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The Heegaard splitting of S^3 and the Hopf fibration

Clarisson Rizzie Canlubo
Institute of Mathematics
University of the Philippines Diliman
Quezon City, Philippines
crpcanlubo@math.upd.edu.ph

Abstract

Decomposition and fibrations are among the usual tools when working with 3-manifolds. One of the most famous decompositions is the Heegaard splitting while the Hopf fibration is the most revered among the fibrations. In this article, we give an alternative proof of the fact that the 3-sphere S^3 can be decomposed into a union of two solid tori intersecting only at a common torus boundary using the Hopf fibration.
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1 The Hopf fibration

In the literature, there are four fibrations that are regarded as the Hopf fibrations, namely

$$S^n \longrightarrow S^{2n+1} \xrightarrow{\ p\ } S^{n+1}$$

for $n = 0, 1, 3$ and 7. The one for $n = 0$ is the trivial instance and the one for $n = 1$ is what Hopf discovered, see for example Hopf [1]. For the remainder of the article we will restrict our attention to the case $n = 1$ and refer to it as *the* Hopf fibration. To be more precise, the Hopf fibration is the map p and S^1 is its typical fiber. Let us concretely describe this fibration.

Let us identify $\mathbb{R}^4$ with $\mathbb{C}^2$ and view S^3 as the unit sphere in $\mathbb{C}^2$ centered at the origin. Also, identify $\mathbb{R}^3$ with $\mathbb{C} \times \mathbb{R}$ and view S^2 as the unit sphere in $\mathbb{C} \times \mathbb{R}$ given as

$$S^2 = \left\{(z, x) \mid x^2 + |z|^2 = 1\right\}.$$

The fibration p is then given as

$$p(z, w) = (2z\bar{w}, |z|^2 - |w|^2).$$

Let us briefly discuss a useful visualization of the Hopf fibration. We start with the fact that the one-point compactification of $\mathbb{R}^3$ is S^3. In Figure 1 below we see a decomposition, as a disjoint union of nested tori, of $\mathbb{R}^3$ with a circle and a line remove say the unit circle in the xy-plane and the z-axis. Let us denote by X the complement in $\mathbb{R}^3$ of the unit circle in the xy-plane and the z-axis. Compactifying $\mathbb{R}^3$ by adding a point makes the z-axis into a circle linked with the unit circle in the xy-plane.

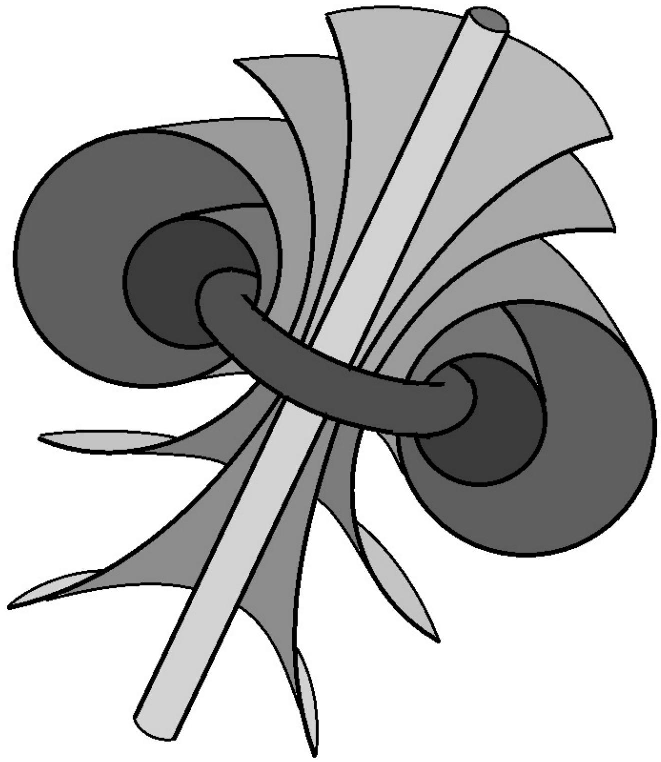


Figure 1

Now, each torus appearing in the decomposition of X is linked with the z-axis and the unit circle in the xy-plane. Each of these tori can be decomposed as a disjoint union of circles as Figure 2 illustrates.

