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On Mixed Generalized Quasi-Einstein Manifolds

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Abstract

A non-flat Riemannian manifold $M = (M^n, g)$, $n \geq 3$, is said to be quasi-Einstein if its Ricci curvature tensor S satisfies $S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y)$, for some scalar functions α and β, and non-zero 1-form A given by $A(X) = g(X, \rho)$ for some unit vector field ρ. Due to its relevance in the general theory of relativity, several generalizations were constructed. In this paper, we consider the class of mixed generalized quasi-Einstein manifolds. We provide three-dimensional and four-dimensional non-trivial examples. We obtain necessary conditions for such manifold to be W_2-flat or W_4-flat. Finally, we look at the sectional and scalar curvatures of these manifolds.

Keywords: mixed generalized quasi-Einstein manifolds, sectional curvature, scalar curvature, W_2-curvature tensor, W_4-curvature tensor
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1 Introduction

A Riemannian manifold $M = (M^n, g)$ is said to be an Einstein manifold if for all smooth vector fields $X, Y \in \mathfrak{X}(M)$, its Ricci curvature tensor S satisfies

$$S(X,Y) = \frac{\tau}{n}\, g(X,Y), \tag{1.1}$$

where τ is the scalar curvature of M. Equation (1.1) is called Einstein metric condition [2]. Due to its relevance in the general theory of relativity, Einstein manifolds were generalized by geometers and mathematical physicists.

In 2000, Chaki and Maity [3] introduced the class of Riemannian manifolds whose Ricci curvature tensor satisfies

$$S(X,Y) - \alpha g(X,Y) = \beta A(X)A(Y), \tag{1.2}$$

for some scalar functions α and β, and where A is a nonzero 1-form given by $g(X, \rho) = A(X)$, for some unit vector field ρ. A manifold satisfying equation (1.2) is called quasi-Einstein and relevance of such was seen during study of exact solutions of Einstein field equations. In particular, quasi-Einstein manifolds can be taken as model of the perfect fluid spacetime [4]. Moreover, it was shown that the Robertson-Walker spacetime are quasi-Einstein manifolds [5]. It cearly follows from the definition that every Einstein manifold is quasi-Einstein.

By imposing conditions on the Ricci curvature tensor, several more generalizations were obtained. This paper focuses on the class of Riemannian manifolds whose Ricci tensor satisfies

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) + \delta\left[A(X)B(Y) + A(Y)B(X)\right],$$

for some scalar function α, β, γ and δ, and nonzero one-forms A and B given by $g(X,\rho) = A(X)$ and $g(X,\mu) = B(X)$ for some orthogonal unit vector fields ρ and μ, called generators of the manifold. Such manifold is called a mixed generalized quasi-Einstein manifold and is denoted by $MGQE$ [1]. The scalar functions α, β, γ and δ are called associated scalars and A and B are called associated one-forms. Suppose $\{e_i : i = 1, 2, \cdots, n\}$ is an orthonormal frame field at any point of the manifold. If we set $X = Y = e_i$ in the Ricci tensor of an $MGQE$ and take the summation over $i, 1 \le i \le n$, we obtain

$$\tau = n\alpha + \beta + \gamma,$$

where τ is the scalar curvature of the manifold. Thus, we see that the scalar curvature τ of the manifold is expressed as a linear combination of the associated scalar functions α, β and γ. Aside from the fact that the scalar curvature of a manifold generalizes its sectional curvature, in manifolds like the $MGQE$, it is expressed in terms of its associated scalar functions. It was this nice relationship between scalar curvature and associated scalars that motivated the investigation of $MGQE$ admitting some curvature tensors.

2 Curvature in Riemmanian Manifolds

In this section, (M^n, g) always denotes a Riemannian manifold with Riemannian metric g and Levi-Civita connection ∇. We will define curvature tensors which measure how a given manifold deviates from being a Euclidean space. From now on, we will use the notation $\partial_i = \frac{\partial}{\partial x^i}$.

Definition 2.1. *Let M be a manifold with linear connection ∇. The curvature tensor of ∇ is the map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by*

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The curvature tensor of the Levi-Civita connection is called the *Riemannian curvature tensor.* We shall denote the components of R by R^k_{lij}. Clearly, R is a tensor field of type $(1,3)$ and the value of $R(X,Y)Z$ at a point $p \in M$ depends only on the values of X_p, Y_p and Z_p.

We now introduce the process of raising or lowering indices of a tensor. Given $1 \le k \le r$ and the matrix of components of g_{ij} of the metric tensor g, one can lower the k^{th} contravariant index of a mixed tensor $\mathbf{T^r_s}(M)$ whose components are $T^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$ by taking the $(k, s+1)$ contraction of the product of g and T; that is,

$$\begin{aligned} g_{i_k j} T^{i_1 i_2 \dots i_{k-1} i_k i_{k+1} \dots i_r}_{j_1 j_2 \dots j_s} &= T^{i_1 i_2 \dots i_{k-1} i_k i_{k+1} \dots i_r}_{j_1 j_2 \dots j_s i_k j} \\ &= T^{i_1 i_2 \dots i_{k-1} i_{k+1} \dots i_r}_{j_1 j_2 \dots j_s j} \in \mathbf{T^{r-1}_{s+1}}(M). \end{aligned}$$

Likewise, given $1 \leq l \leq s$, to raise the l^{th} index of T, we take the $(r+2, l)$ contraction of the product of the components g^{ij} and T:

$$\begin{aligned} g^{ij_l} T^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_{l-1} j_l j_{l+1} \dots j_s} &= T^{i_1 i_2 \dots i_r i j_l}_{j_1 j_2 \dots j_{l-1} j_l j_{l+1} \dots j_s} \\ &= T^{i_1 i_2 \dots i_r i}_{j_1 j_2 \dots j_{l-1} j_{l+1} \dots j_s} \in \mathbf{T^{r+1}_{s-1}}(M). \end{aligned}$$

Thus, given a Riemannian metric g, one may consider a tensor of type $(0,4)$ defined by

$$R(X, Y, Z, W) := g(R(X, Y)Z, W)$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$, called the *covariant curvature tensor* and whose components are of the form R_{klij} and are explicitly given by

$$R_{klij} = g_{km} R^m_{lij}.$$

Remark 2.2. For all $X, Y, Z, W \in \mathfrak{X}(M)$, the covariant curvature tensor satisfies

