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Optimal Zero Ring Labeling Scheme for Trees

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Abstract

A new notion of vertex labeling for graphs, called *zero ring labeling*, is realized by assigning distinct elements of a zero ring to the vertices of the graph such that the sum of the labels of adjacent vertices is not equal to the identity 0 of the zero ring. The *zero ring index* of a graph G is the smallest positive integer $\xi(G)$ such that there exists a zero ring of order $\xi(G)$ for which G admits a zero ring labeling. Any zero ring labeling f of G is *optimal* if it uses a zero ring consisting of $\xi(G)$ elements. In this paper, it is shown that any tree of order n has a zero ring index equal to n. Additionally, a scheme in obtaining an optimal zero ring labeling for any tree of order n using the zero ring $M_2^0(\mathbb{Z}_n)$ is presented.

Keywords: zero ring labeling, zero ring index, labeling scheme, tree, bipartite

2010 MSC: 05C05, 05C78, 05C25

1 Introduction

A graph labeling is an assignment of labels, traditionally by integers, to the vertices or edges, or both of a graph, subject to certain conditions. Several graph labeling techniques such as graceful, harmonious, magic-type and prime labelings have been studied. A dynamic survey on these labelings is regularly updated by Gallian [2].

A new notion of graph labeling, called zero ring labeling, was introduced by Acharya, Pranjali and Gupta [1]. Their empirical study demonstrated that every graph admits a zero ring labeling with respect to some zero ring.

Definition 1 ([3]). *A ring R in which the product of any two elements is 0, where 0 is the additive identity of R, is called a* zero ring *and is denoted by R^0.*

In this paper, we consider one of the standard examples of zero rings. Let R be a commutative ring. Denote by $M_2^0[R]$ the set of all 2×2 matrices of the form

$$\begin{bmatrix} a & -a \\ a & -a \end{bmatrix},\ a \in R.$$

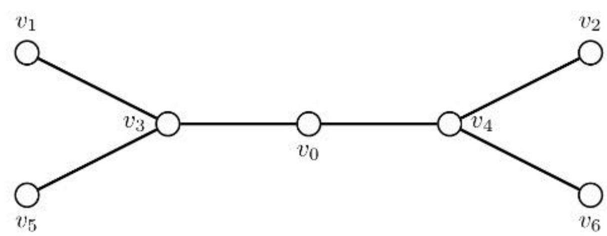
It can be verified that $M_2^0(R)$ is a ring. Moreover, for $a, b \in R, a \neq b$, we have

$$\begin{bmatrix} a & -a \\ a & -a \end{bmatrix} \begin{bmatrix} b & -b \\ b & -b \end{bmatrix} = \begin{bmatrix} ab - ab & -ab + ab \\ ab - ab & -ab + ab \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This shows that $M_2^0(R)$ is a zero ring.

Definition 2 ([1])**.** *Let $G = (V, E)$ be a graph with vertex set $V =: V(G)$ and edge set $E =: E(G)$, and let R^0 be a finite zero ring. An injective function $f : V(G) \to R^0$ is called a* zero ring labeling *of G if $f(u) + f(v) \neq 0$ for every edge $uv \in E(G)$.*

Example 3. *Consider the following graph G:*



Also, consider the zero ring $M_2^0(\mathbb{Z}_7)$ whose elements are

$$y_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, y_1 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, y_2 = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}, \ldots, y_6 = \begin{bmatrix} 6 & -6 \\ 6 & -6 \end{bmatrix}.$$

(a) Define a function $f : V(G) \to M_2^0(\mathbb{Z}_7)$ such that $f(v_i) = y_i$, for all $0 \leqslant i \leqslant 6$. Then, f is an injective function. Moreover, $f(v_1) + f(v_3) = y_4$, $f(v_3) + f(v_0) = y_3$, $f(v_0) + f(v_4) = y_4$, $f(v_2) + f(v_4) = y_6$, $f(v_5) + f(v_3) = y_1$ and $f(v_4) + f(v_6) = y_3$. That is, $f(v_i) + f(v_j) \neq y_0$ for all $v_i v_j \in E(G)$. Thus, f is a zero ring labeling of G with respect to the zero ring $M_2^0(\mathbb{Z}_7)$.

