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Some Approximations on the Probability of Ruin and the Inverse Ruin Function

Lu Kevin S. Ong
Institute of Mathematics
University of the Philippines Diliman
Quezon City, Philippines
lkong@math.upd.edu.ph

Abstract

Ruin theory is a branch of actuarial science that studies an insurer's vulnerability to insolvency. It uses mathematical models to describe the insurer's surplus. The insurance company uses the theory to come up with strategies on how to reduce the probability of ruin and enhance the expected profit or gains. From the insurance regulators' and policyholders' viewpoints, profit and loss are major concerns. In this study, a sequence of independent and identically distributed (*i.i.d.*) mixed exponential random variables is used to approximate the maximal aggregate loss. We then apply some theorems to approximate the probability of ruin. Numerical methods are then used to approximate the inverse ruin function. Mathematics Subject Classification:

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1 Introduction

Ruin Theory is a branch of actuarial science that studies an insurer's vulnerability to insolvency. We say that an insurance company is ruined if it can no longer give the benefit for the insured. The theory is based on mathematical models that are used to describe the insurer's surplus. It permits the derivation and calculation of many ruin-related measures and quantities.

One of these quantities is the probability of ruin. Most insurance companies use strategies on how to reduce the probability of ruin and enhance the expected profit or gains. From the insurance regulators' and policyholders' viewpoints, profit and loss are major concerns. [2]

For an insurance model, the surplus process $\{U(t); t \geq 0\}$ is given by

$$U(t) = U(0) + P(t) - S(t), \tag{1}$$

where $U(0) \geq 0$ is the insurer's initial surplus, $\{P(t); t \geq 0\}$ is the premium process, and $\{S(t); t \geq 0\}$ is the loss process. In symbols, the company is said to be ruined if $U(t) < 0$ for some $t > 0$.

We consider a classical risk model wherein the surplus process $\{U(t); t \geq 0\}$, is given by

$$U(t) = u + ct - S(t), \tag{2}$$

where $u \geq 0$ is the insurer's initial surplus, c is the rate of premium income per unit time, and $S(t)$ represents the aggregate amount of claims up to time t.

The process $S(t)$, is usually a compound Poisson process, with $S(t) = \sum_{i=1}^{N(t)} X_i$, where $\{X_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables, with $X_i > 0$ representing the amount of the i^{th} claim, and $N(t)$ follows a Poisson distribution with parameter λ. The individual claim amount is assumed to have a continuous distribution with cumulative distribution function (cdf) $P(x)$. The expected number of claims in $(0,t]$ is λt, and p_1 is the expected claim size. The premiums are paid continuously at rate $c = (1+\theta)\lambda p_1$, where $\theta > 0$ is the relative security loading, making ct the total premium income in $(0,t]$. The company is said to be ruined if $U(t) < 0$ for some $t > 0$.

In Ramsay's model [4], he came up with an algorithm to approximate the probability of ruin by using the first four moments of the claims distribution. We now use a more general model and derive the formulas needed to estimate the probability of ruin. We also include some numerical results about the inverse ruin function.

2 Methods

The maximal excess of aggregate claims over premiums loss is called the maximal aggregate loss, and it is defined as

$$L = \sup_{t \geq 0}\{S(t) - ct\}. \tag{3}$$

It is known that [1],

$$L = \sum_{k=1}^{N} L_k, \tag{4}$$

where the L_k's are *i.i.d.*-record jumps and N is a geometric random variable. The L_k's are continuous random variables with probability density function (pdf) $h(x)$ and moment generating function (mgf) $M_{L_k}(r)$ given by

$$h(x) = \frac{1 - P(x)}{p_1} \tag{5}$$

and

$$M_{L_k}(r) = \frac{1}{p_1 r}\left[M_X(r) - 1\right], \tag{6}$$

where $P(x)$ is the cdf of the claim amount random variable X_i, p_1 is the expected claim size, and $M_X(r)$ is the mgf of X_i. Also, the mgf of L is related to the mgf of X_i, as given by this theorem,

Theorem 1.

