MATIMYÁS MATEMATIKA Journal of the Mathematical Society of the Philippines
ISSN 0115-6926 Vol. 38 Nos. 1-2 (2015) pp. 51-75

A Lie algebra related to the universal Askey-Wilson algebra

Rafael Reno S. Cantuba
Mathematics Department
De La Salle University Manila
Taft Ave., Manila, Philippines
rafael_ cantuba@dlsu.edu.ph

Abstract

Let $\mathbb{F}$ denote an algebraically closed field. Denote the three-element set by $\mathcal{X} = \{A, B, C\}$, and let $\mathbb{F}\langle\mathcal{X}\rangle$ denote the free unital associative $\mathbb{F}$-algebra on $\mathcal{X}$. Fix a nonzero $q \in \mathbb{F}$ such that $q^4 \neq 1$. The universal Askey-Wilson algebra Δ is the quotient space $\mathbb{F}\langle\mathcal{X}\rangle/\mathbb{I}$, where $\mathbb{I}$ is the two-sided ideal of $\mathbb{F}\langle\mathcal{X}\rangle$ generated by the nine elements $UV - VU$, where U is one of A, B, C, and V is one of

$$(q+q^{-1})A + \frac{qBC - q^{-1}CB}{q - q^{-1}},$$

$$(q+q^{-1})B + \frac{qCA - q^{-1}AC}{q - q^{-1}},$$

$$(q+q^{-1})C + \frac{qAB - q^{-1}BA}{q - q^{-1}}.$$

Turn $\mathbb{F}\langle\mathcal{X}\rangle$ into a Lie algebra with Lie bracket $[X, Y] = XY - YX$ for all $X, Y \in \mathbb{F}\langle\mathcal{X}\rangle$. Let $\mathcal{L}$ denote the Lie subalgebra of $\mathbb{F}\langle\mathcal{X}\rangle$ generated by $\mathcal{X}$, which is also the free Lie algebra on $\mathcal{X}$. Let L denote the Lie subalgebra of Δ generated by A, B, C. Since the given set of defining relations of Δ are not in $\mathcal{L}$, it is natural to conjecture that L is freely generated by A, B, C. We give an answer in the negative by showing that the kernel of the canonical map $\mathbb{F}\langle\mathcal{X}\rangle \to \Delta$ has a nonzero intersection with $\mathcal{L}$. Denote the span of all Hall basis elements of $\mathcal{L}$ of length n by $\mathcal{L}_n$, and denote the image of $\sum_{i=1}^{n} \mathcal{L}_i$ under the canonical map $\mathcal{L} \to L$ by L_n. We show that the simplest nontrivial Lie algebra relations on L occur in L_5. We exhibit a basis for L_4, and we also exhibit a basis for L_5 if q is not a sixth root of unity.

Mathematics Subject Classification: 17B60, 16S15, 33D80
Keywords: universal Askey-Wilson algebra; Hall basis; Lie algebra relations.

1 Introduction

Let $\mathbb{F}$ be an algebraically closed field and fix a nonzero $q \in \mathbb{F}$ such that $q^4 \neq 1$. Given $a, b, c \in \mathbb{F}$, the *Askey-Wilson algebra* with parameters a, b, c is the unital associative $\mathbb{F}$-algebra $AW := AW_q(a, b, c)$ defined as having generators A, B, C and relations

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \frac{a}{q + q^{-1}},$$

$$B+\frac{qCA-q^{-1}AC}{q^2-q^{-2}}=\frac{b}{q+q^{-1}},$$
$$C+\frac{qAB-q^{-1}BA}{q^2-q^{-2}}=\frac{c}{q+q^{-1}}.$$

The algebra AW was introduced in [8] in order to desrcibe the Askey-Wilson polynomials [2]. A wide range of applications of the Askey-Wilson algebra is discussed in [7, Section 1]. These applications include integrable systems, quantum mechanics, the theory of quadratic algebras, Leonard pairs and Leonard triples, and quantum groups. A central extension of the Askey-Wilson algebra AW is introduced in [7], which is called the *universal Askey-Wilson algebra.*

Definition 1 ([7, Definition 1.2])**.** *The* universal Askey-Wilson algebra *is the unital associative* $\mathbb{F}$*-algebra, which we denote by* Δ*, defined as having generators* A, B, C*, and relations which assert that the following are central in* Δ*:*

$$A+\frac{qBC-q^{-1}CB}{q^2-q^{-2}}, \tag{1}$$
$$B+\frac{qCA-q^{-1}AC}{q^2-q^{-2}}, \tag{2}$$
$$C+\frac{qAB-q^{-1}BA}{q^2-q^{-2}}, \tag{3}$$

where q *is a nonzero scalar that is not a fourth root of unity.*

Our main object of study is the Lie subalgebra L of Δ generated by A, B, C. We show that a set of defining relations for Δ cannot be expressed in terms of Lie algebra operations only, and yet this does not imply that L is freely generated by A, B, C. Denote the free unital associative $\mathbb{F}$-algebra on the three-element set $\mathcal{X} = \{A, B, C\}$ by $\mathbb{F}\langle\mathcal{X}\rangle$, and the free Lie algebra on $\mathcal{X}$ by $\mathcal{L}$. Recall that $\mathcal{L}$ is the Lie subalgebra of $\mathbb{F}\langle\mathcal{X}\rangle$ generated by A, B, C. We use the basis of $\mathcal{L}$ which was introduced by Hall [5]. Let us call the images of the Hall basis elements under the canonical map $\mathcal{L} \to L$ as the standard Lie monomials of L. We show that the kernel of the canonical map $\mathbb{F}\langle\mathcal{X}\rangle \to \Delta$ has a nonzero intersection with $\mathcal{L}$. The generators A, B, C are the standard Lie monomials of length 1. The standard Lie monomials of lengths ≥ 1 are constructed according to some rules, which we shall discuss in later sections. We show that the simplest Lie algebra relations on L occur at length 5, and we determine a maximal linearly independent set of standard Lie monomials of length at most 5.