(b) Define a function $g : V(G) \to M_2^0(\mathbb{Z}_7)$ such that $g(v_i) = y_{i+2}$, for all $0 \leqslant i \leqslant 6$. Note that g is also an injective function. Consider the edge v_0v_3 of the graph G. Then, $g(v_0) + g(v_3) = y_2 + y_5 = y_0$. Since $g(v_i) + g(v_j) = y_0$ for some $v_i v_j \in E(G)$, we conclude that g is not a zero ring labeling of the graph G.

Remark 4. *If f is a zero ring labeling of a graph G, then f is also a zero ring labeling of every subgraph of G.*

Definition 5 ([1])**.** *Let $G = (V, E)$ be a finite graph of order n. The* zero ring index *of G is the least positive integer $\xi(G)$ such that there exists a zero ring R^0 of order $\xi(G)$, for which G admits a zero ring labeling. Any zero ring labeling f of G is* optimal *if it uses a zero ring consisting of $\xi(G)$ elements.*

In [1], lower and upper bounds for the zero ring index of a graph were obtained, that is,

$$n \leqslant \xi(G) \leqslant 2^k, \tag{1}$$

where n is the order of the graph G and k is the ceiling of $log_2 n$. Moreover, the zero ring index of a complete graph was found to be equal to its order n if and only if $n = 2^{k_0}$ for some positive integer k_0. The zero ring index of cycle graphs and the Petersen graph was studied in [4] and both were likewise found to have zero ring indices equal to their orders.

2 Zero Ring Index of Trees

In this section, we establish that trees, a special class of bipartite graphs, have zero ring index equal to the order of the graph.

A connected graph without any cycle is called a *tree*. In a tree, a vertex of degree 1 is called a *leaf*.

One interesting class of graphs akin to trees are bipartite graphs. A graph $G = (V_1, V_2, E)$ is said to be *bipartite* if its vertex set can be partitioned into nonempty disjoint subsets V_1 and V_2, called *partite sets*, in such a way that each edge $e \in E$ connects a vertex in V_1 to one vertex in V_2.

Bipartite graphs have several different characterizations. One popular characterization is given in the following lemma:

Lemma 6 ([5])**.** *A graph G is bipartite if and only if G does not have any odd cycle.*

One special kind of bipartite graph is a complete bipartite graph (V_1, V_2, E) in which every vertex in V_1 is adjacent to each vertex in V_2. If $|V_1| = m$ and $|V_2| = q$, then the complete bipartite graph is denoted by $K_{m,q}$.

The following theorem indicates the existence of a zero ring labeling of a complete bipartite $K_{m,q}$ with respect to a zero ring of order $m + q$.

Theorem 7. *For positive integers m and q such that $m + q = n$, $\xi(K_{m,q}) = n$.*

Proof: Without loss of generality, we assume $m \leqslant q$. Let V_1 and V_2 denote the partite sets of the vertices of $K_{m,q}$ such that $V_1 = \{v_0, v_1, v_2, \dots, v_{m-1}\}$ and $V_2 = \{v_m, v_{m+1}, v_{m+2}, \dots, v_{n-1}\}$. An edge in $K_{m,q}$ is of the form $v_i v_j$ where $0 \leqslant i \leqslant m-1$ and $m \leqslant j \leqslant n-1$. To prove the desired result, we need to tackle three cases.

Case 1: m=1. Define $f : V(K_{m,q}) \to M_2^0(\mathbb{Z}_n)$ such that

$$f(v_i) = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix}, \quad \text{for all } 0 \leqslant i \leqslant n-1.$$

Then, f is an injective function. Moreover, for all $v_0 v_j \in E(K_{1,q})$, $1 \leqslant j \leqslant n-1$,

$$f(v_0) + f(v_j) = \begin{bmatrix} j & -j \\ j & -j \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (\text{mod } n).$$

Thus, f is a zero ring labeling of $K_{1,q}$ with respect to the ring $M_2^0(\mathbb{Z}_n)$. In view of Definition 1.3, we conclude that $\xi(K_{1,q}) = n$.

