MATIMYÁS MATEMATIKA
ISSN 0115-6926

Journal of the Mathematical Society of the Philippines
Vol. 39 No. 2 (2016) pp. 1-16

Randomness of Interest Rates in Microcredit

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Abstract

Microcredit is a process that allows efficient lending without collateral. It is characterized by a large number of frequent settlements with possible delays in repayment, creating randomness of the actual interest rate. We take the example of Yunus equation to examine the probabilistic law of this rate, as a measure of risk in microcredit. In particular, we aim to study the case where a borrower has a single random delay in repayment and we provide an asymptotic expansion of the involved rate.

Keywords: Microcredit, random interest rate, Yunus equation, asymptotic expansion
Mathematics Subject Classification: 91G40, 91G60, 60G99

1 Introduction

Microcredit is the provision of small amounts of loan to very poor or low-income people. Its origins can be traced back to the late 70's to the early 80's. One of the pioneers of microcredit is Muhammad Yunus who founded what is considered as the first modern microcredit institution, named Grameen Bank, in 1983. The Nobel Peace Prize 2006 was jointly awarded to Yunus and Grameen bank for their efforts to "create economic and social development from below" [1]. Microcredit is just one of the many financial and social services provided by Microfinance institutions (MFIs) for their clients. Other programs and services include savings, housing, insurance, investments, livelihood trainings, etc.

Microcredit can be classified as either individual or group lending. The borrowers, being traditionally poor, have no access to regular credit. This is because either they have no stable job, no verifiable credit history, or even collateral. Unlike commercial banks, MFIs cannot screen borrowers and secure material collateral. Hence the introduction of the group lending model. It uses social collateral to ensure monitoring of borrowers in order to prevent default and use of loans for consumption needs. A group of people borrow collectively and peer pressure is used to guarantee repayment. When one borrower does not pay, the others

[1] www.nobelprize.org

will be responsible to pay for it. However, group lending bears additional risk, leads to loss of privacy, and is more time-consuming in terms of group meetings [5].

A comprehensive review of existing literature in Microfinance was presented in a paper by Brau and Woller [1]. It was mentioned that since 1997, academic journals have been publishing hundreds of peer-reviewed articles on topics of microcredit and microfinance in general. These articles have more to do with economics or social dimension, and not on the mathematical aspects. The authors believe that mathematical research would be of strong potential to improve microfinance theory and practice worldwide. Also, in strong contrast to finance of stock derivatives or credit risk, the probabilistic approach to microcredit has not benefited much from the tools in stochastic calculus. G. Tedeschi, in her research [11], introduced a model of individual microlending. The model makes use of dynamic incentives to get away from borrowers' strategic default, that is refusing to pay a loan despite the financial capability. Inspired by this model, a formalized model was introduced using a Markov chain representation in a paper by Diener, Khodr and Protter [3]. This model basically involves two states, being an applicant and a beneficiary of a loan. Moreover, a borrower enters credit exclusion for a certain time if she is not able to fully repay a loan. She can then apply once again after such period. The paper provided a good starting point for the stochastics of microcredit, but the authors say that it is too simple to apply. The model does not take into account innovations in the repayment process, say for example partial settlements.

In this paper, we will use and extend another stochastic model introduced in [6] which addresses one of the main risks in microcredit, namely the delays that can occur in scheduled installments. Indeed, in microcredit activity, the risk of default is mainly replaced by the risk of delay and the consequent randomness of the actual interest rate. Installments postponed by one or possibly more time periods would definitely affect all subsequent installments and the interest rate received by the microcredit or microfinance institution. In [6], a basic individual microloan repayment process was introduced as a mathematical model of random delays in repayment. Simulations of the actual interest rate for a portfolio of 10000 borrowers illustrated how delays may affect its value.

As the first step in identifying the probabilistic distribution of the random interest rate in this model, we focus on the case where there is only one delay in repayment. The main idea is to take into account that in microcredit activity, the number of settlements for a single loan is usually very large. Thus, to compute for the actual rate in the case of a single delay, an asymptotic expansion of the rate with respect to this large parameter is obtained.

In the next section, we will describe our stochastic model and explain how non standard analysis will help to take into account, in a simple way, its asymptoptic properties. In section 3, we study the distribution of the actual interest rate through Monte Carlo simulations. In section 4, we explain the results we obtained in the case of a single delay in microloan repayment. The last section gives the proof of our results.

2 A stochastic and asymptotic model for interest rates

2.1 An introductory example: the Yunus Equation

Our model and our choices of parameters in the simulations are inspired by an example described by Yunus in [13], of how he could lend an amount of 1000 Bangladesh Taka (BDT) [2], that will be repaid in 50 weekly settlements of 22 BDT[3]. This means that the borrower makes each installment at week j, that is, at time $t_j = \frac{j}{52}$ (time unit is one year).