2 Preliminaries

Let $\mathbb{F}$ be an algebraically closed field. Throughout, by an $\mathbb{F}$-algebra we mean a unital associative $\mathbb{F}$-algebra. Let $\mathfrak{A}$ be an $\mathbb{F}$-algebra. Recall that an anti-automorphism of $\mathfrak{A}$ is a bijective $\mathbb{F}$-linear map $\psi : \mathfrak{A} \to \mathfrak{A}$ such that $\psi(fg) = \psi(g)\psi(f)$ for all $f, g \in \mathfrak{A}$. We turn $\mathfrak{A}$ into a Lie algebra with Lie bracket $[f, g] = fg - gf$ for $f, g \in \mathfrak{A}$.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers. Given a nonzero $n \in \mathbb{N}$, let $\mathcal{X}$ denote an n-element set. We shall refer to any element of $\mathcal{X}$ as a *letter.* For $t \in \mathbb{N}$, by a *word of length* t on $\mathcal{X}$ we mean a sequence of the form

$$X_1X_2\cdots X_t, \tag{4}$$

where $X_i \in \mathcal{X}$ for $1 \leq i \leq t$. Given a word W on $\mathcal{X}$, denote the length of W by $|W|$. The word of length 0 will be denoted by 1. Let $\langle \mathcal{X} \rangle$ denote the set of all words on $\mathcal{X}$. Given words $X_1 X_2 \cdots X_s$ and $Y_1 Y_2 \cdots Y_t$ on $\mathcal{X}$, their *concatenation product* is

$$X_1 X_2 \cdots X_s Y_1 Y_2 \cdots Y_t.$$

We now recall the free $\mathbb{F}$-algebra $\mathbb{F}\langle \mathcal{X} \rangle$. The $\mathbb{F}$-vector space $\mathbb{F}\langle \mathcal{X} \rangle$ has basis $\langle \mathcal{X} \rangle$. Multiplication in the $\mathbb{F}$-algebra $\mathbb{F}\langle \mathcal{X} \rangle$ is the concatenation product. We endow $\mathbb{F}\langle \mathcal{X} \rangle$ with a symmetric bilinear form $(\ , \ )$ with respect to which $\langle \mathcal{X} \rangle$ is an orthonormal basis. For any $f \in \mathbb{F}\langle \mathcal{X} \rangle$ and any word W, the coefficient of W in f is (f, W).

Given $n \in \mathbb{N}$, the subspace of $\mathbb{F}\langle \mathcal{X} \rangle$ spanned by all the words of length n is the *n-homogenous component* of $\mathbb{F}\langle \mathcal{X} \rangle$. Observe that $\mathbb{F}\langle \mathcal{X} \rangle$ is the direct sum of all the n-homogenous components for $n \in \mathbb{N}$. If f is an element of the m-homogenous component and g is an element of the n-homogenous component, then fg is an element of the $(m+n)$-homogenous component. It follows that the set of all n-homogenous components of $\mathbb{F}\langle \mathcal{X} \rangle$ for all $n \in \mathbb{N}$ is a grading of $\mathbb{F}\langle \mathcal{X} \rangle$.

The following notation will be useful. Let $W = X_1 X_2 \cdots X_t$ denote a word on $\mathcal{X}$. We define W^* to be the word $X_t X_{t-1} \cdots X_1$ on $\mathcal{X}$. Let θ denote the $\mathbb{F}$-linear map

$$\begin{aligned} \theta : \mathbb{F}\langle \mathcal{X} \rangle &\rightarrow \mathbb{F}\langle \mathcal{X} \rangle, \\ W &\mapsto (-1)^{|W|} W^*, \end{aligned} \tag{5}$$

for any word W. By [6, p. 19], the map θ is the unique anti-automorphism of the $\mathbb{F}$-algebra $\mathbb{F}\langle \mathcal{X} \rangle$ that sends X to $-X$ for any letter X.

Let $\mathcal{L}$ denote the Lie subalgebra of the Lie algebra $\mathbb{F}\langle \mathcal{X} \rangle$ generated by $\mathcal{X}$. Following [6, Theorem 0.5], we call $\mathcal{L}$ the *free Lie algebra on $\mathcal{X}$*.

Proposition 2 ([6, Lemma 1.7])**.** *For $f \in \mathcal{L}$, we have $\theta(f) = -f$.*

We now recall the notion of a *Lie monomial* on $\mathcal{X}$. The set of all Lie monomials on $\mathcal{X}$ is the minimal subset of $\mathbb{F}\langle \mathcal{X} \rangle$ that contains $\mathcal{X}$ and is closed under the Lie bracket. Observe that 0 is a Lie monomial. Let U be a Lie monomial. Then U is an element of some n-homogenous component of $\mathbb{F}\langle \mathcal{X} \rangle$. We define the *length* of the Lie monomial U to be n. Observe that 0 has length n for any $n \in \mathbb{N}$. Any nonzero Lie monomial has a unique length. Observe that the set of all Lie monomials of length 1 is $\mathcal{X}$. We now consider an ordering of Lie monomials.