Case 2: $1 < m \leqslant q$ and m is even. Define $f : V(K_{m,q}) \to M_2^0(\mathbb{Z}_n)$ such that

$$f(v_i) = \begin{cases} \begin{bmatrix} i+1 & -(i+1) \\ i+1 & -(i+1) \end{bmatrix}, & 0 \leqslant i < \frac{m}{2}, \\ \begin{bmatrix} n-i-1+\frac{m}{2} & -(n-i-1+\frac{m}{2}) \\ n-i-1+\frac{m}{2} & -(n-i-1+\frac{m}{2}) \end{bmatrix}, & \frac{m}{2} \leqslant i < m, \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & i = m, \\ \begin{bmatrix} i-\frac{m}{2} & -(i-\frac{m}{2}) \\ i-\frac{m}{2} & -(i-\frac{m}{2}) \end{bmatrix}, & m < i \leqslant n-1. \end{cases}.$$

Note that f is an injective function. Let $v_iv_j \in E(K_{m,q})$. We have $0 \leqslant i \leqslant m-1$ and $m \leqslant j \leqslant n-1$.

If $0 \leqslant i \leqslant \frac{m}{2}-1$ and $j=m$, then

$$f(v_i)+f(v_j)=\begin{bmatrix} i+1 & -(i+1) \\ i+1 & -(i+1) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

since $1 \leqslant i+1 \leqslant \frac{m}{2}$. If $\frac{m}{2} \leqslant i \leqslant m-1$ and $j=m$, then

$$f(v_i)+f(v_j)=\begin{bmatrix} n-i-1+\frac{m}{2} & -(n-i-1+\frac{m}{2}) \\ n-i-1+\frac{m}{2} & -(n-i-1+\frac{m}{2}) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

since $n-\frac{m}{2} \leqslant n-i-1+\frac{m}{2} \leqslant n-1$. If $0 \leqslant i \leqslant \frac{m}{2}-1$ and $m+1 \leqslant j \leqslant n-1$, then

$$f(v_i)+f(v_j)=\begin{bmatrix} i+j+1-\frac{m}{2} & -(i+j+1-\frac{m}{2}) \\ i+j+1-\frac{m}{2} & -(i+j+1-\frac{m}{2}) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

since $\frac{m}{2}+2 \leqslant i+j+1-\frac{m}{2} \leqslant n-1$. Finally, if $\frac{m}{2} \leqslant i \leqslant m-1$ and $m+1 \leqslant j \leqslant n-1$, then

$$f(v_i)+f(v_j)=\begin{bmatrix} n-1-i+j & -(n-1-i+j) \\ n-1-i+j & -(n-1-i+j) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

since $1 \leqslant n-1-i+j \leqslant n-2-\frac{m}{2} \pmod n$.

Case 3: $1 < m \leqslant q$ and m is odd. Define $f : V(K_{m,q}) \to M_2^0(\mathbb{Z}_n)$ such that

$$f(v_i)=\begin{cases} \begin{bmatrix} i+1 & -(i+1) \\ i+1 & -(i+1) \end{bmatrix}, & 0 \leqslant i < \frac{m-1}{2}, \\ \begin{bmatrix} n-i-1+\frac{m-1}{2} & -(n-i-1+\frac{m-1}{2}) \\ n-i-1+\frac{m-1}{2} & -(n-i-1+\frac{m-1}{2}) \end{bmatrix}, & \frac{m-1}{2} \leqslant i < m-1 \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & i=m-1, \\ \begin{bmatrix} i-\frac{m-1}{2} & -(i-\frac{m-1}{2}) \\ i-\frac{m-1}{2} & -(i-\frac{m-1}{2}) \end{bmatrix}, & m \leqslant i \leqslant n-1. \end{cases}.$$