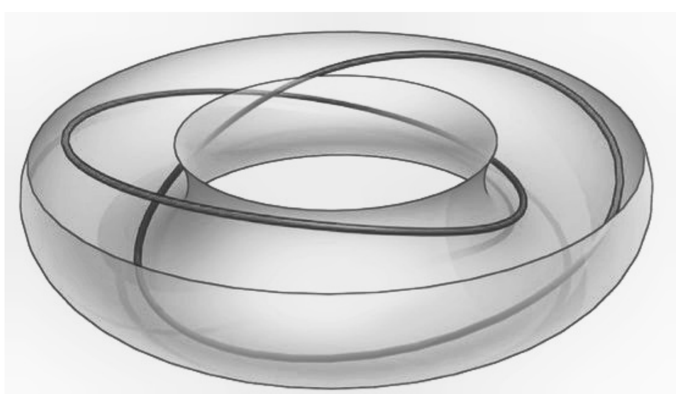


Figure 2

Thus, each of the circles appearing in this decomposition of the torus is linked with the unit circle in the xy-plane and the z-axis. Since the tori are nested, circles appearing in different tori are linked as well. Passing through the one-point compactification, this gives a decomposition of S^3 into a disjoint union of circles each of which is linked with any other circles appearing in the decomposition.

Going back to the Hopf map $p : S^3 \longrightarrow S^2$, the preimages of each point $x \in S^2$ are circles and these preimages are linked just as how we described it in the above discussion. For details, see Stillwell [3].

2 The Heegaard splitting

The 3-manifolds are regarded special among manifolds. For one, dimension 3 is the least dimension in which usual visualization does not work for this requires at least 4 dimensions. Secondly, adding dimensions often trivializes geometric difficulties. Lastly, there are tools that are available only in dimension 3. One such example is the so-called *Heegaard splitting*.

Consider the $2g$-sided polygon in Figure 3 (a). Identifying the vertices and edges with the same label according to the orientation indicated gives M_g, the closed, orientable, genus g surface. By the classification theorem for compact surfaces, such a surface can be embedded in $\mathbb{R}^3$. Figure 3 (b) illustrates one such embedding when $g = 2$.

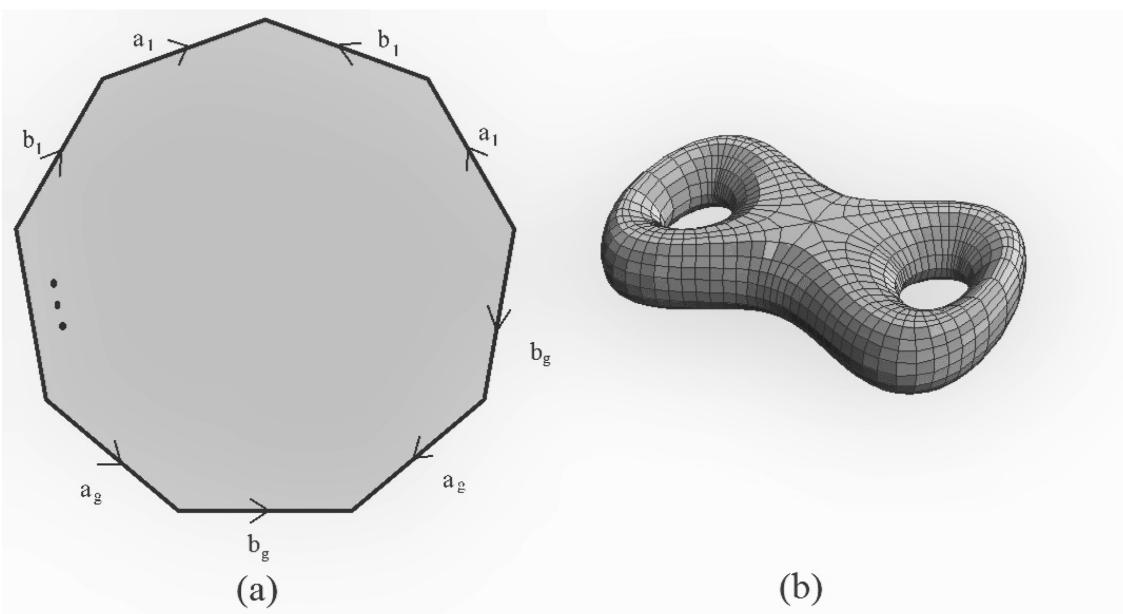


Figure 3

With this embedding, we can consider $\widehat{M_g}$, the space one gets by adding to M_g the region it bounds. We refer to $\widehat{M_g}$ as the *solid* M_g. One can construct a 3-manifold by gluing two copies of $\widehat{M_g}$ along an orientation-reversing homeomorphism ϕ between their boundaries, denoted as

$$\widehat{M_g} \cup_\phi \widehat{M_g}'.$$

In fact, any closed, orientable 3-manifold can be decomposed as such. In this case, the above decomposition of a 3-manifold M is referred to as a *genus g Heegaard splitting* of M.