$$M_L(r) = \frac{\theta p_1 r}{1 + (1+\theta)p_1 r - M_X(r)}. \tag{7}$$

An equivalent formula is given by

$$M_L(r) = \frac{\theta}{1+\theta} + \frac{1}{1+\theta}\left\{\frac{\theta\left[M_X(r) - 1\right]}{1 + (1+\theta)p_1 r - M_X(r)}\right\}. \tag{8}$$

The probability of ruin is defined as the probability that the insurer's surplus becomes negative at some point in time, and it is given by

$$\psi(u) = Pr[U(t) < 0 | U(0) = u]. \tag{9}$$

We have [4],

$$\psi(u) = Pr[L > u] = 1 - Pr[L \leq u] = 1 - \frac{\theta}{1+\theta} \sum_{n=0}^{\infty} \left(\frac{1}{1+\theta}\right)^n H_n^*(u), \tag{10}$$

where $H_0^*(u) = 1$, if $u \geq 0$ and $H_0^*(u) = 0$, if $u < 0$,

$$H_1^*(u) = H(u) = \int_0^u h(x)dx$$

and

$$H_n^*(u) = \int_0^u H_{n-1}^*(u-x)dH(x)$$

is the n-fold convolution of $H(u)$.

One of the main problems in risk theory is the numerical evaluation of $\psi(u)$. In (10), we are able to solve $\psi(u)$ only if we know the claim amount distribution for X_i. In practice, it is uncertain to approximate the claim amounts accurately by a distribution function. Moreover, even if we are able to fit the claims into a distribution, it may be very difficult to use (10) with that distribution. So the more feasible approach in approximating $\psi(u)$ is by using the sample moments from the claims data. We are able to do this by using the mgf. By using (8), we have [1],

$$\int_0^{\infty} e^{ur} \left[-\psi'(u)\right] du = M_L(r) - \frac{\theta}{1+\theta} = \frac{1}{1+\theta} \left\{ \frac{\theta \left[M_X(r) - 1\right]}{1 + (1+\theta)p_1 r - M_X(r)} \right\}. \tag{11}$$

A mixed exponential distribution of order n, where n is a positive integer, has a pdf of the form

$$f(x) = \sum_{i=1}^{n} a_i b_i e^{-b_i x}, \tag{12}$$

where $b_i > 0$ are distinct, $a_i \neq 0$, for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} a_i = 1$. The mgf for a mixed exponential distribution is given by

$$M_X(r) = \sum_{i=1}^{n} \frac{a_i b_i}{b_i - r}. \tag{13}$$

If we substitute (13) into equation (11), and apply partial fractions decomposition, we have [1],

$$\int_0^{\infty} e^{ur} \left(-\psi'(u)\right) du = \sum_{i=1}^{n} \frac{c_i r_i}{r_i - r}, \tag{14}$$

where c_i and r_i are constants, for $i = 1, 2, \ldots, n$. And thus, the probability of ruin is

$$\psi(u) = \sum_{i=1}^{n} c_i e^{-r_i u}. \tag{15}$$

3 Results

3.1 The general model

The model that we are going to use will be similar to the model of Ramsay [4]. In his paper, he replaced the record jump random variable L_1 (which is based on the given data), by a random variable $\hat{L}_1$ for his model, where the pdf of $\hat{L}_1$ is a piecewise function consisting of a second-order mixed exponential and a gamma distribution with shape parameter of 2. We now consider the case when the pdf of $\hat{L}_1$ is a mixed exponential of order n. Let $\hat{h}(x)$ be the pdf of $\hat{L}_1$, given by

$$\hat{h}(x) = \sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} e^{-\frac{x}{\beta_i}}, \tag{16}$$

where $\beta_i > 0$ are distinct, $\alpha_i \neq 0$, and $\sum_{i=1}^{n} \alpha_i = 1$. To determine the values of the parameters β_i, where $i = 1, 2, \ldots, n$, we start by matching the moments of L_1 and $\hat{L}_1$. Let $p_j = E[X_1^j], u_j = E[L_1^j]$, and $\hat{u}_j = E[\hat{L}_1^j]$, where $j = 1, 2, \ldots$.