Clearly, f is injective. It remains to show that if $v_iv_j \in E(K_{m,q})$,

$$f(v_i)+f(v_j) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \pmod n.$$

If $0 \leqslant i \leqslant \frac{m-1}{2}-1$ and $m \leqslant j \leqslant n-1$, then

$$f(v_i)+f(v_j)=\begin{bmatrix} i+j+1-\frac{m-1}{2} & -(i+j+1-\frac{m-1}{2}) \\ i+j+1-\frac{m-1}{2} & -(i+j+1-\frac{m-1}{2}) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

since $\frac{m+3}{2} \leqslant i+j+1-\frac{m-1}{2} \leqslant n-1$. If $\frac{m-1}{2} \leqslant i \leqslant m-2$ and $m \leqslant j \leqslant n-1$, then

$$f(v_i)+f(v_j)=\begin{bmatrix} n-1-i+j & -(n-1-i+j) \\ n-1-i+j & -(n-1-i+j) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

since $1 \leqslant n-1-i+j \leqslant n-2-\frac{m-1}{2} \pmod{n}$. Lastly, if $i = m-1$ and $m \leqslant j \leqslant n-1$, then $j \geqslant m > \frac{m-1}{2}$ and

$$f(v_i) + f(v_j) = \begin{bmatrix} j-\frac{m-1}{2} & -(j-\frac{m-1}{2}) \\ j-\frac{m-1}{2} & -(j-\frac{m-1}{2}) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We have thus shown that f is a zero ring labeling of $K_{m,q}$ and $\xi(K_{m,q}) = m+q = n$. □

Theorem 8. *If G is a bipartite graph of order n, then $\xi(G) = n$.*

Proof: Let $G = (V_1, V_2, E)$ be a bipartite graph of order n such that $|V_1| = m$ and $|V_2| = q$. Consider the complete bipartite $K_{m,q}$ with vertex set $V_1 \cup V_2$. Since $E(G) \subseteq E(K_{m,q})$, we see that G is a spanning subgraph of $K_{m,q}$. It follows that a zero ring labeling of $K_{m,q}$ with respect to the zero ring $M_2^0(\mathbb{Z}_n)$, where $n = m+q$, is also a zero ring labeling of G. Hence, $\xi(G) = n$. □

Theorem 9. *For every tree T of order n, $\xi(T) = n$.*

Proof: Let T be a tree of order n. Since T does not contain any cycle, T is bipartite, by Lemma 6. It then follows from Theorem 8 that $\xi(T) = n$. □

The previous theorem guarantees that any tree of order n admits an optimal zero ring labeling with respect to a zero ring of order n. Interestingly, such optimal zero ring labeling can be constructed using the zero ring $M_2^0(\mathbb{Z}_n)$.

3 Optimal Zero Ring Labeling of Trees

This section presents a scheme for obtaining an optimal zero ring labeling for trees using the zero ring $M_2^0(\mathbb{Z}_n)$.

The basis for the labeling is the zero ring labeling of the complete bipartite $K_{m,q}$ discussed in the proof of Theorem 7. We denote the matrix $\left[\begin{smallmatrix} i & -i \\ i & -i \end{smallmatrix}\right] \in M_2^0(\mathbb{Z}_n)$ by A_i.

Consider a tree T of order n. The labeling scheme starts with partitioning the n vertices of T to form the partite sets V_1 and V_2. Without loss of generality, we choose V_1 such that $|V_1| \leq |V_2|$.

Case 1. If $|V_1| = 1$, then the lone vertex in V_1 is labeled A_0 and we assign the labels A_i, where $i = 1, 2, ..., n-1$, to the vertices in V_2.

Case 2. If $1 < |V_1| \leqslant |V_2|$ and $|V_1| = m$ is even, then the vertices in V_1 are labeled by A_i, where $i = 1, 2, ..., \frac{m}{2}, n-1, n-2, ..., n-\frac{m}{2}$. On the other hand, the vertices in V_2 are labeled by A_i, where $i = 0, \frac{m}{2}+1, \frac{m}{2}+2, ..., n-1-\frac{m}{2}$.

Case 3. If $1 < |V_1| \leqslant |V_2|$ and $|V_1| = m$ is odd, then the vertices in V_1 are labeled by A_i, where $i = 0, 1, 2, ..., \frac{m-1}{2}, n-1, n-2, ..., n-\frac{m-1}{2}$, while the vertices in V_2 are labeled by A_i, where $i = \frac{m+1}{2}, \frac{m+1}{2}+1, ..., n-1-\frac{m-1}{2}$.