Definition 3 ([4, p. 581])**.** *Fix an ordering $<$ on $\mathcal{X}$. Suppose that the set of all Lie monomials of lengths $1, 2, \ldots, t-1$ have been ordered such that $U < V$ if the length of U is strictly less than that of V. If U, V both have length t, and can be written as $U = [X_1, Y_1], V = [X_2, Y_2]$, then we compare U, V using the following rules:*

1. *If $Y_1 \neq Y_2$, then $U < V$ iff $Y_1 < Y_2$.*
2. *If $Y_1 = Y_2$, then $U < V$ iff $X_1 < X_2$.*

We now introduce a basis for $\mathcal{L}$ consisting of Lie monomials.

Proposition 4 ([5, Theorem 3.1])**.** *Let $\mathbb{H}$ be the set of Lie monomials such that $\mathcal{X} \subset \mathbb{H}$, and that for any $U, V \in \mathbb{H}$, the Lie monomial $[U, V]$ is also in $\mathbb{H}$ whenever the following conditions hold.*

1. *$U > V$.*

2. *If* $U = [X, Y]$ *for some Lie monomials* X, Y*, then* $Y \leq V$.

Then $\mathbb{H}$ *is a basis for* $\mathcal{L}$*, often referred to as the* Hall basis *of* $\mathcal{L}$.

Example 5. *Suppose* $\mathcal{X} = \{A, B, C\}$ *and* $A < B < C$. *Then the elements of* $\mathbb{H}$ *of length at most* 4 *are:*

$$
\begin{aligned}
&A, B, C,\ [B, A],\ [C, A],\ [C, B],\ [[B, A], A],\ [[C, A], A],\ [[B, A], B],\\
&[[C, A], B],\ [[C, B], B],\ [[B, A], C],\ [[C, A], C],\ [[C, B], C],\\
&[[[B, A], A], A],\ [[[C, A], A], A],\ [[[B, A], A], B],\ [[[C, A], A], B],\\
&[[[B, A], B], B],\ [[[C, A], B], B],\ [[[C, B], B], B],\ [[[B, A], A], C],\\
&[[[C, A], A], C],\ [[[B, A], B], C],\ [[[C, A], B], C],\ [[[C, B], B], C],\\
&[[[B, A], C], C],\ [[[C, A], C], C],\ [[[C, B], C], C],\\
&[[C, A], [B, A]],\ [[C, B], [B, A]],\ [[C, B], [C, A]]. \qquad (6)
\end{aligned}
$$

Observe that the above Lie monomials are listed according to the ordering in Definition 3.

Given a Lie algebra $\mathfrak{L}$ and $x, y \in \mathfrak{L}$, recall the adjoint linear map

$$\text{ad } x : \mathfrak{L} \to \mathfrak{L}$$

that sends $y \mapsto [x, y]$. Denote an arbitrary word on $\mathcal{X}$ by $W = X_1 X_2 \cdots X_t$. The *Lie bracketing from left to right* is the linear map $\mathbb{F}\langle\mathcal{X}\rangle \to \mathcal{L}$ that sends $1 \mapsto 0$ and sends the word W into a Lie monomial according to the following rules:

1. If $|W| = 1$, then $W \mapsto W$.

2. Suppose that the images of all words of length $< |W|$ have been defined. Denote the image of $X_1 X_2 \cdots X_{t-1}$ by V. Then

$$W \mapsto (-\text{ad } X_t)(V) = [V, X_t].$$

That is, $X_1 X_2 \cdots X_t \mapsto [[[X_1, X_2], \cdots], X_t]$ for $t \geq 2$. A Lie monomial that is an image of some word under Lie bracketing from left to right is said to be *left-normed*.

Notation 6. *Given a word* W*, we denote the image of* W *under Lie bracketing from left to right by* $[W]$.

Example 7. *With reference to Example 5, we rewrite* (6) *using Notation 6.*

$$
\begin{aligned}
&A, B, C,\ [BA],\ [CA],\ [CB],\ [BA^2],\ [CA^2],\ [BAB],\\
&[CAB],\ [CB^2],\ [BAC],\ [CAC],\ [CBC],\\
&[BA^3],\ [CA^3],\ [BA^2B],\ [CA^2B],\\
&[BAB^2],\ [CAB^2],\ [CB^3],\ [BA^2C],\\
&[CA^2C],\ [BABC],\ [CABC],\ [CB^2C],\\
&[BAC^2],\ [CAC^2],\ [CBC^2],\\
&[[CA],[BA]],\ [[CB],[BA]],\ [[CB],[CA]]. \qquad (7)
\end{aligned}
$$

Throughout, by an *ideal* of an $\mathbb{F}$-algebra $\mathfrak{A}$ we mean a two-sided ideal of $\mathfrak{A}$. By a *Lie ideal* of a Lie algebra $\mathfrak{L}$ we mean an ideal of $\mathfrak{L}$ under the Lie algebra structure. We now recall the notion of algebras having generators and relations (i.e., having a presentation). Denote the elements of $\mathcal{X}$ by $G_1, G_2, \ldots, G_n$.

