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Some Identities on Generalized Poly-Euler and Poly-Bernoulli Polynomials

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Abstract

In this paper, more identities for generalized poly-Euler and poly-Bernoulli polynomials with three parameters are obtained in connection with the Stirling numbers of the second kind. Moreover, symmetrized generalization is introduced to establish certain duality relation.

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1 Introduction

Leonard Euler (1707-1783) introduced Euler numbers and polynomials in his desire to evaluate the alternating sum

$$A_n(m) = m^n - (m-1)^n + \ldots + (-1)^{m-1}1^n.$$

Euler numbers are usually defined as coefficients of the Taylor series expansion of the reciprocal of hyperbolic cosine function. More precisely,

$$\frac{1}{\cosh t} = \frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{1.1}$$

One can easily verify that Euler numbers with odd indices are all zero. That is,

$$E_{2n-1} = 0, \ \ \forall \ n = 1, 2, \ldots$$

On the other hand, the even-indexed ones are non-zero with alternating signs such as

$$E_0 = 1, \ E_2 = -1, \ E_4 = 5, \ E_6 = -61, \ E_8 = 1385, \ E_{10} = -50521.$$

Almost all generalizations of Euler numbers are based on the above expansion (1.1). For instance, a simple generalization in polynomial form is given by

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}$$

where $E_n(x)$ are the so-called *Euler polynomials*. Consequently, this definition gives the following identity

$$E_n(x+1) + E_n(x) = 2x^n. \tag{1.2}$$

As mentioned above, the main objective of introducing Euler polynomials is to evaluate the alternating sum $A_n(m)$. Then, adding and subtracting identity (1.2) with $x = m-1, x = m-2, \ldots, x = 1$ yields

$$A_n(m) = \frac{1}{2}\left(E_n(m+1) - (-1)^m E_n(1)\right)$$

for integers $m, n \geq 0$.

Another generalization is due to Ohno and Sasaki [17] which is defined by

$$\frac{\mathrm{Li}_k(1-e^{-4t})}{4t\cosh t} = \sum_{n=0}^{\infty} E_n^{(k)} \frac{t^n}{n!}$$

where

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}, \tag{1.3}$$

are the so-called *polylogarithm* and the numbers $E_n^{(k)}$ are called *poly-Euler numbers*. Recently, H. Jolany et al. [16] have extended these numbers in polynomial form, denoted by $E_n^{(k)}(x; a, b, c)$, which are called *generalized polyEuler polynomials*. That is,

$$\frac{2\mathrm{Li}_k(1-(ab)^{-t})}{a^{-t}+b^t} c^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x; a, b, c) \frac{t^n}{n!}.$$

These polynomials satisfy the following identities

$$E_n^{(k)}(x; a, b, c) = \sum_{i=0}^{n} \binom{n}{i} (\ln c)^{n-i} E_i^{(k)}(a, b) x^{n-i}$$

$$E_n^{(k)}(x; a, b, c) = (\ln a + \ln b)^n E_n^{(k)}\left(\frac{x\ln c + \ln a}{\ln a + \ln b}\right)$$

$$\frac{d}{dx} E_{n+1}^{(k)}(x; a, b, c) = (n+1)(\ln c) E_n^{(k)}(x; a, b, c)$$

where $E_i^{(k)}(a,b) = E_i^{(k)}(0;a,b,c)$ and $E_i^{(k)}(x) = E_i^{(k)}(x;1,e,e)$, which can be defined by

$$\frac{2\mathrm{Li}_k(1-(ab)^{-t})}{a^{-t}+b^t} = \sum_{n=0}^{\infty} E_n^{(k)}(a,b)\frac{t^n}{n!}$$

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{1+e^t}e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x)\frac{t^n}{n!}$$

(see [9, 16]).

On the other hand, Bernoulli numbers, denoted by B_n, first appeared in a famous treatise published in 1713, eight years after the death of Jacob Bernoulli (1654-1705). These numbers appeared as part of the coefficients when Bernoulli expressed the sums of powers of consecutive integer

$$s_p(n) = \sum_{k=1}^{n-1} k^p$$

as polynomial in n, which is given by

$$s_p(n) = \frac{1}{p+1}\sum_{k=0}^{p}\binom{p+1}{k}B_k\, n^{p+1-k}$$

where p and n are positive integers and $B_1 = 1/2$ is used. Bernoulli numbers also appeared in the computation of Riemann zeta function

$$\zeta(2p) = \sum_{k=1}^{\infty}\frac{1}{k^{2p}} = \frac{4^p|B_{2p}|\pi^{2p}}{2(2p)!}$$

and as coefficients in the Taylor series expansion of the hyperbolic tangent function

$$\frac{z}{e^z-1} = \sum_{k=0}^{\infty} B_k\frac{z^k}{k!}, \quad |z| < 2\pi.$$

Bernoulli numbers have also been extended in polynomial form as coefficients of the following generating function

$$\frac{ze^{xz}}{e^z-1} = \sum_{k=0}^{\infty} B_k(x)\frac{z^k}{k!}$$

where $B_k(x)$ are called the Bernoulli polynomials. These polynomials have been extended further by Kaneko [13] as coefficients, denoted by $B_n^{(k)}$, of the following series expansion

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)}\frac{x^n}{n!},$$

which are called poly-Bernoulli polynomials. Several properties of $B_n^{(k)}$ are established by Kaneko in [13] including some explicit and recursive formulas.

In this paper, more identities for generalized poly-Euler polynomials are established containing Stirling numbers of the second kind and other known combinatorial numbers. Moreover, the generalized poly-Bernoulli polynomials are defined and some necessary properties are obtained parallel to those of generalized poly-Euler polynomials.

2 Generalized Poly-Euler Polynomials

We start our discussion in this section by introducing first the involved combinatorial numbers.