For special kinds of trees, the method of partitioning the vertices into partite sets V_1 and V_2 is discussed in the following subsections. In the illustrations, vertices belonging to the partite set V_1 are colored white while the vertices in V_2 are colored black.

3.1 Star Graph

The *star graph* S_n of order n is a special kind of tree with one central vertex having degree $n-1$ and the other $n-1$ vertices, called *leaves*, having degree 1. The star graph S_n is therefore isomorphic to the complete bipartite $K_{1,n-1}$.

Let v_0 denote the central vertex of S_n and let v_i, $1 \leqslant i \leqslant n-1$, be its adjacent vertices. Then, $V_1 = \{v_0\}$ and $V_2 = \{v_1, v_2, ..., v_{n-1}\}$.

For the labeling of vertices of S_n, Case 1 applies. Figure 1 shows a zero ring labeling of a star graph of order 6.

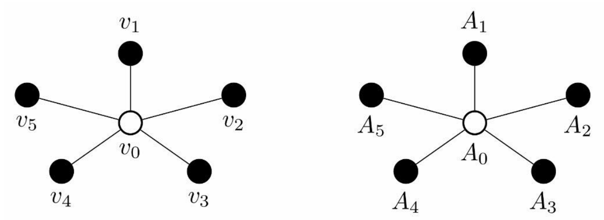


Figure 1: Zero Ring Labeling of Star Graph S_6 Using $M_2^0(\mathbb{Z}_6)$.

3.2 Caterpillar and Double Star Graphs

A *caterpillar graph* is a tree derived from a path by hanging any number of leaves from the vertices of the path.

Let $[w_0, w_1, w_2, ..., w_r]$ denote the path in the caterpillar and let n_i, $i = 0, 1, 2, ..., r$, denote the number of leaves hanging from w_i. The leaves hanging from w_i are denoted by $w_i^1, w_i^2, ..., w_i^{n_i}$.

If $r = 1$, then the caterpillar is called a *double star graph.*

To determine the partite sets V_1 and V_2 of the vertex set of the caterpillar graph, we classify first the vertices into two sets A and B. To do this, we place w_k, where k is even, in set A, all the leaves $w_k^1, w_k^2, ..., w_k^{n_k}$ hanging from w_k in set B. Further, we place w_l, where l is odd, in set B, all the leaves $w_l^1, w_l^2, ..., w_l^{n_l}$ hanging from w_l in set A. Rename these sets as V_1 and V_2 such that $|V_1| = m \leqslant q = |V_2|$. Relabel the vertices in V_1 (white vertices) using v_0, v_1,..., v_{m-1} in any order and relabel the vertices in V_2 (black vertices) arbitrarily using v_m, v_{m+1},..., v_{n-1} where $n = m + q$. Provide the zero ring labeling according to the case in Theorem 7 where this falls.

Figure 2 shows the zero ring labeling of a caterpillar of order 15 with $|V_1| = 6$ and $|V_2| = 9$. Here, Case 2 applies.

3.3 Lobster

A *lobster graph* is a tree in which all the vertices are within distance 2 of a central path. It has the property that the removal of leaf nodes results in a caterpillar graph.

Let $[w_0, w_1, w_2, ..., w_r]$ denote the central path in the lobster graph. Excluding the vertices of the central path, let $w_i^1, w_i^2, ..., w_i^{n_i}$ be the vertices of distance 1 from w_i, $i = 0, 1, 2, ..., r$.

Let p_{i,n_i} denote the number of leaves hanging from $w_i^{n_i}$. The leaves hanging from $w_i^{n_i}$ are denoted by $w_i^{n_i,1}, w_i^{n_i,2}, ..., w_i^{n_i,(p_{i,n_i})}$.

We now classify the vertices into two sets A and B.