Let $f_1, f_2, \ldots, f_m \in \mathbb{F}\langle\mathcal{X}\rangle$ and let I be the ideal of $\mathbb{F}\langle\mathcal{X}\rangle$ generated by $f_1, f_2, \ldots, f_m$. We define $\mathbb{F}\langle\mathcal{X}\rangle/I$ as the $\mathbb{F}$-algebra with generators $G_1, G_2, \ldots, G_n$ and relations $f_1 = 0, f_2 = 0, \ldots, f_m = 0$. The Lie subalgebra of $\mathbb{F}\langle\mathcal{X}\rangle/I$ generated by $\mathcal{X}$ is $\mathcal{L}/(I \cap \mathcal{L})$.

Let $g_1, g_2, \ldots, g_m \in \mathcal{L}$ and let J be the Lie ideal of $\mathcal{L}$ generated by $g_1, g_2, \ldots, g_m$. We define $\mathcal{L}/J$ as the Lie algebra with generators $G_1, G_2, \ldots, G_n$ and relations $g_1 = 0, g_2 = 0, \ldots, g_m = 0$.

Suppose $\mathfrak{L}$ is a Lie algebra (over $\mathbb{F}$) generated by $\mathcal{X}$. Then there exists an ideal $\mathcal{K}$ of $\mathcal{L}$ such that $\mathfrak{L} = \mathcal{L}/\mathcal{K}$. Let $\phi : \mathcal{L} \to \mathcal{L}/\mathcal{K}$ be the canonical Lie algebra homomorphism. Then the following span $\mathfrak{L}$:

$$\phi(U), \quad \text{for } U \in \mathbb{H}. \tag{8}$$

We call (8) the *standard Lie monomials* of the Lie algebra $\mathfrak{L}$. Observe that the list of the standard Lie monomials of $\mathfrak{L}$ is identical to the list of elements of $\mathbb{H}$. This is because the Lie algebra homomorphism ϕ fixes generators. We order the list of standard Lie monomials of $\mathfrak{L}$ in a manner analogous to that given in Definition 3.

3 The universal Askey-Wilson algebra

Hereon, let $\mathbb{F}$ be an algebraically closed field, and fix a nonzero $q \in \mathbb{F}$ such that $q^4 \neq 1$. We fix $\mathcal{X} = \{A, B, C\}$. Let $\mathbb{F}\langle\mathcal{X}\rangle$ be the free associative algebra on $\mathcal{X}$. We use the ordering $A < B < C$ to construct the Hall basis $\mathbb{H}$ of the free Lie algebra $\mathcal{L}$ on $\mathcal{X}$. Define the following elements of the free algebra $\mathbb{F}\langle\mathcal{X}\rangle$.

$$\alpha := (q+q^{-1})A + \frac{qBC - q^{-1}CB}{q-q^{-1}}, \tag{9}$$

$$\beta := (q+q^{-1})B + \frac{qCA - q^{-1}AC}{q-q^{-1}}, \tag{10}$$

$$\gamma := (q+q^{-1})C + \frac{qAB - q^{-1}BA}{q-q^{-1}}. \tag{11}$$

We also define the following Lie products in $\mathbb{F}\langle\mathcal{X}\rangle$.

$$\begin{aligned} r_0 &:= [A,\alpha], & r_3 &:= [B,\alpha], & r_6 &:= [C,\alpha], \\ r_1 &:= [B,\beta], & r_4 &:= [C,\beta], & r_7 &:= [A,\beta], \\ r_2 &:= [C,\gamma], & r_5 &:= [A,\gamma], & r_8 &:= [B,\gamma]. \end{aligned}$$

Define $\mathbb{I}$ as the ideal of $\mathbb{F}\langle\mathcal{X}\rangle$ generated by $r_0, r_1, \ldots, r_8$.

With reference to Definition 1, we express Δ as a quotient space of $\mathbb{F}\langle\mathcal{X}\rangle$, and as a consequence make explicit the defining relations of Δ.

Proposition 8. $\Delta = \mathbb{F}\langle\mathcal{X}\rangle/\mathbb{I}$.

Proof: Recall Δ has relations which assert that each of (1),(2),(3) commutes with every element of Δ. Equivalently, each of (1),(2),(3) commutes with every generator A, B, C. Observe that each of α, β, γ is a scalar multiple of (1),(2),(3), respectively. Then it suffices to define Δ as having nine defining relations of the form $[X, \delta]$, where $X \in \{A, B, C\}$ and $\delta \in \{\alpha, \beta, \gamma\}$. By the definition of $\mathbb{I}$, we get the desired result. $\square$

We denote the images of α, β, γ under the canonical map $\mathbb{F}\langle\mathcal{X}\rangle \rightarrow \Delta$ by the same symbols.

Proposition 9. *$r_0, r_1, \ldots, r_8 \notin \mathcal{L}$.*

Proof: Let θ denote the $\mathbb{F}$-linear map in (5). It is routine to show that in the free algebra $\mathbb{F}\langle\mathcal{X}\rangle$, we have $\theta(r_i) + r_i \neq 0$ for $0 \leq i \leq 8$. Use Proposition 2. $\square$

By a *a word in* Δ we mean the image of an element of $\langle\mathcal{X}\rangle$ under the canonical map $\mathbb{F}\langle\mathcal{X}\rangle \rightarrow \Delta$. Observe that the list of all words in Δ is identical to the list of all the words on $\mathcal{X}$ in the free algebra $\mathbb{F}\langle\mathcal{X}\rangle$ since the canonical map $\mathbb{F}\langle\mathcal{X}\rangle \rightarrow \Delta$ is an $\mathbb{F}$-algebra homomorphism that fixes generators. We also preserve the ordering of generators $A < B < C$ in Δ. By a Δ-*word*, we mean all elements of Δ of the form

$$W\alpha^r\beta^s\gamma^t \tag{12}$$

where W is a word in Δ, and $r, s, t \in \mathbb{N}$.