1. $R(X, Y, Z, W) = -R(Y, X, Z, W)$
2. $R(X, Y, Z, W) = -R(X, Y, W, Z)$
3. $R(X, Y, Z, W) = R(Z, W, X, Y)$
4. $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$

Definition 2.3. *Let $p \in M$ and let π be a two-dimensional subspace of T_pM spanned by vectors X and Y. The sectional curvature of M at p associated to π, denoted by $K(\pi)$, is defined by*

$$K(\pi) = \frac{R(X, Y, Y, X)}{g(X, X)g(Y, Y) - (g(X, Y))^2}.$$

We may also denote $K(\pi)$ by $K(X, Y)$. It must be noted that $K(\pi)$ is independent of the choice of basis for π. In particular, if $\{X, Y\}$ is an orthonormal basis for π, then

$$K(\pi) := R(X, Y, Y, X) = g(R(X, Y)Y, X).$$

Now, let (M^n, g) be a Riemannian manifold and consider a basis $\{e_1, e_2, \dots, e_n\}$ for T_pM. Note that since $\nabla_{[e_i, e_j]} e_l = 0$,

$$R(e_i, e_j)e_l = \nabla_{e_i} \nabla_{e_j} e_l - \nabla_{e_j} \nabla_{e_i} e_l.$$

So that

$$\begin{aligned} R(e_i, e_j)e_l &= \nabla_{e_i} \nabla_{e_j} e_l - \nabla_{e_j} \nabla_{e_i} e_l \\ &= \nabla_{e_i} \Gamma^s_{jl} e_s - \nabla_{e_j} \Gamma^s_{il} e_s \\ &= \frac{\partial}{\partial x^i} \Gamma^k_{jl} e_k - \frac{\partial}{\partial x^j} \Gamma^k_{il} e_k + \Gamma^s_{jl} \Gamma^k_{is} e_k - \Gamma^s_{il} \Gamma^k_{js} e_k \\ dx^k(R(e_i, e_j)e_l) &= \frac{\partial}{\partial x^i} \Gamma^k_{jl} - \frac{\partial}{\partial x^j} \Gamma^k_{il} + \Gamma^s_{jl} \Gamma^k_{is} - \Gamma^s_{il} \Gamma^k_{js} \\ &= \begin{vmatrix} \frac{\partial}{\partial x^i} & \frac{\partial}{\partial x^j} \\ \Gamma^k_{il} & \Gamma^k_{jl} \end{vmatrix} + \begin{vmatrix} \Gamma^k_{im} & \Gamma^k_{jm} \\ \Gamma^m_{il} & \Gamma^m_{jl} \end{vmatrix}, \end{aligned}$$

where $|\cdot|$ denotes the determinant. With this, and using properties of the covariant curvature tensor, the independent components of R are of the form R_{klij}, R_{ikij} and R_{ijij}.

By applying a contraction on indices $\{j, l\}$ of the Riemannian curvature tensor, we get a symmetric covariant $(0, 2)$-tensor S given by

$$S_{ik} = g^{jl} R_{ijkl}.$$

This is known as the Ricci tensor.

Definition 2.4. *The Ricci tensor is the map $S : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$, where at each point $p \in M$,*

$$S(X, Y) = \mathrm{tr}[Q : Z \mapsto R(X, Z)Y].$$

We call Q the Ricci tensor transformation and the trace of S is known as the scalar curvature τ.

The Ricci tensor is independent of the choice of orthonormal basis. If $\{e_1, \ldots, e_n\}$ is an orthonormal basis for T_pM then

$$S(X, Y) = \sum_{i=1}^{n} R(e_i, X, Y, e_i).$$

Definition 2.5. *The Ricci curvature of a Riemannian manifold (M^n, g) is the map $S : \mathfrak{X}(M) \to C^\infty(M)$ mapping $X \mapsto S(X) := S(X, X)$.*

For any unit vector field $X \in T_pM$, $S(X, X)$ is the sum of the sectional curvatures of planes spanned by X and other elements of an orthonormal basis for T_pM. Since S is symmetric and bilinear, it is compltely determined by its values of the form $S(X, X)$ for unit vector fields X.

Finally, if we take the sum of all sectional curvatures of planes spanned by pairs of orthonormal basis elements, we get the *scalar curvature* τ of the manifold. That is

$$\tau = \sum_{j \neq k} K(e_j, e_k).$$

By raising an index of S to obtain the Ricci tensor transformation and then apply contraction to get its trace, the scalar curvature will be given by

$$\tau = g^{ij} S_{ij}.$$

3 Examples of MGQE

We provide examples of mixed generalized quasi-Einstein manifolds of dimension three and dimension four.

Example 3.1. Define a metric g on $\mathbb{R}^3$ with components

$$g_{11} = g_{22} = g_{33} = e^{x^1} \text{ and } g_{ij} = 0, \forall i \neq j.$$

The non-vanishing components of the Christoffel symbols are

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \tfrac{1}{2}$$
$$\Gamma^1_{22} = \Gamma^1_{33} = -\tfrac{1}{2}$$

and those which can be obtained using symmetries of the Christoffel symbols. Since the non-vanishing components of the Christoffel symbols are constants, then the partial derivatives of the Christoffel symbols are all zero. The Riemannian curvature tensor then becomes

$$R^k_{lij} = \begin{vmatrix} \frac{\partial}{\partial x^i} & \frac{\partial}{\partial x^j} \\ \Gamma^k_{il} & \Gamma^k_{jl} \end{vmatrix} + \begin{vmatrix} \Gamma^k_{im} & \Gamma^k_{jm} \\ \Gamma^m_{il} & \Gamma^m_{jl} \end{vmatrix} = \begin{vmatrix} \Gamma^k_{im} & \Gamma^k_{jm} \\ \Gamma^m_{il} & \Gamma^m_{jl} \end{vmatrix}.$$

The only nonzero components of the Riemannian curvature tensor are

$$R^2_{323} = R^3_{232} = -\frac{1}{4}.$$

Thus, the only nonzero components of the covariant curvature tensor are

$$R_{3232} - \frac{1}{4}e^{x^1},$$

and those which can be obtained from the symmetry properties.