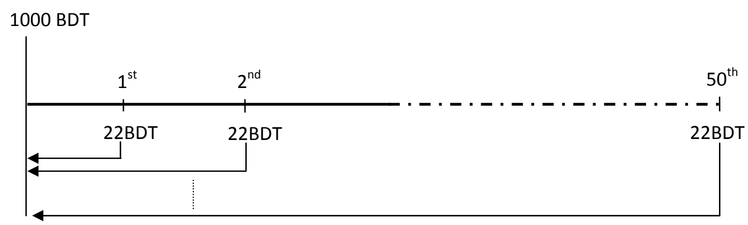

Figure 1: The Yunus example

We denote by r the annual interest rate, compounded continuously. This rate is defined implicitly by the *Deterministic Yunus Equation* [6]:

$$1000 = 22 \sum_{j=1}^{50} q^j, \ q = e^{-\frac{r}{52}}. \tag{1}$$

This equation is easy to solve numerically as its solution can be seen as the unique real root ($\neq 1$) of a polynomial of degree 51. It is equal to $q^* = 0.9962107\ldots$, and consequently, the actual interest rate of this loan is $r^* = 0.1974175\ldots \approx 20\%$.

We now consider dividing a year into N intervals, where $N \in \mathbb{Z}^+$ and $N > 0$. This implies that the loan will be repaid in N installments. Denoting L the loan amount and r_f the flate rate, (1) can be rewritten in a more general way as follows.

$$L = \frac{L}{N}(1 + r_f) \sum_{j=1}^{N} q^j,$$

which is equivalent to

$$N = (1 + r_f) \sum_{j=1}^{N} q^j.$$

2.2 Introduction of random settlement dates

Yunus's example does not take into account the fact that a borrower has a possibility of having a problem of not being able to pay at the scheduled time, say at time j, which will then cause delay in all subsequent settlements. To address this situation, we think more realistically to see the settlement date j as a random variable T_j (see [6]).

[2]1 BDT $\approx$ 0.60 PHP
[3]Installment amount is based on a 10% flat rate.

We introduce a Bernoulli process $\mathcal{B} = (B_m)_{m\geq 1}$, that is, a sequence of independent Bernoulli random variables $B_m \sim \mathcal{B}(1,p)$. The variable $B_m = 1$ represents the event that the borrower is able to pay at time m with probability p, and $B_m = 0$ in case of delay. Hence, the jth settlement takes place, at the stopping-time $T_j = \min\{t|B_1+\ldots+B_t = j\}$, $j = 1,\ldots,N$. The sequence of gaps between two successive installments is denoted by $X_j = T_j - T_{j-1}$, with $T_0 = 0$ (refer to Figure 2). It is easy to see that $\mathbb{P}(X_j = k) = (1-p)^{k-1}p$, so X_j is geometrically distributed, denoted $X_j \rightsquigarrow \mathcal{G}(p)$.

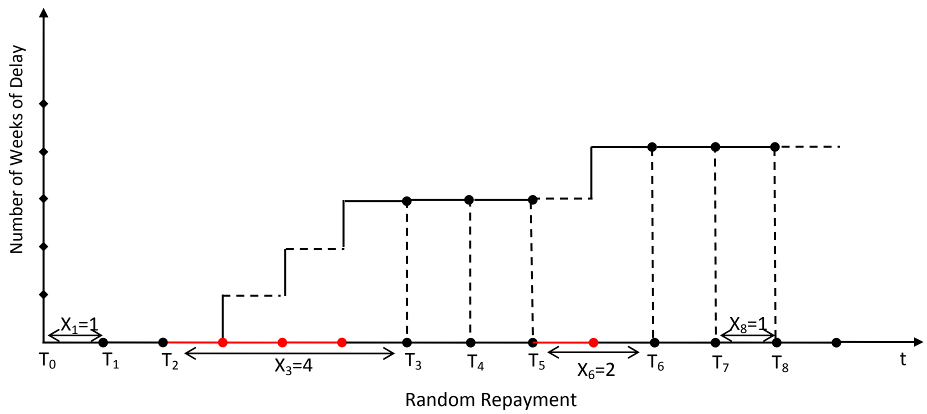

Figure 2: Repayment of a microloan at random times

In this stochastic model, the actual interest rate, which we now denote by R, is implicitly defined by what we call the *Generalized Yunus Equation* (GYE):

$$N = (1+r_f)\sum_{j=1}^{N} Q^{T_j},\ Q = e^{-R/N}. \quad (2)$$

The actual rate is no longer a number only related to the loan (through the two parameters N and r_f), but a random variable which also depends on the borrower. The least value of this random variable can be seen as a measure of the risk for microcredit activity due to the possibility of delays in repayment.

2.3 An asymptotic version of the model for large N

Even if the stochastic model we introduced is built on a simple Bernoulli process, the distribution of the random interest rate R defined by GYE is not as easy as it appears. It is not easy to compute precisely because it is defined in an implicit way. As the simulations will show in the next section, one way to deal with this difficulty is to take into account that the number of settlements N is very large. This allows one to compute not the law of R itself, but its asymptotic properties when N is large enough. This is what we will do using a non-standard version of our stochastic model. That is, we will assume that the parameter N is infinitely large. It could have been done in classical (standard) analysis by assuming that N tends to infinity. However, this would seem to be a strange hypothesis for a fixed number of settlements.