We now recall some properties of Δ as studied in [7]. Let $U = X_1X_2\cdots X_t$ be a Δ-word, where X_i is either a generator of Δ or one of α, β, γ for $1 \leq i \leq t$. Without loss of generality, we assume U is of the form (12) since α, β, γ are central in Δ. By an *inversion* for W we mean an ordered pair $(j, k) \in \mathbb{N}^2$ such that $1 \leq j < k \leq t$ and $X_j, X_k \in \{A, B, C\}$ such that $X_j > X_k$. Any Δ-word with no inversions is said to be *irreducible*. For instance, $CABA$ has 4 inversions and CB^2A has 5, while the Δ-words A^2BC, AB^2C are irreducible. The shortest words for which inversions exist are BA, CA, CB and using (9) to (11), the following hold in both $\mathbb{F}\langle\mathcal{X}\rangle$ and Δ.

$$\begin{aligned} BA &= q^2AB + q(q+q^{-1})(q-q^{-1})C - q(q-q^{-1})\gamma, &(13)\\ CA &= q^{-2}AC - q^{-1}(q+q^{-1})(q-q^{-1})B + q^{-1}(q-q^{-1})\beta, &(14)\\ CB &= q^2BC + q(q+q^{-1})(q-q^{-1})A - q(q-q^{-1})\alpha. &(15)\end{aligned}$$

Consider the word $CABA$, one of the 4 inversions in which is caused by the first two letters C, A. Substituting for CA using (14), the result is a linear combination of $ACBA$, B^2A, $BA\beta$, each having fewer inversions than $CABA$.

Remark 10. *Following [7, p. 7] and [3, Theorem 1.2], for any Δ-word W, there exists a finite number of steps of substituting for inversions using* (13) *to* (15) *such that the final result is a unique linear combination of irreducible Δ-words. It follows that a basis for Δ consists of the vectors*

$$A^iB^jC^k\alpha^r\beta^s\gamma^t, \quad i, j, k, r, s, t, \in \mathbb{N}. \tag{16}$$

Given subspaces H, K of Δ, define $HK := \text{Span}\,\{hk \mid h \in H, k \in K\}$. If K is a subspace of H, we say that a subspace K' of H is a *complement of K in H* whenever

$$H = K + K'. \qquad \text{(direct sum)}$$

We now recall a filtration for Δ as given in [7, Section 5]. This filtration is a sequence $\{\Delta_n\}_{n\in\mathbb{N}}$ of subspaces of Δ defined by

$$\begin{aligned} \Delta_0 &:= \mathbb{F}1,\\ \Delta_1 &:= \Delta_0 + \text{Span}\,\{A, B, C, \alpha, \beta, \gamma\},\\ \Delta_n &:= \Delta_1\Delta_{n-1}, \quad n \geq 1,\end{aligned}$$

and has the following properties for all $i, j \in \mathbb{N}$.

$$\Delta_i \subseteq \Delta_{i+1}, \tag{17}$$
$$\Delta = \bigcup_{n\in\mathbb{N}} \Delta_n,$$
$$\Delta_i\Delta_j = \Delta_{i+j}. \tag{18}$$

Given $n \in \mathbb{N}$, a basis for Δ_n consists of the vectors

$$A^iB^jC^k\alpha^r\beta^s\gamma^t, \quad i,j,k,r,s,t,\in\mathbb{N}, \quad i+j+k+r+s+t \leq n, \tag{19}$$

while the following vectors form a basis for a complement of Δ_n in Δ_{n+1}

$$A^iB^jC^k\alpha^r\beta^s\gamma^t, \quad i,j,k,r,s,t,\in\mathbb{N}, \quad i+j+k+r+s+t = n+1. \tag{20}$$

We denote the span of the vectors (20) by Δ_n^c.

By [7, Lemma 6.1], the following elements of Δ coincide and are central.

$$qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma, \tag{21}$$
$$qBCA + q^2A^2 + q^2B^2 + q^{-2}C^2 - qA\alpha - qB\beta - q^{-1}C\gamma, \tag{22}$$
$$qCAB + q^{-2}A^2 + q^2B^2 + q^2C^2 - q^{-1}A\alpha - qB\beta - qC\gamma, \tag{23}$$
$$q^{-1}CBA + q^{-2}A^2 + q^2B^2 + q^{-2}C^2 - q^{-1}A\alpha - qB\beta - q^{-1}C\gamma, \tag{24}$$
$$q^{-1}ACB + q^{-2}A^2 + q^{-2}B^2 + q^2C^2 - q^{-1}A\alpha - q^{-1}B\beta - qC\gamma, \tag{25}$$
$$q^{-1}BAC + q^2A^2 + q^{-2}B^2 + q^{-2}C^2 - qA\alpha - q^{-1}B\beta - q^{-1}C\gamma. \tag{26}$$

Denote this element by Ω, which is called in [7] as the *Casimir element of* Δ. As shown in [7, Section 7], we have other bases for Δ, Δ_n (for $n \in \mathbb{N}$) that involve Ω. First, the following vectors form a basis for Δ.