Stirling numbers are defined in pair, denoted by $\left\{ \begin{bmatrix} n \\ m \end{bmatrix}, \begin{Bmatrix} n \\ m \end{Bmatrix} \right\}$, by means of the following relation

$$(x)_n = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} x^m \tag{2.1}$$

$$x^n = \sum_{m=0}^{n} \begin{Bmatrix} n \\ m \end{Bmatrix} (x)_m \tag{2.2}$$

where the numbers $\begin{bmatrix} n \\ m \end{bmatrix}$ and $\begin{Bmatrix} n \\ m \end{Bmatrix}$ are called Stirling numbers of the first and second kind, respectively, and $(x)_n$ is the well-known Pochammer symbol for the falling factorial defined by

$$(x)_n = x(x-1)(x-2)\dots(x-n+1).$$

One can easily write (2.1) as

$$x^{(n)} = \sum_{m=0}^{n} \left| \begin{bmatrix} n \\ m \end{bmatrix} \right| x^m$$

where the numbers $\left| \begin{bmatrix} n \\ m \end{bmatrix} \right| = (-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}$ are called the signless Stirling numbers of the first kind and $x^{(n)}$ is the well-known Pochammer symbol for the rising factorial defined by

$$x^{(n)} = x(x+1)(x+2)\dots(x+n-1).$$

The signless Stirling numbers of the fisrt kind $\left| \begin{bmatrix} n \\ m \end{bmatrix} \right|$ can be interpreted as the number of possible permutations of an n-set with m nonempty cycles and the Stirling numbers of the second kind $\begin{Bmatrix} n \\ m \end{Bmatrix}$ can be interpreted as the number of ways to partition an n-set into m nonempty subsets. These numbers satisfy the following exponential generating function

$$\frac{\log^m(1+t)}{m!} = \sum_{n=0}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix} \frac{t^n}{n!} \tag{2.3}$$

$$\frac{(e^t-1)^m}{m!} = \sum_{n=0}^{\infty} \begin{Bmatrix} n \\ m \end{Bmatrix} \frac{t^n}{n!}. \tag{2.4}$$

For a detailed discussion of these numbers, one may see [5].

Before we introduce results in this section, let us define first formally the generalized poly-Euler polynomials.

Definition 2.1. [16, H. Jolany et al.] For any positive numbers a, b, c and any real number x with $k \in \mathbb{Z}$ and $n \geq 0$, the generalized poly-Euler polynomials are defined by

$$\frac{2\mathrm{Li}_k(1-(ab)^{-t})}{a^{-t}+b^t} c^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x; a, b, c) \frac{t^n}{n!}. \tag{2.5}$$

Some identities on generalized poly-Euler polynomials are expressed in terms of Stirling numbers of the second kind. Such identities have appeared in Theorem 2.6 of [8] but with $c = e$. That is, $E_n^{(k)}(x; a, b) = E_n^{(k)}(x; a, b, e)$. More precisely,

$$\frac{2\mathrm{Li}_k(1-(ab)^{-t})}{a^{-t}+b^t}e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x; a, b)\frac{t^n}{n!}. \tag{2.6}$$

The said identities are given in the following theorem.

Theorem 2.2. [8] *For any positive numbers a, b and any real numbers x with $k \in \mathbb{Z}$ and $n \geq 0$, the polynomials $E_n^{(k)}(x; a, b)$ satisfy the following explicit formulas*

$$E_n^{(k)}(x; a, b) = \sum_{m=0}^{\infty}\sum_{l=m}^{n} \left\{ {l \atop m} \right\} \binom{n}{l} E_{n-l}^{(k)}(-m; a, b)x^{(m)} \tag{2.7}$$

$$E_n^{(k)}(x; a, b) = \sum_{m=0}^{\infty}\sum_{l=m}^{n} \left\{ {l \atop m} \right\} \binom{n}{l} E_{n-l}^{(k)}(0; a, b)(x)_m \tag{2.8}$$

$$E_n^{(k)}(x; a, b) = \sum_{m=0}^{\infty}\binom{n}{m}\sum_{l=0}^{n-m} \frac{\binom{n-m}{l}}{\binom{l+s}{l}} \left\{ {l+s \atop s} \right\} E_{n-m-l}^{(k)}(0; a, b)B_m^{(s)}(x) \tag{2.9}$$

$$E_n^{(k)}(x; a, b) = \sum_{m=0}^{\infty}\frac{\binom{n}{m}}{(1-\lambda)^s}\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}E_{n-m}^{(k)}(j; a, b)H_m^{(s)}(x; \lambda), \tag{2.10}$$

where

$$\left(\frac{t}{e^t-1}\right)^s e^{xt} = \sum_{n=0}^{\infty} B_n^{(s)}(x)\frac{t^n}{n!} \quad \textit{and} \quad \left(\frac{1-\lambda}{e^t-\lambda}\right)^s e^{xt} = \sum_{n=0}^{\infty} H_n^{(s)}(x; \lambda)\frac{t^n}{n!}.$$

Here, we derive some identities for $E_n^{(k)}(x; a, b, c)$ which are parallel to those in Theorem 2.2. The identities are given in the following theorem.