The most well-known example of such a splitting is the one for $M = S^3$. Note that M_1 is homeomorphic to the usual torus $\mathbb{T}^2$. In this case, $\widehat{M_1}$ is simply the solid torus. Consider

$$\begin{array}{ccc} \mathbb{T}^2 \cong \partial\widehat{M_1} & \xrightarrow{\ \phi\ } & \partial\widehat{M_1}' \cong \mathbb{T}^2 \\ (z,w) & \longmapsto & (-z,-w) \end{array}$$

an orientation-reversing homeomorphism between the bounding tori of the component solid tori. With this homeomorphism, we get the following theorem.

Theorem 1. *With the notations above, we have*

$$S^3 \cong \widehat{M_1} \bigcup_{\phi} \widehat{M_1}'.$$

In the remainder of this section, we will describe a visualization of this splitting. In the next section, we will relate this splitting with the Hopf fibration giving an alternative proof of the above homeomorphism.

It is an obvious fact that the quotient of the closed unit ball D^n in $\mathbb{R}^n$ modulo its boundary is homeomorphic to S^n. Let us start with the standard embedding of $\mathbb{T}^2$ in a sufficiently large closed ball B in $\mathbb{R}^3$ whose boundary does not intersect $\mathbb{T}^2$. Remove the open region inside the torus $\mathbb{T}^2$. Collapsing the boundary of B gives a space homeomorphic to a solid torus. However, collapsing the boundary of this closed ball before removing the open region inside the torus gives S^3. This implies that S^3 can be decomposed as a union of two solid tori intersecting only at their boundary. Note that collapsing the boundary of B gives a space homeomorphic to the one-point compactification of R^3. This remark will be of particular importance once we move to the setting of the Hopf fibration where S^3 is viewed as the one-point compactification of $\mathbb{R}^3$ rather than a quotient of a closed 3-ball.

3 An alternative view

In this section, appealing to the Hopf fibration, we will give a different visualization of the genus 1 Heegaard splitting of S^3 as described in Theorem 1.

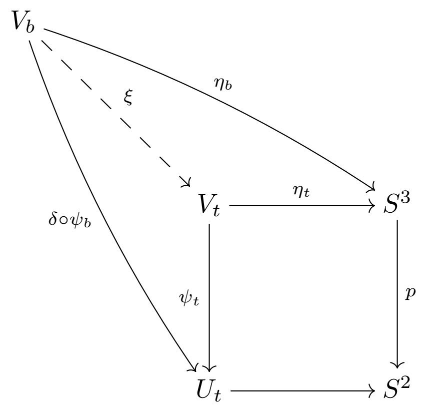
Let us start from the aforementioned fact that the preimages of points in S^2 along the Hopf map p are linked circles in S^3. This tells us that the preimage of a circle in S^2 is a torus inside S^3. Denote by U_t and U_b the top and bottom closed hemispheres of S^2 viewed as subsets of $\mathbb{R}^3$. Note that with the convention of Section 1, we have

$$U_t = \{(z,x) \mid x \geqslant 0\} \quad \text{and} \quad U_b = \{(z,x) \mid x \leqslant 0\}.$$

Also, note that U_t and U_b are homeomorphic. Pulling back $S^3 \xrightarrow{p} S^2$ along the inclusion map $U_t \longrightarrow S^2$ gives a space V_t fitting into a diagram

$$\begin{array}{ccc} V_t & \xrightarrow{\ \eta_t\ } & S^3 \\ \downarrow{\scriptstyle \psi_t} & & \downarrow{\scriptstyle p} \\ U_t & \longrightarrow & S^2 \end{array} \qquad \begin{array}{ccc} V_b & \xrightarrow{\ \eta_b\ } & S^3 \\ \downarrow{\scriptstyle \psi_b} & & \downarrow{\scriptstyle p} \\ U_b & \longrightarrow & S^2. \end{array}$$

Similarly, we have a space V_b fitting into a pull-back diagram. Post-composing ψ_b with a homeomorphism $U_b \xrightarrow{\delta} U_t$ gives the following diagram.



By the universal property of pull-backs, we have a continuous map $V_b \xrightarrow{\xi} V_t$. Reversing the roles of V_t and V_b gives a continuous map $V_t \longrightarrow V_b$ inverse to ξ. Thus, V_t and V_b are homeomorphic subspaces of S^3.

A question remains: how exactly do these subspaces look like? We claim that V_b is a solid torus. To see this, note that U_b consists of those points (z, x) for which $x \leqslant 0$. From the definition of the map p, the union of the primages of the points from U_b are all contained in tori nested inside the torus associated to the boundary circle of U_b. But this is precisely V_b and so, V_b is a solid torus. From the homeomorphism ξ, we see that V_t is a solid torus as well. Note that U_b and U_t only intersect at their boundary, which is a circle in S^2. The preimage of this circle is precisely the common boundary torus of V_t and V_b. This illustrates the genus 1 Heegaard splitting of S^3 as the union of the sections of the Hopf map p intersecting at the section lying above the equatorial circle of S^2.

4 References

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