$$A^iB^jC^k\Omega^l\alpha^r\beta^s\gamma^t, \quad i,j,k,l,r,s,t\in\mathbb{N}, \quad ijk=0. \tag{27}$$

Given $n \in \mathbb{N}$, a basis for Δ_n consists of the vectors

$$A^iB^jC^k\Omega^l\alpha^r\beta^s\gamma^t, \quad i,j,k,l,r,s,t\in\mathbb{N},\ ijk=0,\ i+j+k+3l+r+s+t \leq n, \tag{28}$$

while the following vectors form a basis for a complement of Δ_n in Δ_{n+1}.

$$A^iB^jC^k\Omega^l\alpha^r\beta^s\gamma^t, \quad i,j,k,l,r,s,t\in\mathbb{N},\ ijk=0,\ i+j+k+3l+r+s+t = n+1.$$

Recall that Δ is a Lie algebra with Lie bracket $[X,Y] := XY - YX$ for $X, Y \in \Delta$. Denote the derived algebra of Δ by $[\Delta,\Delta]$, and the ideal of Δ generated by $[\Delta,\Delta]$ by $\Delta\,[\Delta,\Delta]\,\Delta$. It follows that the Lie subalgebra of Δ generated by A, B, C is $L := \mathcal{L}/(\mathbb{I}\cap\mathcal{L})$. Given nonzero $n \in \mathbb{N}$, denote the span of all Hall basis elements of $\mathcal{L}$ of length n by $\mathcal{L}_n$. Denote the image of $\sum_{i=1}^n \mathcal{L}_i$ under the canonical map $\mathcal{L} \to L$ by L_n. It follows that all standard Lie monomials of L of length at most n span L_n.

Proposition 11. $L \subseteq \mathbb{F}A + \mathbb{F}B + \mathbb{F}C + \Delta\,[\Delta,\Delta]\,\Delta$.

Proof: By the definition of L, we have $L \subseteq \mathbb{F}A + \mathbb{F}B + \mathbb{F}C + [\Delta,\Delta]$. Since Δ has a multiplicative identity, we have $[\Delta,\Delta] \subseteq \Delta\,[\Delta,\Delta]\,\Delta$. From these we get the desired set inclusion. □

Proposition 12. *If q is not a root of unity, then L has zero center.*

Proof: Let q be not a root of unity. Suppose that $Z(L)$ has a nonzero element f. Since the generators A, B, C of Δ are also in L, we have $Z(L) \subseteq Z(\Delta)$. By [7, Corollary 8.3], $Z(\Delta)$ is generated by $\alpha, \beta, \gamma, \Omega$. Observe that there exists a filtration subspace Δ_n such that $f \in \Delta_n$, and that $\Delta_n \cap Z(\Delta)$ has a basis consisting of the vectors

$$\Omega^l \alpha^r \beta^s \gamma^t, \quad l, r, s, t \in \mathbb{N}, \quad 3l + r + s + t \leq n. \tag{29}$$

Since f is nonzero, there exists a nonzero $c \in \mathbb{F}$ and a vector $\Omega^w \alpha^x \beta^y \gamma^z$ in (29) such that

$$f - c\Omega^w \alpha^x \beta^y \gamma^z = g, \tag{30}$$

where g is a linear combination of vectors in (29) other than $\Omega^w \alpha^x \beta^y \gamma^z$. Let $\mathbb{F}\left[\bar{A}, \bar{B}, \bar{C}\right]$ denote the $\mathbb{F}$-algebra of polynomials in three mutually commuting indeterminates $\bar{A}, \bar{B}, \bar{C}$, with coefficients from $\mathbb{F}$. As shown in [7, p. 17], there exists a unique surjective $\mathbb{F}$-algebra homomorphism $\Psi : \Delta \to \mathbb{F}\left[\bar{A}, \bar{B}, \bar{C}\right]$ with kernel $\Delta\left[\Delta, \Delta\right]\Delta$ that sends

$$A \mapsto \bar{A}, \ B \mapsto \bar{B}, \ C \mapsto \bar{C}. \tag{31}$$

Under this homomorphism, denote the images of $\alpha, \beta, \gamma, \Omega$ by $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\Omega}$, respectively. As shown in [7, Lemma 11.3], we have

$$\bar{\alpha} = (q + q^{-1})\bar{A} + \bar{B}\bar{C}, \tag{32}$$
$$\bar{\beta} = (q + q^{-1})\bar{B} + \bar{C}\bar{A}, \tag{33}$$
$$\bar{\gamma} = (q + q^{-1})\bar{C} + \bar{A}\bar{B}, \tag{34}$$
$$\bar{\Omega} = (q + q^{-1})\bar{A}\bar{B}\bar{C} - \bar{A}^2 - \bar{B}^2 - \bar{C}^2. \tag{35}$$

It is routine to show that the vectors

$$\bar{A}, \bar{B}, \bar{C}, \bar{\Omega}^l \bar{\alpha}^r \bar{\beta}^s \bar{\gamma}^t, \quad l, r, s, t \in \mathbb{N}, \quad 3l + r + s + t \leq n, \tag{36}$$

are linearly independent in $\mathbb{F}\left[\bar{A}, \bar{B}, \bar{C}\right]$. Observe also that by Proposition 11,

$$f \in \mathbb{F}A + \mathbb{F}B + \mathbb{F}C + \ker \Psi.$$

Applying Ψ to both sides of (30), we have

$$c_1\bar{A} + c_2\bar{B} + c_3\bar{C} - c\bar{\Omega}^w \bar{\alpha}^x \bar{\beta}^y \bar{\gamma}^z = \bar{g}, \tag{37}$$

where $\bar{g}$ is a linear combination of the vectors in (36) except $\bar{\Omega}^w \bar{\alpha}^x \bar{\beta}^y \bar{\gamma}^z$. We get a contradiction from (37). Therefore, $Z(L) = 0$. □