Theorem 2.3. *For any positive numbers a, b, c and any real numbers x with $k \in \mathbb{Z}$ and $n \geq 0$, the generalized poly-Euler polynomials satisfy the following relation*

$$E_n^{(k)}(x; a, b, c) = \sum_{m=0}^{\infty}\sum_{l=m}^{n} (\log c)^l \left\{ {l \atop m} \right\} \binom{n}{l} E_{n-l}^{(k)}(-m\log c; a, b)x^{(m)} \tag{2.11}$$

$$E_n^{(k)}(x; a, b, c) = \sum_{m=0}^{\infty}\sum_{l=m}^{n} (\log c)^l \left\{ {l \atop m} \right\} \binom{n}{l} E_{n-l}^{(k)}(0; a, b)(x)_m \tag{2.12}$$

$$E_n^{(k)}(x; a, b, c) = \sum_{m=0}^{\infty}\binom{n}{m}\sum_{l=0}^{n-m} \frac{\binom{n-m}{l}}{\binom{l+s}{l}} \left\{ {l+s \atop s} \right\} E_{n-m-l}^{(k)}(0; a, b)B_m^{(s)}(x\log c) \tag{2.13}$$

$$E_n^{(k)}(x; a, b, c) = \sum_{m=0}^{n}\frac{\binom{n}{m}}{(1-\lambda)^s}\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}E_{n-m}^{(k)}(j; a, b)H_m^{(s)}(x\log c; \lambda). \tag{2.14}$$

Proof: For relation (2.11), note that (2.5) can be written as

$$\sum_{n=0}^{\infty} E_n^{(k)}(x; a, b, c)\frac{t^n}{n!} = \frac{2\mathrm{Li}_{(k)}(1-(ab)^{-t})}{a^{-t}+b^t}(1-(1-e^{-t\log c}))^{-x}$$

Using Newton's Binomial Theorem and the exponential generating function for $\left\{ {n \atop m} \right\}$ in (2.4), we have

$$\sum_{n=0}^{\infty} E_n^{(k)}(x;a,b,c)\frac{t^n}{n!} = \frac{2\mathrm{Li}_{(k)}(1-(ab)^{-t})}{a^{-t}+b^t}\sum_{m=0}^{\infty}\binom{x+m-1}{m}(1-e^{-t\log c})^m$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} x^{(m)}\frac{(e^{t\log c}-1)^m}{m!}\frac{2\mathrm{Li}_{(k)}(1-(ab)^{-t})}{a^{-t}+b^t}e^{-mt\log c}\\
&= \sum_{m=0}^{\infty} x^{(m)}\left(\sum_{n=0}^{\infty}\left\{ {n \atop m} \right\}\frac{(t\log c)^n}{n!}\right)\left(\sum_{n=0}^{\infty}E_n^{(k)}(-m\log c;a,b)\frac{t^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{\infty}\sum_{l=m}^{n}(\log c)^l\left\{ {l \atop m} \right\}\binom{n}{l}E_{n-l}^{(k)}(-m\log c;a,b)x^{(m)}\right\}\frac{t^n}{n!}
\end{aligned}$$

Comparing coefficients completes the proof of (2.11).

For relation (2.12), we can write (2.5) as

$$\sum_{n=0}^{\infty} E_n^{(k)}(x;a,b,c)\frac{t^n}{n!} = \frac{2\mathrm{Li}_{(k)}(1-(ab)^{-t})}{a^{-t}+b^t}((e^{t\log c}-1)+1)^x$$

Using Newton's Binomial Theorem and the exponential generating function for $\left\{ {n \atop m} \right\}$ in (2.4), we have

$$\sum_{n=0}^{\infty} E_n^{(k)}(x;a,b,c)\frac{t^n}{n!} = \frac{2\mathrm{Li}_{(k)}(1-(ab)^{-t})}{a^{-t}+b^t}\sum_{m=0}^{\infty}\binom{x}{m}(e^{t\log c}-1)^m$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty}(x)_m\frac{(e^{t\log c}-1)^m}{m!}\frac{2\mathrm{Li}_{(k)}(1-(ab)^{-t})}{a^{-t}+b^t}\\
&= \sum_{m=0}^{\infty}(x)_m\left(\sum_{n=0}^{\infty}\left\{ {n \atop m} \right\}\frac{(t\log c)^n}{n!}\right)\left(\sum_{n=0}^{\infty}E_n^{(k)}(0;a,b)\frac{t^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{\infty}\sum_{l=m}^{n}(\log c)^l\left\{ {l \atop m} \right\}\binom{n}{l}E_{n-l}^{(k)}(0;a,b)(x)_m\right\}\frac{t^n}{n!}
\end{aligned}$$

Comparing coefficients completes the proof of (2.12).

For relation (2.13), equation (2.5) can be written as

$$\sum_{n=0}^{\infty} E_n^{(k)}(x;a,b,c)\frac{t^n}{n!} = \left(\frac{(e^t-1)^s}{s!}\right)\left(\frac{t^s e^{xt\log c}}{(e^t-1)^s}\right)\left(\frac{2\mathrm{Li}_{(k)}(1-(ab)^{-t})}{a^{-t}+b^t}\right)\frac{s!}{t^s}$$

$$= \left(\sum_{n=0}^{\infty} \left\{ {n+s \atop s} \right\} \frac{t^{n+s}}{(n+s)!} \right) \left(\sum_{m=0}^{\infty} B_m^{(s)}(x \log c) \frac{t^m}{m!} \right) \left(\sum_{n=0}^{\infty} E_n^{(k)}(0;a,b) \frac{t^n}{n!} \right) \frac{s!}{t^s}$$
$$= \left(\sum_{n=0}^{\infty} \left\{ {n+s \atop s} \right\} \frac{t^{n+s}}{(n+s)!} \right) \left(\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} B_m^{(s)}(x \log c) \frac{t^m}{m!} E_{n-m}^{(k)}(0;a,b) \frac{t^{n-m}}{(n-m)!} \right) \frac{s!}{t^s}$$
$$= \sum_{m=0}^{\infty} \left\{ \sum_{n=m}^{\infty} \sum_{l=0}^{n-m} \left\{ {l+s \atop s} \right\} \frac{t^{l+s}}{(l+s)!} B_m^{(s)}(x \log c) E_{n-m-l}^{(k)}(0;a,b) \frac{t^{n-m-l}}{(n-m-l)!} \frac{t^m}{m!} \frac{s!}{t^s} \right\}$$
$$= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \binom{n}{m} \sum_{l=0}^{n-m} \frac{\binom{n-m}{l}}{\binom{l+s}{s!}} \left\{ {l+s \atop s} \right\} E_{n-m-l}^{(k)}(0;a,b) B_m^{(s)}(x \log c) \right\} \frac{t^n}{n!}$$

Comparing coefficients completes the proof of (2.13).