It can be shown that

$$u_j = \frac{p_{j+1}}{(j+1)p_1}, \qquad j = 1, 2, \ldots \tag{17}$$

and

$$\hat{u}_j = M_{\hat{L}_1}^{(j)}(0) = \sum_{i=1}^{n} j! \alpha_i \beta_i^j. \tag{18}$$

Equating u_j and $\hat{u}_j$ for $j = 1, 2, \ldots, n$, we have

$$\sum_{i=1}^{n} j! \alpha_i \beta_i^j = u_j \tag{19}$$

The problem here is to solve for the values of β_i, given the values of α_i and $u_i, i = 1, 2, \ldots, n$. For $n = 1$, we have the trivial case. That is, $\alpha_1 = 1$ and $\beta_1 = u_1$, because the pdf of $\hat{L}_1$ is exponential. For $n = 2$, we have [4]

$$\beta_1 = u_1 \left(1 - \frac{\sqrt{q(v-1)}}{2p}\right), \quad \beta_2 = u_1 \left(1 + \frac{p}{q} \frac{\sqrt{q(v-1)}}{2p}\right) \tag{20}$$

where $\{p, q\} = \{\alpha_1, \alpha_2\}$, and $v = \left(u_2 - u_1^2\right)/u_1^2$.

For $n \geq 3$, it is very difficult to express β_i in terms of α_i and u_i, $i = 1, 2, \ldots, n$ explicitly. Also, some assumptions on the α_i are needed so that the resulting values of β_i are real numbers.

3.2 Derivation of formulas needed to estimate the probability of ruin

It can be shown that

$$M_X(r) = p_1 r \left(\sum_{i=1}^{n} \frac{\alpha_i}{1 - \beta_i r}\right) + 1. \tag{21}$$

Substituting (21) in the right hand side of (11), then

$$\int_0^\infty e^{ur}\left(-\psi'(u)\right)du = \frac{\theta}{1+\theta}\left\{\frac{\left(\sum_{i=1}^n \dfrac{\alpha_i}{1-\beta_i r}\right)}{(1+\theta)-\left(\sum_{i=1}^n \dfrac{\alpha_i}{1-\beta_i r}\right)}\right\} \tag{22}$$

where $\hat{u}_1 = \sum_{i=1}^{n} \alpha_i\beta_i$, from (18).

Now, we define the k-th elementary symmetric sum of a set A.

Definition 2. *Let A be a set of n numbers. Let k be a natural number, where $1 \leq k \leq n$. The k-th elementary symmetric sum of A, denoted by $S_k(A)$, is the sum of all products of k distinct numbers in A.*

We want to simplify the right hand side of (22) for any given natural number n. We are able to do this by using the definition above to observe the pattern for $n = 1, 2, 3$.

Theorem 3. *Let $A = \{\beta_1, \beta_2, \ldots, \beta_n\}$ and $A_k = A - \{\beta_k\}$. Then*

$$\int_0^\infty e^{ur}\left(-\psi'(u)\right)du = \frac{\theta}{1+\theta}\left[\frac{V(r)}{W(r)}\right] \tag{23}$$

where $V(r) = \sum_{i=0}^{n-1} v_i r^i$, $W(r) = \sum_{i=0}^{n} w_i r^i$ and the values of the coefficients v_i and w_i are given by: $v_0 = 1$, $v_1 = \hat{u}_1 - S_1(A)$, and

$$v_i = (-1)^i \sum_{k=1}^{n} \left(\alpha_k S_i(A_k)\right)$$

for $i = 2, \ldots, n-1$. Moreover, we have $w_0 = \theta$, $w_1 = -\theta S_1(A) - \hat{u}_1$,

$$w_i = (-1)^i \left((1+\theta)S_i(A) - \sum_{k=1}^{n} \alpha_k S_i(A_k)\right),$$

for $i = 2, \ldots, n-1$, and $w_n = (-1)^n (1+\theta) S_n(A)$.