Non Standard Analysis (NSA) is a branch of Mathematics which provides a logical foundation for the use of infinitesimals and infinitely large numbers. It was introduced by Abraham Robinson in the early 1960's (see [4], [9], [10]). In this research, we will use an axiomatic approach to Robinson's theory, which rather requires less logic. It is called the Internal Set Theory (IST), which was developed by Edward Nelson in the mid-1970's (see [7]). It is a conservative extension of the Zermelo-Fraenkel set theory and the axiom of choice, which are considered as the most common foundations of mathematics (see the tutorial in [2]). IST introduced a new predicate *standard* and three additional axioms namely Idealization, Standardization and Transfer. Intuitively, every object uniquely defined in classical mathematical terms is standard, otherwise it is non-standard. The axioms govern the use of this new predicate. Idealization ensures the existence of non-standard numbers. Standardization implies that any real number not infinitely large is infinitely close to a unique standard number. Transfer states that if a standard statement holds for all standard values of a variable, then it holds for all values, standard or not, of this variable.

NSA is effective especially when doing technical computation using limits as we will see in the last section. Here are some useful definitions:

Definition 1. *A real number x is said to be*

1. ***infinitesimal*** *if* $|x| < \dfrac{1}{m}$ *for any standard* $m \in \mathbb{N}$.
2. ***infinitely large****, denoted i.l., if* $|x| > m$ *for any standard* $m \in \mathbb{N}$.

Definition 2. *A real number x is called*

1. ***limited*** *if it is not infinitely large, that is if* $|x| < m$*, for some standard* $m \in \mathbb{N}$.
2. ***appreciable*** *if it is neither infinitely large nor infinitesimal.*

Remark 3. Any limited real number is either infinitesimal or appreciable. As a consequence of Standardization, if x is limited, there exists a standard x_0 such that $x - x_0$ is infinitesimal. This number x_0 is called the *standard part of* x, denoted $st(x)$.

Definition 4. *Two real numbers x and y are said to be* ***infinitely close*** *(to each other), denoted $x \simeq y$ if their difference $x - y$ is infinitesimal. Hence, if x is infinitesimal, $x \simeq 0$.*

Let ε and ρ be infinitesimals, a and b appreciable and ω_1, ω_2 infinitely large (i.l.) numbers. We assume that these numbers are fixed. The following rules are easy consequences of these definitions:

(i.) If $\varepsilon \neq 0$, then $\dfrac{1}{\varepsilon}$ is i.l.

(ii.) $\dfrac{1}{\omega_1} \simeq 0$

(iii.) $\varepsilon + \rho \simeq 0$

(iv.) $\varepsilon + b \simeq b$

(v.) $\varepsilon + \omega_1$ and $a + \omega_1$ are i.l.

(vi.) $\varepsilon \cdot \rho \simeq 0$ and $\varepsilon \cdot a \simeq 0$

(vii.) $\omega_1 \cdot a$ and $\omega_1 \cdot \omega_2$ are i.l.

(viii.) $\dfrac{\varepsilon}{a} \simeq 0$, $\dfrac{\varepsilon}{\omega_1} \simeq 0$, and $\dfrac{a}{\omega_1} \simeq 0$

(ix.) $\dfrac{a}{\varepsilon}$ and $\dfrac{\omega_1}{a}$ are i.l.

Additional notations, $\oslash$, £, @, and $\not\infty$, are frequently used when doing asymptotic calculus, in order to denote an infinitesimal, limited, appreciable and infinitely large number, respectively when the exact value of the real number is not important, but only its order of magnitude (see [12] for an extended study of these new symbols). It is important to notice that two occurences of such a symbol in the same formula are not necessarily equal.

Here is an example of a typical result in NSA which can be easily proven using the three axioms of IST:

Proposition 5. *If f is a standard function, then $\lim_{N\to\infty} f(N) = 0$ if and only if for any N infinitely large, $f(N)$ is infinitesimal.*

Ideas of NSA were applied to different branches of mathematics - in Banach and Hilbert spaces, probability theory, ordinary and partial differential equations, number and group theory, economics and mathematical finance, mathematical physics, etc. [8] The reader is referred to [2] and [12] for a more detailed discussion on different applications of NSA.

3 Simulation of the interest rate's distribution

The model defines implicitly from (2) the random variables Q and $R = -N\log(Q)$. In order to study its distribution, we consider a Monte Carlo sample of the N random settlements of 10000 microloans. We obtained a distribution of the actual rates received by the microcredit institution having these loans in its portfolio. For the simulation, we consider $N = 50$ and $r_f = 10\%$, as considered in the Yunus example. In [6], Mauk showed that the probability p can be expressed in terms of the repayment rate. We adapt Mauk's computed value p=0.84. This probability is related to a 97% repayment rate which is just the complement of a 3% default rate, where default means that $X_j > 4$ for some j. Figure 3 shows the histogram of such a simulation. Observe that the maximum is $R \approx 0.189$, which corresponds to $r \approx 0.197$. This is precisely the actual interest rate when there is no delay in repayment. We can compute for the weighted average of R in this simulation and see that the value is more than 3% below this "no delay" value of r. This explains why we think that the delay is as important as the default risk.