For relation (2.14), note that (2.5) can be written as

$$\sum_{n=0}^{\infty} E_n^{(k)}(x;a,b,c) \frac{t^n}{n!} = \left(\frac{(1-\lambda)^s}{(e^t-\lambda)^s} e^{xt \log c} \right) \left(\frac{(e^t-\lambda)^s}{(1-\lambda)^s} \right) \left(\frac{2\mathrm{Li}_{(k)}(1-(ab)^{-t})}{a^{-t}+b^t} \right)$$

$$= \left(\sum_{n=0}^{\infty} H_n^{(s)}(x \log c; \lambda) \frac{t^n}{n!} \right) \left(\sum_{j=0}^{s} \binom{s}{j} (-\lambda)^{s-j} \frac{2\mathrm{Li}_{(k)}(1-(ab)^{-t})}{a^{-t}+b^t} e^{jt} \right)$$
$$= \frac{1}{(1-\lambda)^s} \sum_{j=0}^{s} \binom{s}{j} (-\lambda)^{s-j} \left(\sum_{n=0}^{\infty} H_n^{(s)}(x \log c; \lambda) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} E_n^{(k)}(j;a,b) \frac{t^n}{n!} \right)$$
$$= \frac{1}{(1-\lambda)^s} \sum_{j=0}^{s} \binom{s}{j} (-\lambda)^{s-j} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} H_m^{(s)}(x \log c; \lambda) E_{n-m}^{(k)}(j;a,b) \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{\binom{n}{m}}{(1-\lambda)^s} \sum_{j=0}^{s} \binom{s}{j} (-\lambda)^{s-j} E_{n-m}^{(k)}(j;a,b) H_m^{(s)}(x \log c; \lambda) \right) \frac{t^n}{n!}.$$

Comparing coefficients completes the proof of (2.14).

In particular, when $c = e$ (2.11), (2.12), (2.13) and (2.14) yields (2.7), (2.8), (2.9) and (2.10).

3 Generalized Poly-Bernoulli Polynomials

Parallel to the definition of generalized poly-Euler polynomials in (2.5), we have the following generalization of poly-Bernoulli numbers.

Definition 3.1. For any positive numbers a, b, c and any real numbers x with $k \in \mathbb{Z}$ and $n \geq 0$, the generalized poly-Bernoulli polynomials are defined by

$$\frac{\mathrm{Li}_{(k)}(1-(ab)^{-t})}{b^t - a^{-t}} c^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x;a,b,c) \frac{t^n}{n!}. \tag{3.1}$$

One can easily prove the following theorem using the same argument in deriving the identities in Theorem 2.3.

Theorem 3.2. *For any positive numbers a, b, c and any real numbers x with $k \in \mathbb{Z}$ and $n \geq 0$, the generalized poly-Bernoulli polynomials satisfy the following identities.*

$$B_n^{(k)}(x; a, b, c) = \sum_{m=0}^{\infty} \sum_{l=m}^{n} (\log c)^l \left\{ \begin{matrix} l \\ m \end{matrix} \right\} \binom{n}{l} B_{n-l}^{(k)}(-m \log c; a, b) x^{(m)}$$

$$B_n^{(k)}(x; a, b, c) = \sum_{m=0}^{\infty} \sum_{l=m}^{n} (\log c)^l \left\{ \begin{matrix} l \\ m \end{matrix} \right\} \binom{n}{l} B_{n-l}^{(k)}(0; a, b)(x)_m$$

$$B_n^{(k)}(x; a, b, c) = \sum_{m=0}^{\infty} \binom{n}{m} \sum_{l=0}^{n-m} \frac{\binom{n-m}{l}}{\binom{l+s}{l}} \left\{ \begin{matrix} l+s \\ s \end{matrix} \right\} B_{n-m-l}^{(k)}(0; a, b) B_m^{(s)}(x \log c)$$

$$B_n^{(k)}(x; a, b, c) = \sum_{m=0}^{n} \frac{\binom{n}{m}}{(1-\lambda)^s} \sum_{j=0}^{s} \binom{s}{j} (-\lambda)^{s-j} B_{n-m}^{(k)}(j; a, b) H_m^{(s)}(x \log c; \lambda)$$

The next theorem contains an explicit formula for $B_n^{(k)}(x; a, b, c)$.

Theorem 3.3. (**Explicit Formula**) *For any positive numbers a, b, c and any real numbers x with $k \in \mathbb{Z}$ and $n \geq 0$,, we have*

$$B_n^{(k)}(x; a, b, c) = \sum_{m>0} \frac{1}{m^k} \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} (x \ln c - j \ln a - (j+1) \ln b)^n. \tag{3.2}$$

Proof: First, we need to expand the following expression using the definition of polylogarithm in (1.3) as follows

$$\begin{aligned} \frac{\mathrm{Li}_{(k)}(1-(ab)^{-t})}{b^t - a^{-t}} &= \left(\sum_{m>0} \frac{(1-(ab)^{-t})^m}{b^t(1-(ab)^{-t}) m^k} \right) \\ &= b^{-t} \left(\sum_{m>0} \frac{(1-(ab)^{-t})^{m-1}}{m^k} \right) \end{aligned}$$