We end this section by discussing some properties of Δ related to the group $PSL_2(\mathbb{Z})$. We denote by $PSL_2(\mathbb{Z})$ the free product of the cyclic group of order two and the cyclic group of order three [1]. Let ρ, σ denote the generators of $PSL_2(\mathbb{Z})$ such that $\rho^3 = 1$ and $\sigma^2 = 1$. By [7, Theorem 3.1], the group $PSL_2(\mathbb{Z})$ acts faithfully on Δ as a group of automorphisms in the following way:

u	A	B	C	α	β	γ
$\rho(u)$	B	C	A	β	γ	α
$\sigma(u)$	B	A	$C + (q - q^{-1})^{-1}[A, B]$	β	α	γ

By [7, Theorem 6.4], Ω is fixed by ρ, σ. It is routine to show that given $n \in \mathbb{N}$, the filtration subspace Δ_n is invariant under ρ.

Proposition 13. *The Lie algebra L is $PSL_2(\mathbb{Z})$-invariant.*

Proof: Let $\tau \in \{\rho, \sigma\}$. It suffices to argue in the following way. Show that the images of the generators A, B, C under τ are in L, and show that if the images of $f, g \in L$ under τ are in L, then so is the image of $[f, g]$ under τ. By the above table, we are done with the first step. For the second step, assume that the images of $f, g \in L$ under τ are in L. Since τ is an $\mathbb{F}$-algebra automorphism, we have $\tau([f, g]) = \tau(fg - gf) = [\tau(f), \tau(g)] \in L$. $\square$

4 L is not free

In this section, all computations are done in the free algebra $\mathbb{F}\langle\mathcal{X}\rangle$. Our goal is to show that $\mathbb{I} \cap \mathcal{L}$ contains a nonzero element.

Proposition 14. *In the free algebra $\mathbb{F}\langle\mathcal{X}\rangle$,*

$$\frac{[BA]}{q(q-q^{-1})} - (q+q^{-1})C = AB - \gamma, \tag{38}$$

$$-\frac{[CA]}{q^{-1}(q-q^{-1})} - (q+q^{-1})B = AC - \beta. \tag{39}$$

Proof: Use (13),(14) to get (38),(39), respectively. $\square$

Proposition 15. *In the free algebra $\mathbb{F}\langle\mathcal{X}\rangle$,*

$$\frac{[BA^2]}{q^2(q-q^{-1})^2} - \frac{(q+q^{-1})[CA]}{q(q-q^{-1})} = A^2B + (q+q^{-1})AC - A\gamma + \frac{r_5}{q(q-q^{-1})}, \tag{40}$$

$$\frac{[BAC]}{(q+q^{-1})(q-q^{-1})^2} = B^2 - A^2 + \frac{A\alpha - B\beta + r_1}{q+q^{-1}} + \frac{r_2}{q^{-1}(q+q^{-1})(q-q^{-1})} \tag{41}$$

Proof: Apply $-\text{ad}\, A$ to both sides of (38). The linear combination in the right side of the resulting equation contains ABA which can be further simplified using (13). From this, we get (40). We show (41) holds. Apply $-\text{ad}\, C$ to both sides of (38). This results to a right side that involves CAB. We substitute for CA in CAB using (14). The result involves ACB and βB in the right side, which can be further simplified using (15) and $r_1 = B\beta - \beta B$. From this, we get (41). $\square$

Definition 16. *We define the following elements of $\mathcal{L}$.*

$$H_0 := \frac{[[CB],[BA]] - [BABC]}{(q+q^{-1})^2(q-q^{-1})^2} + \frac{[BA^2]}{q-q^{-1}} - 2[CA] \tag{42}$$

$$I_0 := [H_0, [BA]] \tag{43}$$

Lemma 17. *In the free algebra $\mathbb{F}\langle\mathcal{X}\rangle$,*

$$\frac{[BA]\,\alpha}{(q+q^{-1})^2} = H_0 + \frac{r_1B - Ar_3}{(q+q^{-1})^2} - \frac{2r_5}{q+q^{-1}} + \frac{[r_2, B]}{q^{-1}(q+q^{-1})^2(q-q^{-1})}. \tag{44}$$

Proof: Apply $-\mathrm{ad}\ B$ to both sides of (41). The left side involves BA^2 which can be written uniquely as a linear combination of $A^2B, AC, A\gamma, B, \beta, r_5$ by repeated use of the relations (13),(14). We get

$$\begin{aligned}\frac{[BACB]}{q^2(q+q^{-1})^2(q-q^{-1})^3} &= A^2B + q^{-3}(q^4+1)AC - A\gamma \\ &\quad -q^{-2}(q+q^{-1})(q-q^{-1})B + q^{-2}(q-q^{-1})\beta \\ &\quad + \frac{r_5}{q(q+q^{-1})} - \frac{[BA]\,\alpha + Ar_3 - r_1B}{q^2(q+q^{-1})^2(q-q^{-1})} \\ &\quad + \frac{[r_2, B]}{q(q+q^{-1})^2(q-q^{-1})^2} \end{aligned} \tag{45}$$