Now, clearly, if $i \neq k$,

$$S_{ik} = g^{jl}R_{ijkl} = 0.$$

Moreover,

$$S_{11} = g^{jl}R_{1j1l} = 0$$
$$S_{22} = g^{jl}R_{2j2l} = g^{33}R_{2323} = \left(e^{-x^1}\right)\left(-\frac{1}{4}e^{x^1}\right) = -\frac{1}{4}$$
$$S_{33} = g^{jl}R_{3j3l} = g^{33}R_{3232} = \left(e^{-x^1}\right)\left(-\frac{1}{4}e^{x^1}\right) = -\frac{1}{4}.$$

and the scalar curvature is given by

$$\tau = g^{ij}S_{ij} = g^{22}S_{22} + g^{33}S_{33} = -\frac{1}{2}e^{-x^1},$$

which is nonzero and nonconstant. Let us now consider the following vector fields:

$$\rho = \frac{1}{\sqrt{2e^{x^1}}}\partial_1 + \frac{1}{\sqrt{2e^{x^1}}}\partial_2 \text{ and } \mu = \frac{1}{\sqrt{2e^{x^1}}}\partial_1 - \frac{1}{\sqrt{2e^{x^1}}}\partial_2,$$

which are orthogonal unit vector fields with respect to the defined metric g. Associated to these vector fields are the 1-forms A and B with components

$$A_1 = A_2 = \sqrt{\frac{e^{x^1}}{2}},\ A_3 = 0$$

$$B_1 = \sqrt{\frac{e^{x^1}}{2}},\ B_2 = -\sqrt{\frac{e^{x^1}}{2}},\ B_3 = 0.$$

Take the associated scalar functions

$$\alpha = -\frac{1}{4}e^{-x^1} \quad \text{and} \quad \beta = \gamma = \delta = \frac{1}{8}e^{-x^1}.$$

We now check that for all $i, j = 1, 2, 3, 4$,

$$S_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma B_i B_j + \delta(A_i B_j + A_j B_i).$$

For $i = j = 1$,

$$\begin{aligned}\alpha g_{11} + \beta A_1 A_1 + \gamma B_1 B_1 + \delta(A_1 B_1 + A_1 B_1) &= \left(-\frac{1}{4}e^{-x^1}\right) e^{x^1} + \left(\frac{1}{8}e^{-x^1}\right)\left(\frac{1}{2}e^{x^1}\right) \\ &\quad + \left(\frac{1}{8}e^{-x^1}\right)\left(\frac{1}{2}e^{x^1}\right) + \left(\frac{1}{8}e^{-x^1}\right) e^{x^1} \\ &= 0 \\ &= S_{11}.\end{aligned}$$

For $i = j = 2$,

$$\begin{aligned}\alpha g_{22} + \beta A_2 A_2 + \gamma B_2 B_2 + \delta(A_2 B_2 + A_2 B_2) &= \left(-\frac{1}{4}e^{-x^1}\right) e^{x^1} + \left(\frac{1}{8}e^{-x^1}\right)\left(\frac{1}{2}e^{x^1}\right) \\ &\quad + \left(\frac{1}{8}e^{-x^1}\right)\left(\frac{1}{2}e^{x^1}\right) + \left(\frac{1}{8}e^{-x^1}\right)\left(-e^{x^1}\right) \\ &= -\frac{1}{4} \\ &= S_{22}.\end{aligned}$$

For $i = j = 3$,

$$\begin{aligned}\alpha g_{33} + \beta A_3 A_3 + \gamma B_3 B_3 + \delta(A_3 B_3 + A_3 B_3) &= \left(-\frac{1}{4}e^{-x^1}\right) e^{x^1} \\ &= -\frac{1}{4} \\ &= S_{33}.\end{aligned}$$

Note that because of the symmetry of S_{ij}, same results will be obtained when $i = 1, j = 2$ and when $i = 2, j = 1$. For these cases,

$$\begin{aligned}\alpha g_{12} + \beta A_1 A_2 + \gamma B_1 B_2 + \delta(A_1 B_2 + A_2 B_1) &= \left(\frac{1}{8}e^{-x^1}\right)\left(\frac{1}{2}e^{x^1}\right) + \left(\frac{1}{8}e^{-x^1}\right)\left(-\frac{1}{2}e^{x^1}\right) \\ &\quad + \left(\frac{1}{8}e^{-x^1}\right)\left(-\frac{1}{2}e^{x^1} + \frac{1}{2}e^{x^1}\right) \\ &= 0 \\ &= S_{12} = S_{21}.\end{aligned}$$

Similarly, same values will be obtained when $i \neq j$ and $i = 3$ or $j = 3$. Without loss of generality, let $j = 3$. Then for $i = 1, 2$, we have

$$\alpha g_{i3} + \beta A_i A_3 + \gamma B_i B_3 + \delta(A_i B_3 + A_3 B_i) = 0 = S_{i3} = S_{3i}.$$

Therefore, $(\mathbb{R}^3, g)$ is a mixed generalized quasi-Einstein manifold.

Example 3.2. Consider the metric g on $\mathbb{R}^4$ with components $g_{ij} = 0$, $\forall i \neq j$, and

$$g_{11} = g_{22} = a^2 + \left(x^1\right)^2, \; g_{33} = g_{44} = \frac{1}{a^2 + (x^1)^2},$$

where $a \in \mathbb{R}\backslash\{0\}$. The non-vanishing components of the Christoffel symbols are

$$\Gamma^1_{11} = \Gamma^2_{12} = \frac{x^1}{a^2 + (x^1)^2}$$

$$\Gamma^1_{22} = \Gamma^3_{13} = \Gamma^4_{14} = -\frac{x^1}{a^2 + (x^1)^2}$$

$$\Gamma^1_{33} = \Gamma^1_{44} = \frac{x^1}{\left(a^2 + (x^1)^2\right)^3}$$

and those which can be obtained using symmetry properties. The nonzero derivatives of the Christoffel symbols are

$$\frac{\partial \Gamma^1_{11}}{\partial x^1} = \frac{\partial \Gamma^2_{12}}{\partial x^1} = \frac{a^2 - \left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^2} = -\frac{\partial \Gamma^1_{22}}{\partial x^1} = -\frac{\partial \Gamma^3_{13}}{\partial x^1} = -\frac{\partial \Gamma^4_{14}}{\partial x^1}$$

$$\frac{\partial \Gamma^1_{33}}{\partial x^1} = \frac{\partial \Gamma^1_{44}}{\partial x^1} = \frac{a^2 - 5\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^4}.$$