Using the Binomial Theorem, we have

$$\begin{aligned} \frac{\mathrm{Li}_{(k)}(1-(ab)^{-t})}{b^t - a^{-t}} &= b^{-t} \sum_{m>0} \frac{1}{m^k} \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} e^{-jt \ln(ab)} \\ &= \sum_{m>0} \frac{1}{m^k} \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} e^{-t(j \ln a + (j+1) \ln b)}. \end{aligned}$$

By making use of equation (3.1), we get

$$\begin{aligned} \sum_{n=0}^{\infty} B_n^{(k)}(x; a, b, c) \frac{t^n}{n!} &= \frac{\mathrm{Li}_{(k)}(1-(ab)^{-t})}{b^t - a^{-t}} e^{xt \ln c} \\ &= \sum_{m>0} \frac{1}{m^k} \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} e^{t(x \ln c - j \ln a - (j+1) \ln b)} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m>0} \frac{1}{m^k} \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} (x \ln c - j \ln a - (j+1) \ln b)^n \right) \frac{t^n}{n!} \end{aligned}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides, the proof is completed.

The next theorem contains an expression of $B_n^{(k)}(x;a,b,c)$ as polynomial in x. Before we mention the theorem, let us introduce the notation $B_i^{(k)}(a,b)$ which is equivalent to $B_i^{(k)}(0;a,b,c)$. More precisely, the numbers $B_i^{(k)}(a,b)$ are defined as coefficients of the following generating function

$$\frac{\mathrm{Li}_{(k)}(1-(ab)^{-t})}{b^t - a^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)}(a,b)\frac{t^n}{n!}.$$

Theorem 3.4. *The generalized poly-Bernoulli polynomials satisfy the following relation*

$$B_n^{(k)}(x;a,b,c) = \sum_{i=0}^{n}\binom{n}{i}(\ln c)^{n-i}B_i^{(k)}(a,b)x^{n-i} \tag{3.3}$$

Proof: Using (3.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_n^{(k)}(x;a,b,c)\frac{t^n}{n!} &= \frac{\mathrm{Li}_{(k)}(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt} = e^{xt\ln c}\sum_{n=0}^{\infty}B_n^{(k)}(a,b)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{(xt\ln c)^{n-i}}{(n-i)!}B_i^{(k)}(a,b)\frac{t^i}{i!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}(\ln c)^{n-i}B_i^{(k)}(a,b)x^{n-i}\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result.

Note that, when $a=c=e$ and $b=1$, Definition 3.1 reduces to

$$\frac{\mathrm{Li}_{(k)}(1-e^{-t})}{1-e^{-t}}e^{xt} = \sum_{n=0}^{\infty}B_n^{(k)}(x;e,1,e)\frac{t^n}{n!}. \tag{3.4}$$

where $B_n^{(k)}(x;e,1,e)$ is equivalent to $B_n^{(k)}(x)$, the poly-Bernoulli polynomials. The following theorem gives a relation between $B_n^{(k)}(x;a,b,c)$ and $B_n^{(k)}(x)$.

Theorem 3.5. *The generalized poly-Bernoulli polynomials satisfy the following relation*

$$B_n^{(k)}(x;a,b,c) = (\ln a + \ln b)^n B_n^{(k)}\left(\frac{x\ln c - \ln b}{\ln a + \ln b}\right) \tag{3.5}$$

Proof: Using (3.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}B_n^{(k)}(x;a,b,c)\frac{t^n}{n!} &= \frac{\mathrm{Li}_{(k)}(1-(ab)^{-t})}{b^t(1-(ab)^{-t})}e^{xt\ln c}\\ &= e^{\frac{x\ln c-\ln b}{\ln ab}t\ln ab}\frac{\mathrm{Li}_{(k)}(1-e^{-t\ln ab})}{1+e^{-t\ln ab}}\\ &= \sum_{n=0}^{\infty}(\ln a+\ln b)^n B_n^{(k)}\left(\frac{x\ln c-\ln b}{\ln a+\ln b}\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result.

The following theorem contains a derivative formula for generalized poly-Bernoulli polynomials.

Theorem 3.6. *The generalized poly-Bernoulli polynomials satisfy the following relation*

$$\frac{d}{dx}B_{n+1}^{(k)}(x;a,b,c) = (n+1)(\ln c)B_n^{(k)}(x;a,b,c) \tag{3.6}$$

Proof: Using (3.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{d}{dx}B_n^{(k)}(x;a,b,c)\frac{t^n}{n!} &= \frac{t(\ln c)\mathrm{Li}_{(k)}(1-(ab)^{-t})}{b^t-a^{-t}}e^{xt\ln c}\\ \sum_{n=0}^{\infty}\frac{d}{dx}B_n^{(k)}(x;a,b,c)\frac{t^{n-1}}{n!} &= \sum_{n=0}^{\infty}(\ln c)B_n^{(k)}(x;a,b,c)\frac{t^n}{n!}.\end{aligned}$$

Hence,

$$\sum_{n=0}^{\infty}\frac{1}{n+1}\frac{d}{dx}B_{n+1}^{(k)}(x;a,b,c)\frac{t^n}{n!} = \sum_{n=0}^{\infty}(\ln c)B_n^{(k)}(x;a,b,c)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result.

The following corollary immediately follows from Theorem 3.6 by taking $c = e$. For brevity, let us denote $B_n^{(k)}(x;a,b,e)$ by $B_n^{(k)}(x;a,b)$.

Corollary 3.1. *The generalized poly-Bernoulli polynomials are Appell polynomials in the sense that*

$$\frac{d}{dx}B_{n+1}^{(k)}(x;a,b) = (n+1)B_n^{(k)}(x;a,b) \tag{3.7}$$

Consequently, using the characterization of Appell polynomials [18, 19, 20], the following addition formula can easily be obtained.