The histogram of R in Figure 3 resembles a bell curve which depicts a normal distribution. The values of R obtained range from 12% to 19% and the peak of the distribution of the rate is somewhere near the middle. The histogram is a bit lopsided, that is, the skewness is not zero. Observe also that it is bounded from above by the value approximately equal to 19% which is precisely the rate when there is no delay in repayment. We then ask ourselves what if the probability p has a different value? So, just like what Mauk did in his experiments, we consider other values.

We consider taking values of p closer to 1. Figures 4 and 5 show the histogram of R given $p = 0.95$ and $p = 0.97$, respectively. Observe that the values of the rate are higher than 15%. The distribution of R becomes evidently more skewed to the left, but still bounded from above. Notice also that there are high bars at the rightmost part of the histograms. It is possible to verify on the sample that these represent precisely the borrowers that do not have delays in repayment. On the other hand, if we consider lower values of p, closer to 0, lower values of the rate are obtained. In Figures 6 and 7, observe that values could go as low 2.5% and the peak of the distribution evidently moved to the left. These would indicate that as the value of p decreases, the value of the random variable R also decreases. That is, the smaller the probability of the borrower to pay the installment on the scheduled time, the smaller the effective interest rate gets as compared to the rate when there is no delay in repayment.

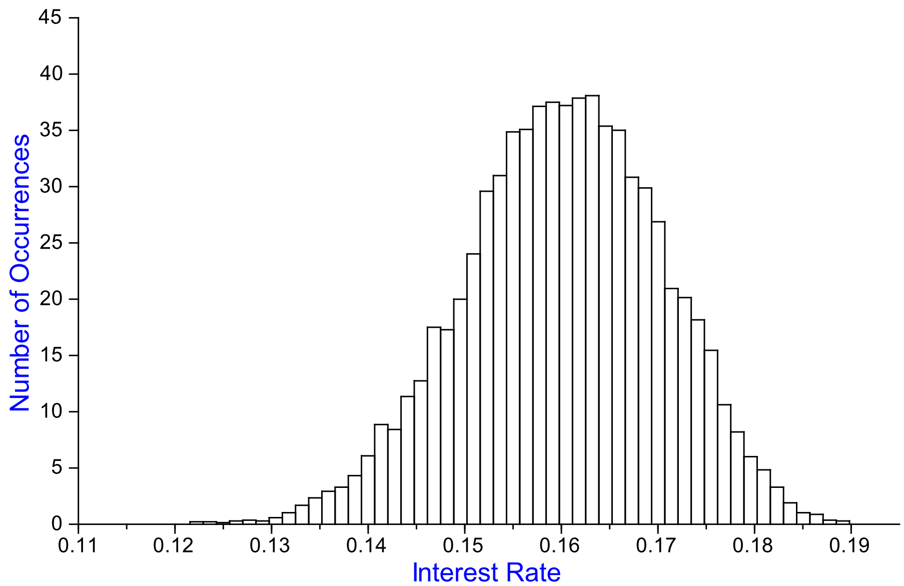

Figure 3: Distribution of R computed from (2) for $N = 50$, $r_f = 10\%$, and $p = 0.84$.