For the Riemannian curvature tensor,

$$R^k_{lij} = \underbrace{\begin{vmatrix} \frac{\partial}{\partial x^i} & \frac{\partial}{\partial x^j} \\ \Gamma^k_{il} & \Gamma^k_{jl} \end{vmatrix}}_{(*)} + \underbrace{\begin{vmatrix} \Gamma^k_{im} & \Gamma^k_{jm} \\ \Gamma^m_{il} & \Gamma^m_{jl} \end{vmatrix}}_{(**)},$$

we consider only the nonzero values of $(*)$ and $(**)$. The nonzero $(*)$-components are:

$$R^1_{212}: \; \frac{\partial \Gamma^1_{22}}{\partial x^1} = \frac{\left(x^1\right)^2 - a^2}{\left(a^2 + (x^1)^2\right)^2}$$

$$R^1_{313}: \; \frac{\partial \Gamma^1_{33}}{\partial x^1} = \frac{a^2 - 5\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^4}$$

$$R^1_{414}: \; \frac{\partial \Gamma^1_{44}}{\partial x^1} = \frac{a^2 - 5\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^4}$$

and the nonzero $(**)$-components are:

$$R^1_{313}: \quad \Gamma^1_{1m}\Gamma^m_{33} - \Gamma^1_{3m}\Gamma^m_{13} = \Gamma^1_{11}\Gamma^1_{33} - \Gamma^1_{33}\Gamma^3_{13} = \frac{2\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^4}$$

$$R^1_{414}: \quad \Gamma^1_{1m}\Gamma^m_{44} - \Gamma^1_{4m}\Gamma^m_{14} = \Gamma^1_{11}\Gamma^1_{44} - \Gamma^1_{44}\Gamma^4_{14} = \frac{2\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^4}$$

$$R^3_{232}: \quad \Gamma^3_{3m}\Gamma^m_{22} - \Gamma^3_{2m}\Gamma^m_{32} = \Gamma^3_{31}\Gamma^1_{22} = \frac{\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^2}$$

$$R^4_{242}: \quad \Gamma^4_{4m}\Gamma^m_{22} - \Gamma^4_{2m}\Gamma^m_{42} = \Gamma^4_{41}\Gamma^1_{22} = \frac{\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^2}$$

$$R^4_{343}: \quad \Gamma^4_{4m}\Gamma^m_{33} - \Gamma^4_{3m}\Gamma^m_{43} = \Gamma^4_{41}\Gamma^1_{33} = -\frac{\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^2}$$

Adding corresponding $(*)$- and $(**)$-components,

$$R^1_{212} = \frac{\left(x^1\right)^2 - a^2}{\left(a^2 + (x^1)^2\right)^2} \quad , \quad R^1_{313} = \frac{a^2 - 3\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^4} = R^1_{414}$$

$$R^4_{343} = -\frac{\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^4} \quad , \quad R^3_{232} = \frac{\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^2} = R^4_{242}$$

Thus, the nonzero components of the covariant curvature tensor, up to symmetry, are

$$R_{1212} = \frac{\left(x^1\right)^2 - a^2}{a^2 + \left(x^1\right)^2} \quad , \quad R_{1313} = \frac{a^2 - 3\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^3} = R_{1414}$$

$$R_{3434} = -\frac{\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^5} \quad , \quad R_{2323} = \frac{\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^3} = R_{2424}$$

It is clear that when $i \neq k$,

$$S_{ik} = g^{jl}R_{ijkl} = 0.$$

Furthermore,

$$S_{11} = g^{jl}R_{1j1l} = g^{22}R_{1212} + g^{33}R_{1313} + g^{44}R_{1414} = \frac{a^2 - 5\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^2}$$

$$S_{22} = g^{jl}R_{2j2l} = g^{11}R_{2121} + g^{33}R_{2323} + g^{44}R_{2424} = \frac{3\left(x^1\right)^2 - a^2}{\left(a^2 + (x^1)^2\right)^2}$$

$$S_{33} = g^{jl}R_{3j3l} = g^{11}R_{3131} + g^{22}R_{3232} + g^{44}R_{3434} = \frac{a^2 - 3\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^4}$$

$$S_{44} = g^{jl}R_{4j4l} = g^{11}R_{4141} + g^{22}R_{4242} + g^{33}R_{4343} = \frac{a^2 - 3\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^4}.$$

The scalar curvature is

$$\tau = g^{11}S_{11} + g^{22}S_{22} + g^{33}S_{33} + g^{44}S_{44} = \frac{2a^2 - 8\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^3}.$$

Now, consider the following orthogonal unit vector fields with respect to the defined metric:

$$\rho = \frac{1}{\sqrt{2\left(a^2 + (x^1)^2\right)}}\partial_1 + \frac{1}{\sqrt{2\left(a^2 + (x^1)^2\right)}}\partial_2$$

$$\mu = \frac{1}{\sqrt{2\left(a^2 + (x^1)^2\right)}}\partial_1 - \frac{1}{\sqrt{2\left(a^2 + (x^1)^2\right)}}\partial_2.$$

The associated 1-forms have components

$$A_1 = A_2 = \sqrt{\frac{a^2 + \left(x^1\right)^2}{2}},\ A_3 = A_4 = 0$$

$$B_1 = -B_2 = \sqrt{\frac{a^2 + \left(x^1\right)^2}{2}},\ B_3 = B_4 = 0.$$

Take the associated scalars

$$\alpha = \frac{a^2 - 3\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^3},\ \beta = \frac{2\left(x^1\right)^2 - a^2}{\left(a^2 + (x^1)^2\right)^3} = \gamma,\ \text{and}\ \delta = \frac{a^2 - 4\left(x^1\right)^2}{\left(a^2 + (x^1)^2\right)^3}.$$