Corollary 3.2. *The generalized poly-Bernoulli polynomials satisfy the following addition formula*

$$B_n^{(k)}(x+y;a,b) = \sum_{i=0}^{n}\binom{n}{i}B_i^{(k)}(x;a,b)y^{n-i} \tag{3.8}$$

However, we can derive the addition formula for $B_n^{(k)}(x;a,b,c)$ as follows

$$\begin{aligned}\sum_{n=0}^{\infty}B_n^{(k)}(x+y;a,b,c)\frac{t^n}{n!} &= \frac{\mathrm{Li}_{(k)}(1-(ab)^{-t})}{b^t-a^{-t}}c^{(x+y)t}\\ &= \frac{\mathrm{Li}_{(k)}(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt}c^{yt}\\ &= \left(\sum_{n=0}^{\infty}B_n^{(k)}(x;a,b,c)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(y\ln c)^n\frac{t^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}(y\ln c)^{n-i}B_i^{(k)}(x;a,b,c)\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields the following result.

Theorem 3.7. *The generalized poly-Bernoulli polynomials satisfy the following addition formula*

$$B_n^{(k)}(x+y;a,b,c) = \sum_{i=0}^{n} \binom{n}{i} (\ln c)^{n-i} B_i^{(k)}(x;a,b,c) y^{n-i}.$$

Note that when $x = 0$, the above addition formula yields

$$B_n^{(k)}(y;a,b,c) = \sum_{i=0}^{n} \binom{n}{i} (\ln c)^{i} B_{n-i}^{(k)}(0;a,b,c) y^{i}$$

which is exactly the formula in Theorem 3.4 that expresses $B_n^{(k)}(y;a,b,c)$ as polynomial in y with the numbers $\binom{n}{i}(\ln c)^i B_{n-i}^{(k)}(0;a,b,c)$ as coefficients.

4 Symmetrized Generalization

The symmetrized generalization of poly-Euler polynomials with parameters a, b and c, denoted by $D_n^{(m)}(x,y;a,b,c)$, has been defined in [8] as follows

$$D_n^{(m)}(x,y;a,b,c) = \frac{1}{(\ln a + \ln b)^n} \sum_{k=0}^{m} \binom{m}{k} E_n^{(-k)}(x;a,b,c) \left(\frac{y \ln c + \ln a}{\ln a + \ln b} \right)^{m-k} \tag{4.1}$$

where $m, n \geq 0$. These polynomials satisfy the following double generating function

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} D_n^{(m)}(x,y;a,b,c) \frac{t^n}{n!} \frac{u^m}{m!} = \frac{2e^{\left(\frac{y \ln c + \ln a}{\ln a + \ln b}\right)u} e^{\left(\frac{x \ln c + \ln a}{\ln a + \ln b}\right)t} e^{t+u} \left(1 - e^{-t}\right)}{(e^t+1)(e^t + e^u - e^{t+u})}, \tag{4.2}$$

and explicit formula

$$\begin{aligned} D_n^{(m)}(x,y;a,b,c) &= 2\sum_{j=0}^{\infty} (j!)^2 \left(\sum_{l=0}^{n} \sum_{i=0}^{\infty} (-1)^i \frac{\left(\ln c^x a^{i+2} b^{i+1}\right)^{n-l} - \left(\ln c^x a^{i+1} b^{i}\right)^{n-l}}{(\ln a + \ln b)^{n-l}} \times \right. \\ &\quad \left. \times \binom{n}{l} \left\{ {l \atop j} \right\} \right) \left(\sum_{r=0}^{m} \left(\frac{y \ln c + 2\ln a + \ln b}{\ln a + \ln b} \right)^{m-r} \binom{m}{r} \left\{ {r \atop j} \right\} \right). \end{aligned}$$

The main purpose of introducing symmetrized generalization is to establish certain duality relation. However, for the symmetrized generalization of poly-Euler polynomials, duality relation is not possible. In this section, we define the symmetrized generalization of poly-Bernoulli polynomials with three parameters, denoted by $C_n^{(m)}(x,y;a,b,c)$, and establish some properties parallel to that of $D_n^{(m)}(x,y;a,b,c)$ including the duality relation.

Now, let us introduce the following definition of $C_n^{(m)}(x,y;a,b,c)$.

Definition 4.1. For $m, n \geq 0$, we define

$$C_n^{(-m)}(x,y;a,b,c) = \frac{1}{(\ln a + \ln b)^n} \sum_{k=0}^{m} \binom{m}{k} B_n^{(-k)}(x;a,b,c) \left(y \ln c - \frac{\ln b}{\ln a + \ln b} \right)^{m-k}. \tag{4.3}$$

The following theorem presents the double generating function for $C_n^{(-m)}(x,y;a,b,c)$ which is parallel to the following double generating function obtained by Kaneko [13]

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B_n^{(-k)}\frac{t^n}{n!}\frac{u^k}{k!} = \frac{e^{t+u}}{e^t+e^u-e^{t+u}}. \tag{4.4}$$

Theorem 4.2. *For $m,n \geq 0$, we have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_n^{(-m)}(x,y;a,b,c)\frac{t^n}{n!}\frac{u^m}{m!} = \frac{e^{\left(x\ln c+\frac{\ln a}{\ln a+\ln b}\right)t}e^{\left(y\ln c+\frac{\ln a}{\ln a+\ln b}\right)u}}{e^t+e^u-e^{t+u}}. \tag{4.5}$$

Proof: By using the definition of $C_n^{(-m)}(x,y;a,b)$, the left-hand side can be written as

$$LHS = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(\ln a+\ln b)^n}\sum_{k=}^{m} B_n^{(-k)}(x;a,b,c)\left(y\ln c-\frac{\ln b}{\ln a+\ln b}\right)^{m-k}\frac{t^n}{n!}\frac{u^m}{k!(m-k)!}$$