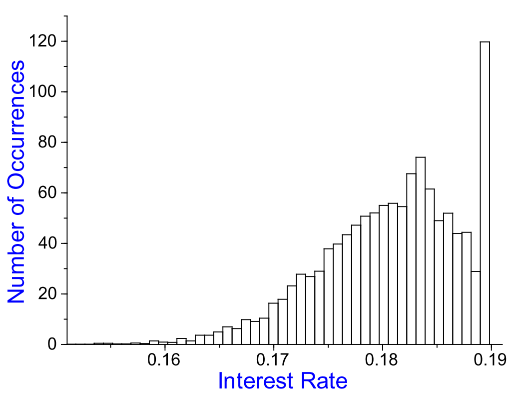

Figure 4: Distribution of R given $p = 0.95$

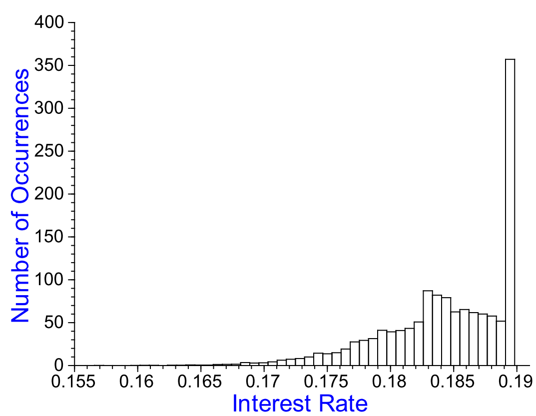

Figure 5: Distribution of R given $p = 0.97$

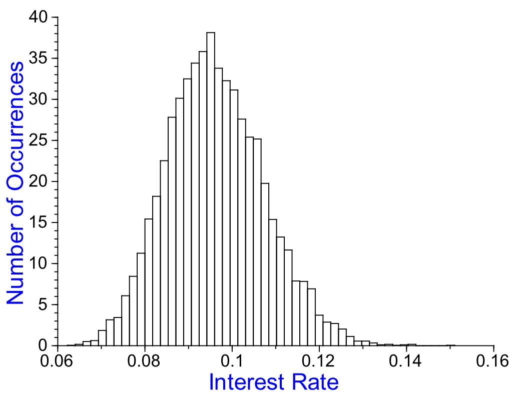

Figure 6: Distribution of R given $p = 0.50$

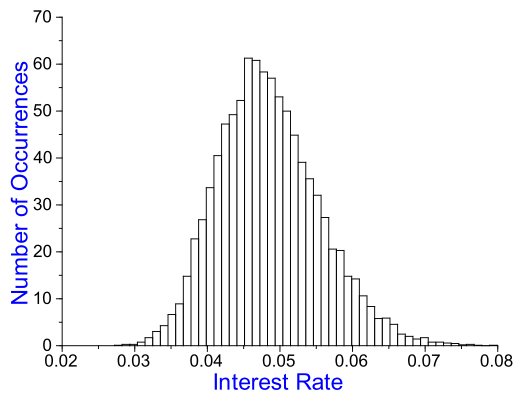

Figure 7: Distribution of R given $p = 0.25$

Now, we look at the rates involved given a specific number of random delays. Figure 8 displays the distribution of the interest rate given 0 to 4 delays. One can check from the sample that the highest bar, in blue, represents the number of borrowers, out of 10000, that do not have delay. The green, red, brown and purple bars represent borrowers with rates obtained given 1, 2, 3 and 4 delays, respectively. Observe that the bars representing rates with one delay seem to be of similar height. We explain this behavior in the next section.

4 The Asymptotics of the Interest Rate for a Single Delay

In our stochastic and asymptotic model, we expect to be able to derive an asymptotic expansion of the distribution of the effective interest rate R,

$$R = \sum_{l=0}^{n} \frac{\alpha_l}{N^l} + \frac{1}{N^n}\varepsilon_n(N),$$

with $\lim_{N\to\infty} \varepsilon_n(N) = 0$ for all $n \geq 0$. This is far from being reached but we succeeded in determining the first terms of an asymptotic expansion of R_1, the implicit interest rate assuming a single delay.

On one hand, the probability distribution of R_1 is just trivial as it corresponds to the case where there is a unique k such that $X_k = 2$ and all the other X_js are equal to 1. Since all the events $\{X_k = 2\}$ have the same probability given by $\mathbb{P}(\{X_k = 2\}) = \frac{1}{N}$, the random variable R_1 is just uniformly distributed. On the other hand, R_1 takes N different values denoted by $R_1 = r(k) = -N \log q(k)$, where $q(k)$ is the unique solution to GYE, given the time delay k. It is easy to see that $T_N = N + 1$ when only one delay occurs at time k,

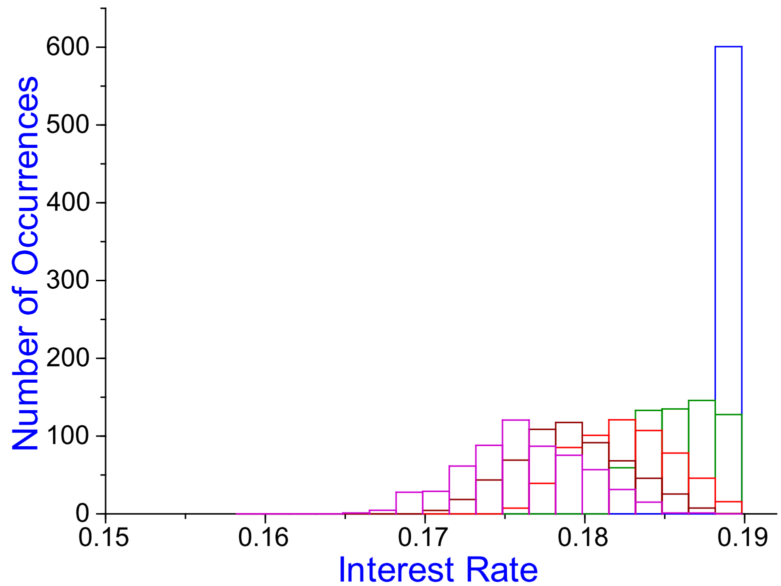

Figure 8: Rate distribution for 0, 1, 2, 3 and 4 delays.

$k \in \{1, 2, \ldots, N\}$. In this case, GYE can be rewritten as follows.

$$N = (1 + r_f) \left(\sum_{j=1}^{N+1} q^j - q^k \right), \ q = \exp\left(-\frac{R_1}{N}\right). \tag{3}$$

Through Scilab, it is easy to compute for the values of $q(k)$ and $r(k)$ numerically and plot them as a function of k. Choosing again $N = 50$ and $r_f = 10\%$, we get a very regular behavior which came as a surprise. The points $(k, q(k))$ and $(k, r(k))$ seem to line up and equally spaced from each other. For example, one can observe in Figure 10 that the value of q_k decreases as k increases and that $q_{k-1} - q_k \approx 0.000003$ for $k = 2, 3, \ldots, N$. This regularity of the numerous and distinct values of the random variable R_1 is also the reason why the histogram of R_1 in Figure 8 (the green bars) is approximately uniform.

From the point of view of the model, the decrease in values of r_k illustrates that when time delay occurs at the start of the repayment process, the interest rate is much less than the expected one. Moreover, if the delay occurs near the end of repayment, the difference of the rate with the expected one is almost negligible.