It is obvious that when either i or j is 3 or 4,

$$\alpha g_{ij} + \beta A_i A_j + \gamma B_i B_j + \delta\left(A_i B_j + A_j B_i\right) = 0 = S_{ij}.$$

For $(i, j) = (1, 2)$ or $(2, 1)$,

$$\begin{aligned}\alpha g_{12} + \beta A_1 A_2 + \gamma B_1 B_2 + \delta\left(A_1 B_2 + A_2 B_1\right) &= \left[\frac{2\left(x^1\right)^2 - a^2}{\left(a^2 + \left(x^1\right)^2\right)^3}\right]\left[\frac{a^2 + \left(x^1\right)^2}{2}\right] \\ &\quad + \left[\frac{2\left(x^1\right)^2 - a^2}{\left(a^2 + \left(x^1\right)^2\right)^3}\right]\left[-\frac{a^2 + \left(x^1\right)^2}{2}\right] \\ &= 0 = S_{12} = S_{21}.\end{aligned}$$

It can be checked that when $i = j$, $i = 1, 2, 3, 4$,

$$\alpha g_{ii} + \beta A_i A_i + \gamma B_i B_i + \delta\left(A_i B_i + A_i B_i\right) = S_{ii}.$$

Thus, $\left(\mathbb{R}^4, g\right)$ is a mixed generalized quasi-Einstein manifold.

4 Flatness with respect to W_2 and W_4-curvature tensors

In 1970, Pokhariyal and Mishra defined several curvature tensors and studied their physical and geometric properties [6], [7]. The W_2 and W_4-curvature tensors are $(0, 4)$-tensors respectively defined by

$$\begin{aligned}W_2(X, Y, Z, T) &= R(X, Y, Z, T) + \frac{1}{n-1}\left[g(X, Z)S(Y, T) - g(Y, Z)S(X, T)\right] \\ W_4(X, Y, Z, T) &= R(X, Y, Z, T) + \frac{1}{n-1}\left[g(X, Z)S(Y, T) - g(X, Y)S(Z, T)\right],\end{aligned}$$

where R is the covariant curvature tensor and S is the Ricci tensor. From this, we can see that if a manifold is W_2-flat, i.e., $W_2(X, Y, Z, T) = 0$ for all X, Y, Z, T, then the covariant curvature tensor satisfies

$$R(X, Y, Z, T) = \frac{1}{n-1}\left\{g(Y, Z)S(X, T) - g(X, Z)S(Y, T)\right\}. \tag{4.1}$$

On the other hand, if a manifold is W_4-flat, i.e., $W_4(X, Y, Z, T) = 0$ for all X, Y, Z, T $\in TM$, then

$$R(X, Y, Z, T) = \frac{1}{n-1}\left\{g(X, Y)S(Z, T) - g(X, Z)S(Y, T)\right\}. \tag{4.2}$$

We look at properties satisfied by manifolds which are W_2-flat or W_4-flat.

Theorem 4.1. *In a W_2-flat mixed generalized quasi-Einstein manifold, the relation*

$$\beta A(X)A(Y) = \gamma B(X)B(Y),$$

holds for all $X, Y \in \mathfrak{X}(M)$.

Proof: Consider a W_2-flat mixed generalized quasi-Einstein manifold. Then

$$\begin{aligned} R(X,Y,\rho,\mu) &= \frac{1}{n-1}\left[g(Y,\rho)S(X,\mu) - g(X,\rho)S(Y,\mu)\right] \\ &= \frac{\alpha}{n-1}\left[A(Y)B(X) - A(X)B(Y)\right] \\ R(\rho,\mu,X,Y) &= \frac{1}{n-1}\left[g(\mu,X)S(\rho,Y) - g(\rho,X)S(\mu,Y)\right] \\ &= \frac{1}{n-1}\left[\alpha(A(Y)B(X) - A(X)B(Y)) + \gamma B(X)B(Y) - \beta A(X)A(Y)\right] \end{aligned}$$

Since $R(X,Y,\rho,\mu) = R(\rho,\mu,X,Y)$, we have $\beta A(X)A(Y) = \gamma B(X)B(Y)$.

Theorem 4.2. *If a mixed generalized quasi-Einstein manifold is W_4-flat then $\delta = 0$.*

Proof: Suppose we have a W_4-flat mixed generalized quasi-Einstein manifold. We have

$$\begin{aligned} R(X,Y,\rho,\mu) &= \frac{1}{n-1}\left\{g(X,Y)S(\rho,\mu) - g(X,\rho)S(Y,\mu)\right\} \\ &= \frac{1}{n-1}\left\{\delta g(X,Y) - A(X)\left[(\alpha+\gamma)B(Y) + \delta A(Y)\right]\right\} \end{aligned} \tag{4.3}$$

$$\begin{aligned} R(\rho,\mu,X,Y) &= \frac{1}{n-1}\left\{g(\rho,\mu)S(X,Y) - g(\rho,X)S(\mu,Y)\right\} \\ &= \frac{1}{n-1}\left\{-A(X)\left[(\alpha+\gamma)B(Y) + \delta A(Y)\right]\right\}. \end{aligned} \tag{4.4}$$

Note that $R(X,Y,\rho,\mu) = R(\rho,\mu,X,Y)$. So from equations (4.3) and (4.4), we obtain

$$\delta g(X,Y) = 0, \text{ for all } X,Y \in TM.$$

Since $g(X,Y)$ is not identically zero, then $\delta = 0$.

It follows from Theorem 4.2 that the Ricci tensor of a W_4-flat mixed quasi-Einstein manifold must be of the form

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y).$$

If we assume further that the manifold is W_4-flat, using Theorem 4.1 we have the following consequence.

Remark 4.3. *A mixed generalized quasi-Einstein manifold that is flat with respect to both W_2 and W_4-curvature tensors is a quasi-Einstein manifold.*

5 Sectional curvature

Theorem 5.1. *Let M be a three-dimensional mixed generalized quasi-Einstein manifold with associated scalars α, β, γ and δ, and associated one-forms A and B. Then*

$$K(\Pi_1) = \frac{1}{2}(\alpha+\beta+\gamma),\ K(\Pi_2) = \frac{1}{2}(\alpha+\beta-\gamma) \text{ and } K(\Pi_3) = \frac{1}{2}(\alpha-\beta+\gamma),$$

for any two-dimensional subspaces Π_1, Π_2 and Π_3 of T_pM, with $\Pi_1 = span\{\rho,\mu\}$, Π_2 orthogonal to μ and Π_3 orthogonal to ρ.