By putting $l=m-k$, we get

$$\begin{aligned}
LHS &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{(\ln a+\ln b)^n}B_n^{(-k)}(x;a,b,c)\left(y\ln c-\frac{\ln b}{\ln a+\ln b}\right)^{l}\frac{t^n}{n!}\frac{u^k}{k!}\frac{u^l}{l!}\\
&= e^{\left(y\ln c-\frac{\ln b}{\ln a+\ln b}\right)u}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(\ln a+\ln b)^n}B_n^{(-k)}(x;a,b,c)\frac{t^n}{n!}\frac{u^k}{k!}\\
&= e^{\left(y\ln c-\frac{\ln b}{\ln a+\ln b}\right)u}\sum_{k=0}^{\infty}\left(e^{xt}\sum_{n=0}^{\infty}B_n^{(-k)}(a,b,c)\frac{\left(\frac{t}{\ln a+\ln b}\right)^n}{n!}\right)\frac{u^k}{k!}\\
&= e^{\left(y\ln c-\frac{\ln b}{\ln a+\ln b}\right)u}\sum_{k=0}^{\infty}\left(e^{xt\ln c}\frac{\mathrm{Li}_{-k}(1-e^{-t})}{1-e^{-t}}e^{\left(\frac{-t\ln b}{\ln a+\ln b}\right)^n}\right)\frac{u^k}{k!}\\
&= e^{\left(y\ln c-\frac{\ln b}{\ln a+\ln b}\right)u}e^{\left(x\ln c-\frac{\ln b}{\ln a+\ln b}\right)t}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}B_n^{(-k)}\frac{t^n}{n!}\frac{u^k}{k!}
\end{aligned}$$

So, by applying the double generating function in (4.4), we obtain the desired result.

As a direct consequence of Theorem 4.2, we have the following corollary containing the well known duality property.

Corollary 4.1. (**Duality Property**) *For $m \geq 0$, we have*

$$C_n^{(-m)}(x,y;a,b,c) = C_n^{(-m)}(y,x;b,a,c). \tag{4.6}$$

Now, we are ready to show a closed formula for $C_n^{(-m)}(x,y;a,b,c)$ which is important and fundamental.

Theorem 4.3. (**Closed Formula**) *For $m \geq 0$, we have*

$$\begin{aligned}
C_n^{(-m)}(x,y;a,b,c) = \sum_{j=0}^{\infty}(j!)^2\left(\sum_{p=0}^{\infty}\left(x\ln c+\frac{\ln a}{\ln a+\ln b}\right)^{n-p}\binom{n}{p}\left\{{p \atop j}\right\}\right)\times \\
\times\left(\sum_{l=0}^{\infty}\left(y\ln c+\frac{\ln b}{\ln a+\ln b}\right)^{m-l}\binom{m}{l}\left\{{l \atop j}\right\}\right)
\end{aligned} \tag{4.7}$$

Proof: By applying Theorem 4.2, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_n^{(-m)}(x,y;a,b,c)\frac{t^n}{n!}\frac{u^m}{m!} = \frac{e^{\left(x\ln c+\frac{\ln a}{\ln a+\ln b}\right)t}e^{\left(y\ln c+\frac{\ln b}{\ln a+\ln b}\right)u}}{e^t+e^u-e^{t+u}}$$
$$= \frac{e^{\left(x\ln c+\frac{\ln a}{\ln a+\ln b}\right)t}e^{\left(y\ln c+\frac{\ln b}{\ln a+\ln b}\right)u}}{1-(e^t-1)(e^u-1)}$$
$$= e^{\left(x\ln c+\frac{\ln a}{\ln a+\ln b}\right)t}e^{\left(y\ln c+\frac{\ln b}{\ln a+\ln b}\right)u}\sum_{j=0}^{\infty}(e^t-1)^j(e^u-1)^j$$
$$= \sum_{j=0}^{\infty} e^{\left(x\ln c+\frac{\ln a}{\ln a+\ln b}\right)t}(e^t-1)^j e^{\left(y\ln c+\frac{\ln b}{\ln a+\ln b}\right)u}(e^u-1)^j$$

By applying the generating function for Stirling numbers of second kind in (2.4), the right-hand side of the last expression becomes

$$= \sum_{j=0}^{\infty}\left(j!\sum_{n=0}^{\infty}\frac{\left(x\ln c+\frac{\ln a}{\ln a+\ln b}\right)^n t^n}{n!}\sum_{m=0}^{\infty}\left\{ m \atop j \right\}\frac{t^m}{m!}\right)\times$$
$$\times\left(j!\sum_{n=0}^{\infty}\frac{\left(y\ln c+\frac{\ln b}{\ln a+\ln b}\right)^n u^n}{n!}\sum_{m=0}^{\infty}\left\{ m \atop j \right\}\frac{u^m}{m!}\right)$$
$$= \sum_{j=0}^{\infty}\left(j!\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left(x\ln c+\frac{\ln a}{\ln a+\ln b}\right)^{l-m}\binom{l}{m}\left\{ m \atop j \right\}\frac{t^l}{l!}\right)\times$$
$$\times\left(j!\sum_{p=0}^{\infty}\sum_{r=0}^{p}\left(y\ln c+\frac{\ln b}{\ln a+\ln b}\right)^{p-r}\binom{p}{r}\left\{ r \atop j \right\}\frac{u^p}{p!}\right)$$
$$= \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{t^l}{l!}\frac{u^p}{p!}\sum_{j=0}^{\infty}(j!)^2\left(\sum_{m=0}^{l}\left(x\ln c+\frac{\ln a}{\ln a+\ln b}\right)^{l-m}\binom{l}{m}\left\{ m \atop j \right\}\right)\times$$
$$\times\left(\sum_{r=0}^{p}\left(y\ln c+\frac{\ln b}{\ln a+\ln b}\right)^{p-r}\binom{p}{r}\left\{ r \atop j \right\}\right)$$

which yields the desired identity.