Taking a closer look at the plots in Figures 9 and 10, we realize that this regularity is not exact but only asymptotically true when N is large enough. The following theorem will explain this asymptotic behavior more precisely. Its proof will be given in the next section.

Theorem 6. *For $k = 1, 2, \ldots, N$, the unique real solution $q(k)$ to equation (3) has an order*

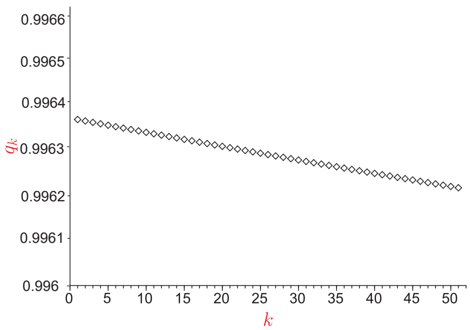

Figure 9: solutions q_k to (3)

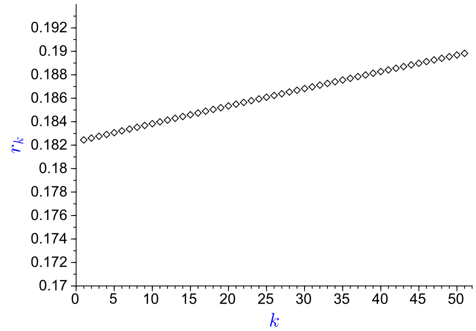

Figure 10: interest rates $r_k = -N\log(q_k)$

three expansion given by

$$q(k) = 1 - \frac{\beta_1}{N} + \frac{\beta_2}{N^2} + \frac{\beta_3(k)}{N^3} + \frac{\varepsilon_k(N)}{N^3}, \tag{4}$$

with $\lim_{N\to\infty} \varepsilon_k(N) = 0$, *where*

$$\begin{aligned}
\beta_1 \quad & \text{is} \quad \text{the real nonzero solution of } 1 - e^{-\beta_1} = \frac{\beta_1}{1+r_f},\\
\beta_2 &= \frac{\beta_1^2(3+\beta_1-r_f)}{2(\beta_1-r_f)},\\
\beta_3(k) &= \lambda k + \mu, \text{ where } \lambda \text{ and } \mu \text{ are constants given by}\\
\lambda &= -\frac{\beta_1^2(1+r_f)}{\beta_1-r_f} \text{ and}\\
\mu &= -\frac{\beta_1(1+r_f)}{\beta_1-r_f}\left\{\frac{\beta_2^2}{\beta_1^2(1+r_f)} - \left(1-\frac{\beta_1}{1+r_f}\right)\left[\beta_2\left(\frac{3}{2}-\frac{\beta_2}{\beta_1^2}-\frac{\beta_2}{2\beta_1}\right) - \beta_1\left(1+\frac{2\beta_1}{3}-\frac{\beta_2}{2}+\frac{\beta_1^2}{8}\right)\right]\right\}.
\end{aligned}$$

The striking fact here is that the first three terms of the expansion of $q(k)$ do not depend on k, and only the fourth, β_3, does and is indeed dependent linearly on k. This verifies our observation that as a function of k, $q(k)$ looks like an affine function when N is fixed and large enough. This is also true for the values of $r(k) = -N\log(q(k))$, as stated in the following theorem.

Theorem 7. *The actual interest rate associated with one delay in repayment at time k has the following order two asymptotic expansion given by*

$$r(k) = \alpha_0 + \frac{\alpha_1}{N} + \frac{\alpha_2(k)}{N^2} + \frac{\epsilon_k(N)}{N^2}, \text{ with } \lim_{N\to\infty} \epsilon_k(N) = 0,$$

where $\alpha_0 = \beta_1$, $\alpha_1 = \frac{1}{2}\beta_1^2 - \beta_2$ *and* $\alpha_2(k) = \frac{1}{3}\beta_1^3 - \frac{1}{2}\beta_1\beta_2 - \beta_3(k)$.

So up to terms small with respect to $\frac{1}{N^2}$, the actual interest rate depends linearly on the time k when the delay occurs.

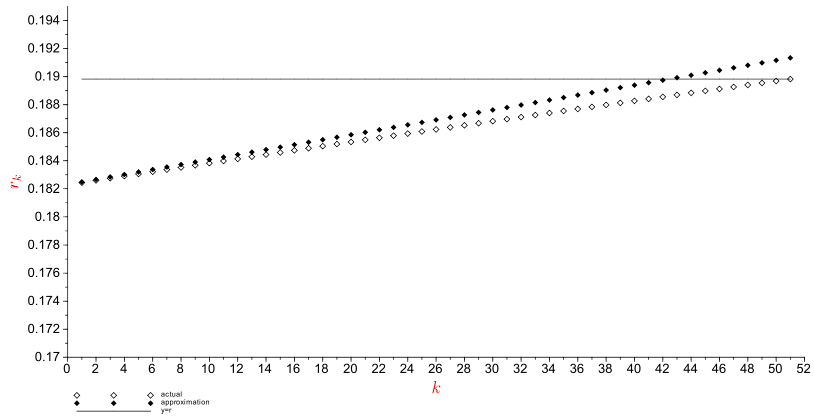

Figure 11: Actual interest rate in case of a single delay, at week k and the corresponding value using asymptotic expansion, for $N = 50$ and $r_f = 10\%$. The horizontal line represents the interest rate r when there is no delay.