Proof: Let $p \in M$ and let $\{e_1, \rho, \mu\}$ be an orthonormal frame of T_pM. We have

$$S(e_1) = K(e_1 \wedge \rho) + K(e_1 \wedge \mu) = \alpha \tag{5.1}$$
$$S(\rho) = K(e_1 \wedge \rho) + K(\rho \wedge \mu) = \alpha + \beta \tag{5.2}$$
$$S(\mu) = K(e_1 \wedge \mu) + K(\rho \wedge \mu) = \alpha + \gamma. \tag{5.3}$$

Without loss of generality, consider the following two dimensional subspaces of T_pM:

$$\Pi_1 = \operatorname{span}\{\rho, \mu\}$$
$$\Pi_2 = \operatorname{span}\{e_1, \rho\}$$
$$\Pi_3 = \operatorname{span}\{e_1, \mu\}.$$

Adding equations (5.2) and (5.3), and subtracting equation (5.1) from the result, we have

$$2K(\rho \wedge \mu) = \alpha + \beta + \gamma.$$

Therefore, $K(\Pi_1) = \frac{1}{2}(\alpha + \beta + \gamma)$. Similarly, summing equations (5.1) and (5.2), and subtracting equation (5.3) from the result, we get

$$2K(e_1 \wedge \rho) = \alpha + \beta - \gamma.$$

Thus, $K(\Pi_2) = \frac{1}{2}(\alpha + \beta - \gamma)$. Finally, adding equations (5.1) and (5.3), and subtracting equation (5.2), yields

$$2K(e_1 \wedge \mu) = \alpha - \beta + \gamma.$$

Hence, $K(\Pi_3) = \frac{1}{2}(\alpha - \beta + \gamma)$.

Theorem 5.2. *Let M be a four-dimensional mixed generalized quasi-Einstein manifold with associated scalars α, β, γ and δ, and associated one-forms A and B. Then*

i. $K(\Pi_1^{\perp}) = K(\Pi_1) + \frac{1}{2}(\beta + \gamma)$, *for any two-dimensional subspace Π_1 of T_pM orthogonal to ρ and μ, and*

ii. $K(\Pi_2^{\perp}) = K(\Pi_2) + \frac{1}{2}(\beta - \gamma)$, *for any two-dimensional subspace Π_2 of T_pM containing μ and orthogonal to ρ.*

Proof: Let $\{e_1, e_2, \rho, \mu\}$ be an orthonormal frame for T_pM. Consider the following two-dimensional subspaces of T_pM:

$$\Pi_1 = \operatorname{span}\{e_1, e_2\} \quad \text{and} \quad \Pi_2 = \operatorname{span}\{e_1, \mu\}.$$

Computing the Ricci curvatures, we have

$$S(e_1) = K(e_1 \wedge e_2) + K(e_1 \wedge \rho) + K(e_1 \wedge \mu) = \alpha \tag{5.4}$$
$$S(e_2) = K(e_1 \wedge e_2) + K(e_2 \wedge \rho) + K(e_2 \wedge \mu) = \alpha \tag{5.5}$$
$$S(\rho) = K(e_1 \wedge \rho) + K(e_2 \wedge \rho) + K(\rho \wedge \mu) = \alpha + \beta. \tag{5.6}$$
$$S(\mu) = K(e_1 \wedge \mu) + K(e_2 \wedge \mu) + K(\rho \wedge \mu) = \alpha + \gamma. \tag{5.7}$$

Adding the respective left-hand and right-hand sids of equations (5.6) and (5.7), we get

$$2K(\rho \wedge \mu) + K(e_1 \wedge \rho) + K(e_2 \wedge \rho) + K(e_1 \wedge \mu) + K(e_2 \wedge \mu) = 2\alpha + \beta + \gamma. \tag{5.8}$$

While adding the respective left-hand and right-hand sids of equations (5.4) and (5.5) results to

$$2K(e_1 \wedge e_2) + K(e_1 \wedge \rho) + K(e_2 \wedge \rho)K(e_1 \wedge \mu) + K(e_2 \wedge \mu) = 2\alpha. \tag{5.9}$$

Now, by subtracting the respective sides of equation (5.7) from (5.8) we get

$$2K(\rho \wedge \mu) - 2K(e_1 \wedge e_2) = \beta + \gamma.$$

Hence,

$$K(\rho \wedge \mu) = K(e_1 \wedge e_2) + \frac{1}{2}(\beta + \gamma).$$

That is,

$$K(\Pi_1^{\perp}) = K(\Pi_1) + \frac{1}{2}\left(\beta + \gamma\right).$$

Now, adding the respective sides of equations (5.5) and (5.6),

$$2K(e_2 \wedge \rho) + K(e_1 \wedge e_2) + K(e_1 \wedge \rho) + K(e_2 \wedge \mu) + K(\rho \wedge \mu) = 2\alpha + \beta. \tag{5.10}$$

Likewise, equations (5.4) and (5.7) give

$$2K(e_1 \wedge \mu) + K(e_1 \wedge e_2) + K(e_1 \wedge \rho) + K(e_2 \wedge \mu) + K(\rho \wedge \mu) = 2\alpha + \gamma. \tag{5.11}$$

Subtracting the respective sides of equations (5.10) and (5.11), we obtain

$$2K(e_2 \wedge \rho) - 2K(e_1 \wedge \mu) = \beta - \gamma.$$

Therefore,

$$K(e_2 \wedge \rho) = K(e_1 \wedge \mu) + \frac{1}{2}(\beta - \gamma).$$

That is,

$$K(\Pi_2^{\perp}) = K(\Pi_2) + \frac{1}{2}\left(\beta - \gamma\right).$$

6 Scalar curvature

For a Riemannian manifold M of dimension n, we generalize the idea of sectional curvatures to subspaces of T_pM of dimension $2 < r \leq n$. Throughout this section, $\forall p \in M$, $P^{\perp}$ denotes the orthogonal complement of P in T_pM.