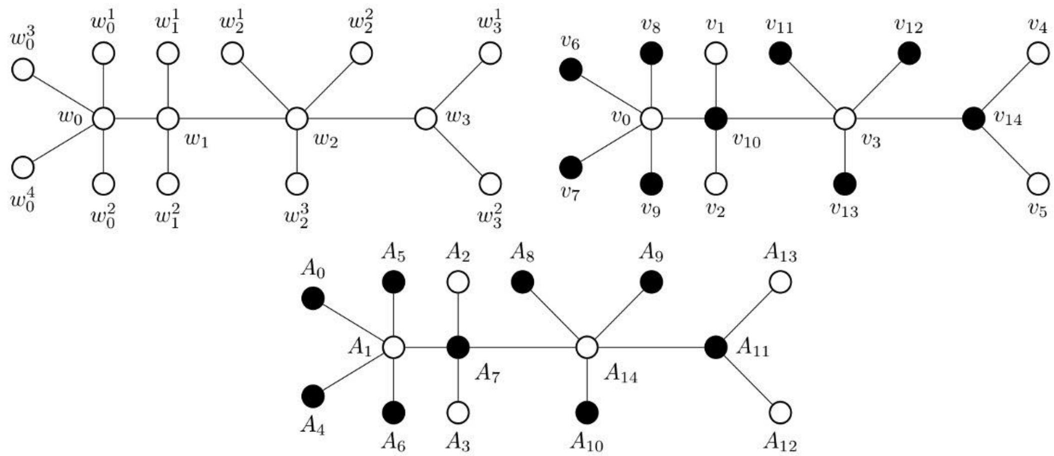


Figure 2: Zero Ring Labeling of a Caterpillar Using $M_2^0(\mathbb{Z}_{15})$.

We start by placing vertex w_k, where k is even, in set A and its adjacent vertices $w_k^1, w_k^2, ..., w_k^{n_k}$ in set B. The leaves hanging from any of $w_k^1, w_k^2, ..., w_k^{n_k}$ join the vertex w_k in set A. On the other hand, we place w_l, where l is odd, in set B and all its adjacent vertices $w_l^1, w_l^2, ..., w_l^{n_l}$ in set A. The leaves hanging from any of $w_l^1, w_l^2, ..., w_l^{n_l}$ are placed back to set B. Rename these sets as V_1 and V_2 such that $|V_1| = m \leqslant q = |V_2|$. Then, the vertices in V_1 (white vertices) and V_2 (black vertices) are relabeled using v_0, v_1,..., v_{m-1} and v_m, v_{m+1},..., v_{n-1}, respectively. The zero ring labeling is obtained according to the case in Theorem 7 where this falls.

Figure 3 shows a zero ring labeling of a lobster of order 22 with $|V_1| = |V_2| = 11$. In this example, Case 3 applies.

3.4 Rooted Tree

A *rooted tree* is a tree in which a particular vertex is designated as the root and every edge is directed away from the root. The *level* of a vertex v is the length of the simple path from the root to v. The *height* of a rooted tree is the maximum level of its vertices.

Consider a rooted tree T of order n. Let A be the set of all vertices in the even-numbered levels and let B be the set of all vertices in the odd-numbered levels. As above, these sets are renamed as V_1 and V_2 such that $|V_1| = m \leqslant q = |V_2|$ and the same procedure is performed to obtain the zero ring labeling of T.

A rooted tree of order 24 is shown in Figure 4. The height of the tree is 4. Observe that there are 13 vertices in the even-numbered levels and there are 11 vertices in the odd-numbered levels. Hence, the vertices of levels 1 and 3 form V_1 and are labeled as v_0, v_1,..., v_{10}. On the other hand, the vertices of levels 2 and 4 join the root in V_2 and are labeled as v_{11}, v_{12},..., v_{23}.

Since $|V_1|$ is an odd integer greater than 1, we apply Case 3. The zero ring labeling of the rooted tree above is given in Figure 5.

In general, any tree T can be redrawn as a rooted tree by designating any one vertex of T as the root. This means that a zero ring labeling of any tree of order n using the the zero ring $M_2^0(\mathbb{Z}_n)$ can be obtained by first transforming it into a rooted tree and applying the scheme described earlier.