Remark 8. As we shall see in the proofs in the next section, we determine the first terms of the asymptotic expansions of $q(k)$ and $r(k)$ assuming that the fixed parameters r_f and k are standard numbers but N is an infinitely large number. Using the Axiom of Tranfer, the expansions obtained will be automatically valid not only for standard k but also for all $k \leq N$, standard or infinitely large (for example for $k = N/2$, which is infinitely large as soon as N is). However, one has to realize that this shows that the error term $\dfrac{\epsilon_k(N)}{N^2}$ tends to 0 when N tends to infinity but not necessarily at a uniform rate with respect to k. Convergence may not hold as k becomes large. Notice that this is precisely what one can observe on Figure 11 where it appears that the approximation of $r(k)$ by its asymptotic expansion is getting worse when k is increasing.

5 Proof of Main Results

The asymptotic expansions obtained in Theorems 6 and 7 and the statement of the theorems themselves are standard, but we may prove them using a non-standard proof. One can of course try proving them in terms of the $\epsilon - \delta$ approach to limits, but the use of NSA in this case is more convenient in computation as we will see.

5.1 Proof of Theorem 6

We claim that to prove Theorem 6, it suffices to show that $\varepsilon_k(N) \simeq 0$ for k standard. It is enough, as N is infinitely large and the fixed parameters r_f and k standard, to prove that $\varepsilon_k(N)$ given in the theorem by

$$\varepsilon_k(N) = q(k) - \left(1 - \frac{\beta_1}{N} + \frac{\beta_2}{N^2} + \frac{\beta_3(k)}{N^3}\right) N^3,$$

is infinitesimal.

Indeed, as k is standard, the function $\varepsilon_k(N)$ is by definition a standard function of N, β_1 and β_2 being standard numbers by definition and $\beta_3(k)$ being a standard function of k, also by definition. Thus, by Proposition 5, we know that $\varepsilon_k(N) \simeq 0$ for all infinitely large N is equivalent to $\lim_{N\to\infty} \varepsilon_k(N) = 0$. Now, as soon as this limit is 0 for all standard values of k, as this statement is a standard property, we can apply the Axiom of Transfer to deduce the property $\lim_{N\to\infty} \varepsilon_k(N) = 0$ for all k, standard or infinitely large.

We will proceed in three steps to show that $\varepsilon_k(N) \simeq 0$ for any infinitely large N given k standard:

1. A first change of variable (or blow up around 1) $\beta := (q(k) - 1)N$ will allow to show that the new unknown β is infinitely close to a standard number β_1.
2. A second change of variable (or blow up of β around $-\beta_1$), $\gamma := (\beta + \beta_1)N$ (or $\gamma := q(k) - (1 - \frac{\beta_1}{N})N^2$) will allow to show that the new unknown γ is infinitely close to a standard number β_2.
3. A third change of variable (or blow up of γ around β_2) $\delta := (\gamma - \beta_2)N$ (or $\delta := (q(k) - (1 - \frac{\beta_1}{N} + \frac{\beta_2}{N^2})N^3)$ will allow to show that the new unknown δ is infinitely close to a standard number denoted $\beta_3(k)$ as it depends on k, and is a linear function of k.

So let k be standard and N be a fixed infinitely large number.

Step 1: Knowing that $q(k)$ is the unique solution of (3), we denote it simply by q and rewrite its equation

$$\frac{N}{1 + r_f} = \left(\sum_{j=1}^{N+1} q^j\right) - q^k = \frac{q(1 - q^{N+1})}{1 - q} - q^k. \tag{5}$$

With the knowledge that $q \in (0, 1)$, it is easy to see from (5) that, as N is infinitely large, q must be infinitely close to 1. Otherwise, the right hand side of the (5) would be limited and the left hand side infinitely large. To identify how small the difference between q and 1 is, we consider a new variable β defined by $q := 1 - \dfrac{\beta}{N}$ or $\beta := (1 - q)N > 0$. Then rewriting (5) with this change of variable yields

$$0 = \left(1 - \frac{\beta}{N}\right)^k \frac{\beta}{N} - \left(1 - \frac{\beta}{N}\right)\left(1 - \left(1 - \frac{\beta}{N}\right)^{N+1}\right) + \frac{\beta}{1 + r_f} \tag{6}$$

Knowing that $\dfrac{\beta}{N} \simeq 0$ and $\left(1 - \dfrac{\beta}{N}\right)^{N+1} = e^{-\beta}(1 + \oslash)$, (6) can be written as follows:

$$0 = \oslash - (1 - \oslash)(1 - e^{-\beta}(1 + \oslash)) + \frac{\beta}{1 + r_f}. \tag{7}$$

This shows that the solution β of this equation must be limited. Thus, it must have a standard part, denoted β_1, and this number must satisfy the equation obtained in taking the standard part of (7) given by

$$1 - e^{-\beta_1} = \frac{\beta_1}{1 + r_f}. \tag{8}$$

Note that the standard part of r_f is r_f itself as it is standard.