Definition 6.1. *Let M be a n-dimensional Riemannian manifold and let $p \in M$. Suppose that $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of T_pM. Let P be a r-dimensional subspace of T_pM with orthonormal basis $\{e_1, e_2, \ldots, e_r\}$, with $2 < r \leq n$. The scalar curvature of P, denoted by $\tau(P)$, is defined by*

$$\tau(P) = \sum_{1 \leq i < j \leq r} K(e_i \wedge e_j).$$

If $r = n$, i.e., $P = T_pM$, the scalar curvature of P is the scalar curvature τ of M at the point p.

For $n > 4$, we can partition the tangent space of M at a point p into two disjoint subspaces, P and $P^{\perp}$, with $\dim P = \dim P^{\perp}$ when n is even and $\dim P = 1 + \dim P^{\perp}$ when n is odd. Thus, Definition 6.1 allows us to relate the scalar curvatures of these subspaces. It turns out that by doing so, we can reduce the Ricci tensor to a form without a "mixed" term.

Theorem 6.2. *Let M be a $2n$-dimensional mixed generalized quasi-Einstein manifold with associated scalars α, β, γ, δ, and associated one-forms A and B. Then*

i. $\tau(P_1^{\perp}) = \tau(P_1) + \frac{1}{2}(\beta+\gamma)$, *for any n-dimensional subspace P_1 of T_pM orthogonal to both ρ and μ, and*

ii. $\tau(P_2^{\perp}) = \tau(P_2) - \frac{1}{2}(\beta-\gamma)$, *for any n-dimensional subspace P_2 of T_pM containing ρ and orthogonal to μ.*

Proof: Let M be a $2n$-dimensional mixed generalized quasi-Einstein manifold, $n \geq 2$, with

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) + \delta\left[A(X)B(Y) + A(Y)B(X)\right].$$

Suppose that $\{e_1, e_2, \ldots, e_n, e_{n+1}, \ldots, e_{2n-1}, e_{2n}\}$ is an orthonormal basis for T_pM, with $e_{2n-1} = \rho$ and $e_{2n} = \mu$. Then for a fixed e_i, $i = 1, 2, \ldots, 2n-2$, the Ricci curvatures are given by

$$S(e_i) = \sum_{j\neq i} K(e_i \wedge e_j) = \alpha. \tag{6.1}$$

For $i = 2n-1, 2n$,

$$S(e_{2n-1}) = S(\rho) = \sum_{j\neq 2n-1} K(e_{2n-1} \wedge e_j) = \alpha + \beta \tag{6.2}$$

$$S(e_{2n}) = S(\mu) = \sum_{j\neq 2n} K(e_{2n} \wedge e_j) = \alpha + \gamma. \tag{6.3}$$

Consider a n-dimensional subspace P_1 of T_pM orthogonal to ρ and μ. Without loss of generality, let $P_1 = \operatorname{span}\{e_1, e_2, \ldots, e_n\}$. For $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
S(e_1) &= \sum_{j=2}^{n} K(e_1 \wedge e_j) + \sum_{j=n+1}^{2n} K(e_1 \wedge e_j) = \alpha \\
S(e_2) &= \sum_{\substack{j=1 \\ j\neq 2}}^{n} K(e_2 \wedge e_j) + \sum_{j=n+1}^{2n} K(e_2 \wedge e_j) = \alpha \\
S(e_3) &= \sum_{\substack{j=1 \\ j\neq 3}}^{n} K(e_3 \wedge e_j) + \sum_{j=n+1}^{2n} K(e_3 \wedge e_j) = \alpha \\
&\vdots \\
S(e_n) &= \sum_{j=1}^{n-1} K(e_n \wedge e_j) + \sum_{j=n+1}^{2n} K(e_n \wedge e_j) = \alpha.
\end{aligned}$$

By adding these equations, we get

$$2\sum_{1\leq i<j\leq n} K(e_i \wedge e_j) + \sum_{1\leq i\leq n<j\leq 2n} K(e_i \wedge e_j) = n\alpha,$$

which is equivalent to

$$2\tau(P_1) + \sum_{1\leq i\leq n<j\leq 2n} K(e_i \wedge e_j) = n\alpha. \tag{6.4}$$

Meanwhile,

$$
\begin{aligned}
S(e_{n+1}) &= \sum_{i=1}^{n} K(e_i \wedge e_{n+1}) + \sum_{j=n+2}^{2n} K(e_{n+1} \wedge e_j) = \alpha \\
S(e_{n+2}) &= \sum_{i=1}^{n} K(e_i \wedge e_{n+2}) + \sum_{\substack{j=n+1 \\ j\neq n+2}}^{2n} K(e_{n+2} \wedge e_j) = \alpha \\
&\vdots \\
S(e_{2n-2}) &= \sum_{i=1}^{n} K(e_i \wedge e_{2n-2}) + \sum_{\substack{j=n+1 \\ j\neq 2n-2}}^{2n} K(e_{2n-2} \wedge e_j) = \alpha \\
S(e_{2n-1}) &= \sum_{i=1}^{n} K(e_i \wedge e_{2n-1}) + \sum_{\substack{j=n+1 \\ j\neq 2n-1}}^{2n} K(e_{2n-1} \wedge e_j) = \alpha + \beta \\
S(e_{2n}) &= \sum_{i=1}^{n} K(e_i \wedge e_{2n}) + \sum_{j=n+1}^{2n-1} K(e_{2n} \wedge e_j) = \alpha + \gamma.
\end{aligned}
$$

The above equations, when added, give

$$
\sum_{1\leq i\leq n<j\leq 2n} K(e_i \wedge e_j) + 2\sum_{n+1\leq i<j\leq 2n} K(e_i \wedge e_j) = n\alpha + \beta + \gamma,
$$

which we can rewrite as

$$
\sum_{1\leq i\leq n<j\leq 2n} K(e_i \wedge e_j) + 2\tau(P_1^{\perp}) = n\alpha + \beta + \gamma. \tag{6.5}
$$

Equations (6.4) and (6.5) now imply that

$$
\tau(P_1^{\perp}) = \tau(P_1) + \frac{1}{2}(\beta + \gamma).
$$

Now consider a n-dimensional subspace P_2 of T_pM that contains ρ and orthogonal to μ. Without loss of generality, take $P_2 = \text{span}\{e_n, e_{n+1}, \ldots, e_{2n-1}\}$. By expansion, (6.1) and

