we place \(w_l\), where \(l\) is odd, in set \(B\) and all its adjacent vertices \(w_1^l, w_2^l, ..., w_{ni}^l\) in set \(A\). Rename these sets as \(V_1\) and \(V_2\) such that \(|V_1| = m \leq q = |V_2|\). Relabel the vertices in \(V_1\) (white vertices) using \(v_0, v_1, ..., v_{m-1}\) in any order and relabel the vertices in \(V_2\) (black vertices) arbitrarily using \(v_m, v_{m+1}, ..., v_{n-1}\) where \(n = m + q\). Provide the zero ring labeling according to the case in Theorem 7 where this falls.

Figure 3.2 shows the zero ring labeling of a caterpillar of order 15 with \(|V_1| = 6\) and \(|V_2| = 9\). Here, case 2 applies.

3.3 Lobster

A lobster graph is a tree in which all the vertices are within distance 2 of a central path. It has the property that the removal of leaf nodes results in a caterpillar graph.

Let \([w_0, w_1, w_2, ..., w_r]\) denote the central path in the lobster graph. Excluding the vertices of the central path, let \(w_1^i, w_2^i, ..., w_{ni}^i\) be the vertices of distance 1 from \(w_i\), \(i = 0, 1, 2, ..., r\).

Let \(p_{i,n_i}\) denote the number of leaves hanging from \(w_{ni}^i\). The leaves hanging from \(w_{ni}^i\) are denoted by \(w_1^{ni}, w_2^{ni}, ..., w_{p_i,n_i}^{ni}\).

We now classify the vertices into two sets \(A\) and \(B\).

We start by placing vertex \(w_k\), where \(k\) is even, in set \(A\) and its adjacent vertices \(w_1^k, w_2^k, ..., w_{ni}^k\) in set \(B\). The leaves hanging from any of \(w_1^k, w_2^k, ..., w_{ni}^k\) join the vertex \(w_k\) in set \(A\). On the other hand, we place \(w_1\), where \(l\) is odd, in set \(B\) and all its adjacent
vertices $w^1_1, w^2_1, ..., w^n_1$ in set $A$. The leaves hanging from any of $w^1_1, w^2_1, ..., w^n_1$ are placed back to set $B$. Rename these sets as $V_1$ and $V_2$ such that $|V_1| = m \leq q = |V_2|$. Then, the vertices in $V_1$ (white vertices) and $V_2$ (black vertices) are relabeled using $v_0, v_1, ..., v_{m-1}$ and $v_m, v_{m+1}, ..., v_n$, respectively. The zero ring labeling is obtained according to the case in Theorem 7 where this falls.

Figure 3.3 shows a zero ring labeling of a lobster of order 22 with $|V_1| = |V_2| = 11$. In this example, case 3 applies.

3.4 Rooted Tree

A **rooted tree** is a tree in which a particular vertex is designated as the root and every edge is directed away from the root. The **level** of a vertex $v$ is the length of the simple path from the root to $v$. The **height** of a rooted tree is the maximum level of its vertices.

Consider a rooted tree $T$ of order $n$. Let $A$ be the set of all vertices in the even-numbered levels and let $B$ be the set of all vertices in the odd-numbered levels. As above, these sets are renamed as $V_1$ and $V_2$ such that $|V_1| = m \leq q = |V_2|$ and the same procedure is performed to obtain the zero ring labeling of $T$.

A rooted tree of order 24 is shown in Figure 3.4. The height of the tree is 4. Observe that there are 13 vertices in the even-numbered levels and there are 11 vertices in the odd-numbered levels. Hence, the vertices of levels 1 and 3 form $V_1$ and are labeled as $v_0, v_1, ..., v_9$. On the other hand, the vertices of levels 2 and 4 join the root in $V_2$ and are labeled as $v_{10}, v_{11}, ..., v_{23}$.
Since $|V_1|$ is an odd integer greater than 1, we apply case 3. The zero ring labeling of the rooted tree above is given in Figure 3.3.

In general, any tree $T$ can be redrawn as a rooted tree by designating any one vertex of $T$ as the root. This means that a zero ring labeling of any tree of order $n$ using the zero ring $M_0^2(Z_n)$ can be obtained by first transforming it into a rooted tree and applying the scheme described earlier.

4 References