Step 2: Notice that the first approximation we found for $q(k)$, given by $1-\dfrac{\beta_1}{N}$, does not depend on k. Hence, we consider a new blow-up of β (or $q(k)$). We want to see how small the difference between q and its approximation is and to check how it depends on k. We make another change of variable by letting $q := 1-\dfrac{\beta_1-\frac{\gamma}{N}}{N}$ or $\beta := \beta_1 - \dfrac{\gamma}{N}$. Rewriting (5) and dividing both sides of the equation by N, we obtain

$$\frac{1}{N}\left(1-\frac{1}{N}\left(\beta_1-\frac{\gamma}{N}\right)\right)^k = \left(1-\frac{1}{N}\left(\beta_1-\frac{\gamma}{N}\right)\right)\frac{1}{\left(\beta_1-\frac{\gamma}{N}\right)}\left[1-\left(1-\frac{1}{N}\left(\beta_1-\frac{\gamma}{N}\right)\right)^{N+1}\right]-\frac{1}{1+r_f}.$$

The goal is to check how γ depends on k. Using that N is infinitely large and k, r_f are standard, we can rewrite the above equation in terms of order of magnitude and take the standard part. Thorough computations will then give that γ is infinitely close to a standard number β_2 which verifies

$$\beta_2 = \frac{\beta_1^2(\beta_1+3-r_f)}{2(\beta_1-r_f)}.$$

Step 3: This shows that β_2, just like β_1, does not depend on k. So we consider a new change of variable by letting $\gamma := \beta_2 + \dfrac{\delta}{N}$, that is, δ is a blow up of γ around β_2. We want to check to what extent does δ possiby depend on k. Carefully solving, we get that δ is infinitely close to a standard function of k, denoted $\beta_3(k)$, given by

$$\beta_3(k) = \frac{-k\beta_1 - \dfrac{\beta_2^2}{\beta_1^2(1+r_f)} - \left(1-\dfrac{\beta_1}{1+r_f}\right)\left[\beta_2\left(\dfrac{3}{2}-\dfrac{\beta_2}{\beta_1^2}-\dfrac{\beta_2}{2\beta_1}\right)-\beta_1\left(1+\dfrac{2\beta_1}{3}-\dfrac{\beta_2}{2}+\dfrac{\beta_1^2}{8}\right)\right]}{\dfrac{\beta_1-r_f}{\beta_1(1+r_f)}}.$$

Notice that since β_1 and β_2 are constants, β_3 will be a linear expression in terms of k. So $\delta = \beta_3(k) + \oslash$.

Therefore, one gets

$$q = 1-\frac{\beta}{N} = 1-(\beta_1-\frac{\gamma}{N})\frac{1}{N} = 1-\frac{\beta_1}{N}+\frac{\gamma}{N^2} = 1-\frac{\beta_1}{N}+(\beta_2+\frac{\delta}{N})\frac{1}{N^2} = 1-\frac{\beta_1}{N}+\frac{\beta_2}{N^2}+(\beta_3(k)+\oslash)\frac{1}{N^3},$$

so that $q(k) = q = 1-\dfrac{\beta_1}{N}+\dfrac{\beta_2}{N^2}+\dfrac{\beta_3(k)}{N^3}+\dfrac{\oslash}{N^3}$. Thus we proved that the error term $\varepsilon_k(N) = \oslash$ in (4) is infinitesimal. □

5.2 Proof of Theorem 7

The second theorem is in fact a simple corollary of the first one. Indeed $r(k) = -N\log(q(k))$. Thus, using the order 3 asymptotic expansion of $q(k)$ given in Theorem 6 and the expansion of $\log(1-x)$ for small x, it is easy to derive the given order 2 asymptotic expansion of $r(k)$. □

6 Conclusion

In this paper, we have examined the risk of microcredit related to delay in repayment of one or more settlements. Having in mind Yunus's example, we have set up a model with possible delays at random times, and thus a random actual interest rate. We have assumed a number of installments N for the loans that is infinitely large in order to be able to compute the distribution of the actual interest rate asymptotically. With the flat rate equal to 10%, the actual rate is a little less then 20% which is the maximum. We have shown that any delay reduces the interest received by the MFI, with an average value of a little more more than 16% when $N = 50$. Moreover, the probability of having a delay is 16%, a value extrapolated within the model, considering the real practice of microcredit. Finally, we have given an order two asymptotic expansion of the rate when restricting the risk to a single delay. The expansion explains why the interest rate $r(k)$, $k = 1, 2, \ldots, N$, $N \in \mathbb{Z}^+$, involved in one delay in repayment of a microloan is, up to terms small with respect to $\frac{1}{N^2}$, an affine function of the time period k when the delay takes place. Similar expansions probably exist for the distribution of the actual interest rate in case of 2 delays, 3 delays, etc. It will then allow us to explain the different shapes of the histograms obtained by simulation: uniform for one delay, triangular for two delays, etc., which would turn out to be similar to the probability distribution of the sum of several dices. But this remains to be done.

Acknowledgment: The work of the second author was partly supported by EU programme Erasmus Mundus - Action 2 - EACEA/42/11.

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