(6.2) give

$$\begin{aligned}
S(e_n) &= \sum_{i=1}^{n-1} K(e_i \wedge e_n) + \sum_{j=n+1}^{2n-1} K(e_n \wedge e_j) + K(e_n \wedge e_{2n}) = \alpha \\
S(e_{n+1}) &= \sum_{i=1}^{n-1} K(e_i \wedge e_{n+1}) + \sum_{\substack{j=n \\ j\neq n+1}}^{2n-1} K(e_{n+1} \wedge e_j) + K(e_{n+1} \wedge e_{2n}) = \alpha \\
S(e_{n+2}) &= \sum_{i=1}^{n-1} K(e_i \wedge e_{n+2}) + \sum_{\substack{j=n \\ j\neq n+2}}^{2n-1} K(e_{n+2} \wedge e_j) + K(e_{n+2} \wedge e_{2n}) = \alpha \\
&\vdots \\
S(e_{2n-1}) &= \sum_{i=1}^{n-1} K(e_i \wedge e_{2n-1}) + \sum_{j=n}^{2n} K(e_{2n-2} \wedge e_j) + K(e_{2n-1} \wedge e_{2n}) = \alpha + \beta.
\end{aligned}$$

Summing the above equations, we have

$$\sum_{1\leq i\leq n-1<j\leq 2n-1} K(e_i \wedge e_j) + 2 \sum_{n\leq i<j\leq 2n-1} K(e_i \wedge e_j) + \sum_{i=n}^{2n-1} K(e_i \wedge e_{2n}) = n\alpha + \beta,$$

or equivalently,

$$\sum_{1\leq i\leq n-1<j\leq 2n-1} K(e_i \wedge e_j) + 2\tau(P_3) + \sum_{i=n}^{2n-1} K(e_i \wedge e_{2n}) = n\alpha + \beta. \tag{6.6}$$

Likewise,

$$\begin{aligned}
S(e_1) &= \sum_{j=2}^{n-1} K(e_1 \wedge e_j) + \sum_{j=n}^{2n-1} K(e_1 \wedge e_j) + K(e_1 \wedge e_{2n}) = \alpha \\
S(e_2) &= \sum_{\substack{j=1 \\ j\neq 2}}^{n-1} K(e_2 \wedge e_j) + \sum_{j=n}^{2n-1} K(e_2 \wedge e_j) + K(e_2 \wedge e_{2n}) = \alpha \\
S(e_3) &= \sum_{\substack{j=1 \\ j\neq 3}}^{n-1} K(e_3 \wedge e_j) + \sum_{j=n}^{2n-1} K(e_3 \wedge e_j) + K(e_3 \wedge e_{2n}) = \alpha \\
&\vdots \\
S(e_{n-1}) &= \sum_{j=1}^{n-2} K(e_{n-1} \wedge e_j) + \sum_{j=n}^{2n-1} K(e_{n-1} \wedge e_j) + K(e_{n-1} \wedge e_{2n}) = \alpha.
\end{aligned}$$

Summing the above equations, we get

$$2 \sum_{1\leq i<j\leq n-1} K(e_i \wedge e_j) + \sum_{1\leq i\leq n-1<j\leq 2n-1} K(e_i \wedge e_j) + \sum_{i=1}^{n-1} K(e_i \wedge e_{2n}) = (n-1)\alpha. \tag{6.7}$$

Combining (6.3) and (6.7),

$$2\sum_{1\le i<j\le n-1} K(e_i\wedge e_j)+2\sum_{i=1}^{n-1} K(e_i\wedge e_{2n})+\sum_{1\le i\le n-1<j\le 2n-1} K(e_i\wedge e_j)+\sum_{i=n}^{2n-1} K(e_i\wedge e_{2n}) = n\alpha+\gamma.$$

Equivalently,

$$2\tau(P_3^{\perp}) + \sum_{1\le i\le n-1<j\le 2n-1} K(e_i \wedge e_j) + \sum_{i=n}^{2n-1} K(e_i \wedge e_{2n}) = n\alpha + \gamma. \qquad (6.8)$$

Finally, (6.6) and (6.8) imply that

$$\tau(P_2^{\perp}) = \tau(P_2) - \frac{1}{2}(\beta - \gamma).$$

In a similar manner, we obtain the following result for odd dimension.

Theorem 6.3. *Let M be a $(2n+1)$-dimensional mixed generalized quasi-Einstein manifold with associated scalars α, β, γ, δ, and associated one-forms A and B. Then*

i. $\tau(P_1^{\perp}) = \tau(P_1) + \frac{1}{2}(\alpha + \beta + \gamma)$ *for any n-dimensional subspace P_1 of T_pM orthogonal to ρ and μ,*

ii. $\tau(P_2^{\perp}) = \tau(P_2) + \frac{1}{2}(\alpha - \beta - \gamma)$ *for any n-dimensional subspace P_2 of T_pM containing ρ and μ,*

iii. $\tau(P_3^{\perp}) = \tau(P_3) + \frac{1}{2}(\alpha - \beta + \gamma)$ *for any n-dimensional subspace P_3 of T_pM containing ρ and orthogonal to μ, and*

iv. $\tau(P_4^{\perp}) = \tau(P_4) + \frac{1}{2}(\alpha + \beta - \gamma)$ *for any n-dimensional subspace P_4 of T_pM containing μ and orthogonal to ρ.*

Notice that the scalar curvatures in Theorem 6.2 and Theorem 6.3 are independent of the associated scalar δ. This leads to an assertion that the Ricci tensor of a mixed generalized quasi-Einstein manifold can actually be reduced to a form

$$S(X,Y) = \overline{\alpha}\; g(X,Y) + \overline{\beta}\; \overline{A}(X)\; \overline{A}(Y) + \overline{\gamma}\; \overline{B}(X)\; \overline{B}(Y),$$

for some scalar functions $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$, orthogonal unit vector fields $\overline{\rho}$ and $\overline{\mu}$, and 1-forms $\overline{A}$ and $\overline{B}$ given by $\overline{A}(X) = g(X,\overline{\rho})$, $\overline{B}(X) = g(X,\overline{\mu})$ for all smooth vector fields X.

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