5 Summary and Conclusion

Preliminary investigation for generalized poly-Euler and poly-Bernoulli polynomials has been done in the papers [9, 16], which provide some necessary properties for the two polynomials including explicit formulas and recurrence relations. However, there are some interesting properties and formulas that are not considered, especially, for the generalized poly-Euler and poly-Bernoulli polynomials with three parameters.

In Section 2 and the first part of Section 3, the discussion focuses on some formulas that expressed the generalized poly-Euler and poly-Bernoulli polynomials with three parameters in terms of Stirling numbers of the second kind, rising and falling factorials, and certain generalization of Bernoulli polynomials. These would be of great help in visualizing the structure of the said polynomials, particularly, in drawing their combinatorial interpretations as the Stirling numbers of the second kind as well as the rising and falling factorials are known to have combinatorial meanings.

The last part of Section 3 devotes its discussion on the expression of $B_n^{(k)}(y;a,b,c)$ as polynomial in x and their classification as Appell polynomials. The said polynomial expression is given by

$$B_n^{(k)}(x;a,b,c) = \sum_{i=0}^{n} \binom{n}{i} (\ln c)^{n-i} B_i^{(k)}(a,b) x^{n-i}.$$

Note that this polynomial can further be written as

$$\begin{aligned} B_n^{(k)}(x;a,b,c) &= \sum_{i=0}^{\infty} \binom{n}{n-i} (\ln c)^i B_{n-i}^{(k)}(a,b) x^i \\ &= \sum_{i=0}^{\infty} \frac{(n)_i}{i!} (\ln c)^i B_{n-i}^{(k)}(a,b) x^i \\ &= \sum_{i=0}^{\infty} (n)_i (\ln c)^i B_{n-i}^{(k)}(a,b) \frac{x^i}{i!}. \end{aligned}$$

Clearly, the numbers $(n)_i(\ln c)^i B_{n-i}^{(k)}(a,b)$ are coefficients of the Taylor series expansion of $B_n^{(k)}(x;a,b,c)$. This implies that

$$\left.\frac{d^i}{dx^i} B_n^{(k)}(x;a,b,c)\right|_{x=0} = (n)_i (\ln c)^i B_{n-i}^{(k)}(a,b). \tag{5.1}$$

Equation (5.1) can also be obtained by applying the recursive formula in Theorem 3.6 repeatedly. That is,

$$\begin{aligned} \frac{d^2}{dx^2} B_n^{(k)}(x;a,b,c) &= \frac{d}{dx}\left[n(\ln c) B_{n-1}^{(k)}(x;a,b,c)\right] = n(n-1)(\ln c)^2 B_{n-2}^{(k)}(x;a,b,c) \\ \frac{d^3}{dx^3} B_n^{(k)}(x;a,b,c) &= \frac{d}{dx}\left[n(n-1)(\ln c)^2 B_{n-2}^{(k)}(x;a,b,c)\right] \\ &= (n)_3 (\ln c)^3 B_{n-3}^{(k)}(x;a,b,c) \end{aligned}$$

$$\vdots$$

$$\frac{d^i}{dx^i} B_n^{(k)}(x;a,b,c) = (n)_i (\ln c)^i B_{n-i}^{(k)}(x;a,b,c). \tag{5.2}$$

Evaluating the *ith* derivative in (5.2) at $x = 0$ and using the fact that $B_{n-i}^{(k)}(0;a,b,c) = B_{n-i}^{(k)}(a,b)$, we obtain (5.1). The derivative formula in Theorem 3.6 has been used to show that the generalized poly-Bernoulli polynomials with two parameters can be classified as Appell polynomials. Moreover, an addition formula for the generalized poly-Bernoulli polynomials with two parameters has been obtained as a consequence of their being Appell polynomials. However, for three-parameter case, an addition formula is derived using different method, the generating function method.

Finally, this paper has been concluded by introducing the symmetrized generalization of $B_n^{(k)}(x;a,b,c)$, which is given by

$$C_n^{(-m)}(x,y;a,b,c) = \frac{1}{(\ln a + \ln b)^n}\sum_{k=0}^{m}\binom{m}{k}B_n^{(-k)}(x;a,b,c)\left(y\ln c - \frac{\ln b}{\ln a+\ln b}\right)^{m-k}.$$

This symmetrized generalization satisfies the following double generating function

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}C_n^{(-m)}(x,y;a,b,c)\frac{t^n}{n!}\frac{u^m}{m!} = \frac{e^{\left(x\ln c+\frac{\ln a}{\ln a+\ln b}\right)t}e^{\left(y\ln c+\frac{\ln a}{\ln a+\ln b}\right)u}}{e^t+e^u-e^{t+u}}.$$

which, consequently, yields the following duality relation

$$C_n^{(-m)}(x,y;a,b,c) = C_n^{(-m)}(y,x;b,a,c)$$

and closed formula

$$C_n^{(-m)}(x,y;a,b,c) = \sum_{j=0}^{\infty}(j!)^2\left(\sum_{p=0}^{\infty}\left(x\ln c+\frac{\ln a}{\ln a+\ln b}\right)^{n-p}\binom{n}{p}\left\{ {p \atop j} \right\}\right)\times$$
$$\times\left(\sum_{l=0}^{\infty}\left(y\ln c+\frac{\ln b}{\ln a+\ln b}\right)^{m-l}\binom{m}{l}\left\{ {l \atop j} \right\}\right).$$

The above duality relation may be considered as counterpart of the duality relation established by Kaneko [13] for poly-Bernoulli numbers.

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