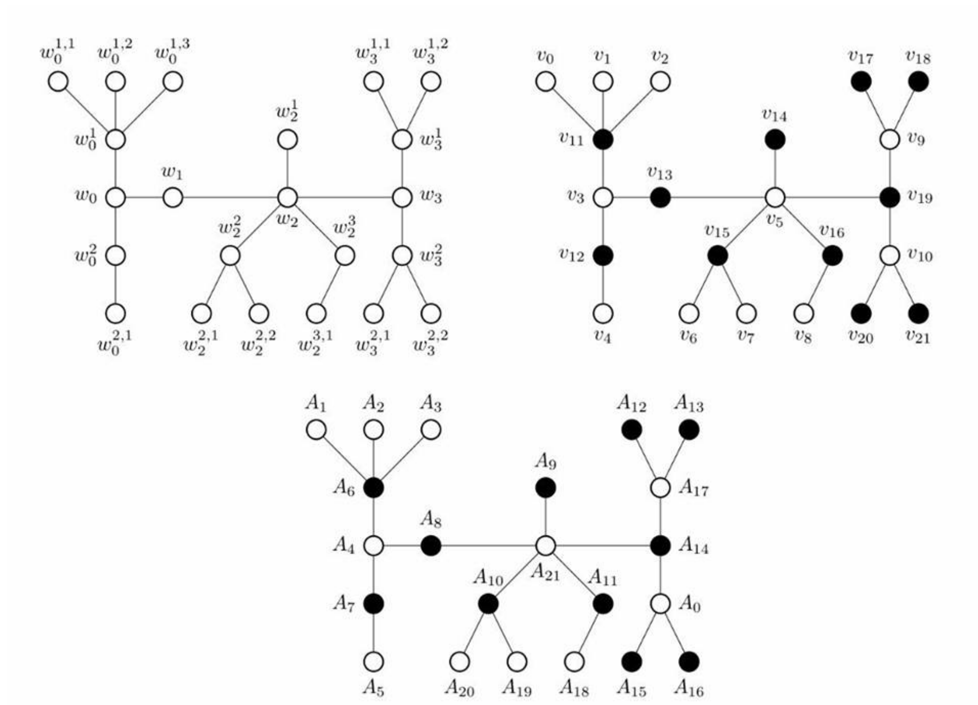


Figure 3: Zero Ring Labeling of a Lobster Over $M_2^0(\mathbb{Z}_{22})$.

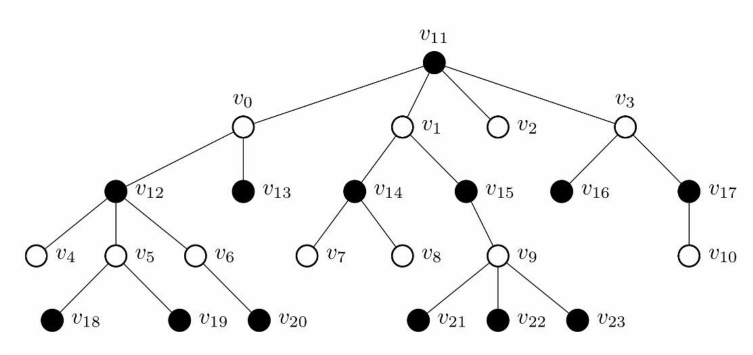


Figure 4: A Rooted Tree of Order 24.

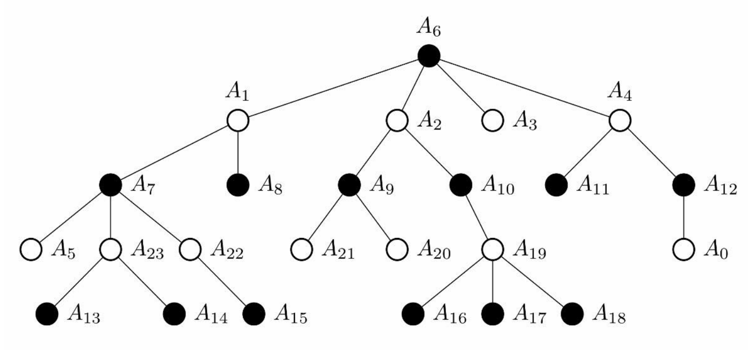


Figure 5: Zero Ring Labeling of a Rooted Tree of Order 24.

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