With these values of v_i and w_i substituted to equation (23), we obtain $V(r)$ and $W(r)$. Equating the right hand side of equation (23) with the right hand side of equation (14), we obtain the values of c_i and r_i. Finally, we apply equation (15) to solve for the probability of ruin.

3.3 Numerical Results on the inverse ruin function

In Ramsay's paper [4], he derived an algorithm wherein: the inputs are the initial surplus u, the relative security loading θ, and the first four moments of the claim distribution p_1, p_2, p_3, p_4, while the output is the probability of ruin ψ. We implemented his algorithm using Matlab®.

In practice, actuaries are interested in the reserve level (contingency surplus) u_ϕ for a given probability of ruin ϕ. That is, if $\psi(u_\phi) = \phi$, then we want to solve for $u_\phi = \psi^{-1}(\phi)$.

We call $\psi^{-1}(\phi)$ the inverse ruin function. We are able to compute u_ϕ by using numerical methods such as the Newton-Rhapson method [3]. This method is used to look for the zeros of a function $g(x)$.

We now define the function $g(u)$ as the difference between the approximated probability of ruin derived from our general model (given an initial surplus u) and the actual probability of ruin ϕ. That is,

$$g(u) = \sum_{i=1}^{n} c_i e^{-r_i u} - \phi$$

where ϕ is a constant, specifically the input for the probability of ruin.

We also have

$$g'(u) = \sum_{i=1}^{n} -c_i r_i e^{-r_i u}$$

We then apply the Newton-Rhapson method to find the zero of $g(u)$, which gives the desired initial surplus u_ϕ. An initial estimate of $u_\phi = 100$ was used for the Newton-Rhapson method. Small changes in the initial estimate of u_ϕ did not affect the output for the program. A comparison of the probability of ruin and the results for the inverse ruin calculated using Matlab® are shown in Table 1. Note that the percentage error $= |u - u_\phi|/u < 0.001$ for $u \neq 0$.

4 Conclusion

In this study, we used a mixed exponential distribution of order n for the maximal aggregate loss to derive the probability of ruin.The main constraint with our result was that the values of the parameters are required to be real numbers. Polynomials of degree n only have n real solutions if the discriminant is greater than zero, and this constraint limited the values of our parameters. As the value of n becomes larger, it is more difficult to write the expressions explicitly. One possible extension for this study is to apply numerical methods on our results. Numerical methods were implemented using a Matlab® program to approximate the inverse ruin function for Ramsay's model. The results showed errors smaller than 0.1%.

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Table 1: Comparison between probability of ruin and inverse ruin calculated numerically. Note that u_ϕ is close to u, the difference being less than 0.1%.

Probability of ruin			Inverse ruin
u	θ	$\psi(u,\theta)$	$u_\phi(\theta,\psi)$
0	0.1	0.9091	−0.0002
0	0.2	0.8333	−0.0010
0	0.3	0.7692	0.0010
0	0.4	0.7143	−0.0005
0	0.5	0.6667	−0.0014
10	0.1	0.5945	10.0009
10	0.2	0.5616	10
10	0.3	0.5312	9.9986
10	0.4	0.5033	9.9977
10	0.5	0.4777	10.0020
20	0.1	0.3992	19.9988
20	0.2	0.3945	20.0018
20	0.3	0.3858	20.0034
20	0.4	0.3751	19.9974
20	0.5	0.3634	19.9946
30	0.1	0.2778	29.9996
30	0.2	0.2915	30.0054
30	0.3	0.2967	29.9969
30	0.4	0.2966	30.0015
30	0.5	0.2934	29.9952
40	0.1	0.2023	39.9973
40	0.2	0.2278	40.0050
40	0.3	0.2416	39.9913
40	0.4	0.2480	40.0010
40	0.5	0.2499	39.9856
50	0.1	0.1552	50.0070
50	0.2	0.1882	49.9868
50	0.3	0.2071	49.9944
50	0.4	0.2173	50.0178
50	0.5	0.2221	50.0044

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