In (45), eliminate $A^2B, A\gamma$ using (40), and eliminate AC, B, β using (39). In the resulting equation, express all Lie monomials in terms of Hall basis elements. From this, we get (44). □

Lemma 18. *In the free algebra $\mathbb{F}\langle\mathcal{X}\rangle$,*

$$\begin{aligned} I_0 &= [BA]\,\frac{[r_0,B]-[r_3,A]}{(q+q^{-1})^2} + \frac{2\,[r_5,[BA]]}{q+q^{-1}} + \frac{[Ar_3 - r_1B, [BA]]}{(q+q^{-1})^2} \\ &\quad - \frac{[[r_2,B],[BA]]}{q^{-1}(q+q^{-1})^2(q-q^{-1})}. \end{aligned} \tag{46}$$

Proof: Apply $-\mathrm{ad}\ [BA]$ to both sides of (44). The resulting left side is

$$\frac{[[BA]\,\alpha, [BA]]}{(q+q^{-1})^2}. \tag{47}$$

It is routine to show that

$$[BA]\,\frac{[r_0,B]-[r_3,A]}{(q+q^{-1})^2}$$

is equal to (47) using $r_0 = A\alpha - \alpha A, r_3 = B\alpha - \alpha B$. The result is (46). □

Theorem 19. *The Lie algebra $L = \mathcal{L}/(\mathbb{I}\cap\mathcal{L})$ is not freely generated by A, B, C.*

Proof: Observe that if we write I_0 in terms of Hall basis elements, we have

$$I_0 = \frac{[[[CB],[BA]],[BA]] - [[BABC],[BA]]}{(q+q^{-1})^2(q-q^{-1})^2} + \frac{\left[\left[BA^2\right],[BA]\right]}{q-q^{-1}} - 2\,[[CA],[BA]],$$

which, by the linear independence of the Hall basis elements, implies that $I_0 \neq 0$. But by (46), we further have $I_0 \in \mathbb{I}\cap\mathcal{L}$. Therefore, $\mathcal{L}/(\mathbb{I}\cap\mathcal{L}) \neq \mathcal{L}$. □

5 Properties of some standard Lie monomials of L

We discuss properties of some standard Lie monomials of L in relation to the filtration $\{\Delta_n\}_{n\in\mathbb{N}}$ of Δ.

Proposition 20. *For any $i, j \in \mathbb{N}$, the following hold in Δ.*

$$\left[BA^i\right] - q^i(q-q^{-1})^iA^iB \in \Delta_i, \tag{48}$$

$$\left[CA^i\right] - (-1)^iq^{-i}(q-q^{-1})^iA^iC \in \Delta_i, \tag{49}$$

$$\left[CB^j\right] - q^j(q-q^{-1})^jB^jC \in \Delta_j. \tag{50}$$

Proof: We show (48) holds by induction on i. The case $i=0$ is trivial. Suppose that for some $i \in \mathbb{N}$, we have

$$\left[BA^{i-1}\right] - q^{i-1}(q-q^{-1})^{i-1}A^{i-1}B \in \Delta_{i-1}. \tag{51}$$

Denote the element in (51) by X. By the properties of the filtration $\{\Delta_n\}_{n\in\mathbb{N}}$, we have $[X,A] \in \Delta_i$. Using (48), we further obtain

$$[X,A] + q^i(q-q^{-1})^i A^{i-1}\left((q+q^{-1})C - \gamma\right) = \left[BA^i\right] - q^i(q-q^{-1})^i A^i B,$$

which proves (48). The relations (49) and (50) are proven similarly. □

Proposition 21 ([7, Lemma 8.1])**.** *Let $i,j,k \in \mathbb{N}$. Then the following hold in Δ.*

$$\left[A, A^iB^jC^k\right] - \left(1-q^{2(j-k)}\right)A^{i+1}B^jC^k \in \Delta_{i+j+k}, \tag{52}$$
$$\left[B, A^iB^jC^k\right] - \left(q^{2i}-q^{2k}\right)A^iB^{j+1}C^k \in \Delta_{i+j+k}, \tag{53}$$
$$\left[C, A^iB^jC^k\right] - \left(q^{2(j-i)}-1\right)A^iB^jC^{k+1} \in \Delta_{i+j+k}. \tag{54}$$

Proposition 22. *For nonzero $i,j,k \in \mathbb{N}$, the following hold in Δ.*

$$\left[BA^iB^j\right] - (-1)^j q^i (q^{2i}-1)^j (q-q^{-1})^i A^iB^{j+1} \in \Delta_{i+j}, \tag{55}$$
$$\left[CA^iC^k\right] - (-1)^i q^{-i(2k+1)} (q^{2i}-1)^k (q-q^{-1})^i A^iC^{k+1} \in \Delta_{i+k}, \tag{56}$$
$$\left[CB^jC^k\right] - (-1)^k q^j (q^{2j}-1)^k (q-q^{-1})^j B^jC^{k+1} \in \Delta_{j+k}. \tag{57}$$

Proof: To show (55), use the relation (48), the relation (53) with k set to zero, and induction on j. The relations (56) and (57) are proven similarly. □

Proposition 23. *The complement Δ_1^c of Δ_1 in Δ_2 contains $[CAB]$ and $[BAC]$.*

Proof: Use the canonical map $\mathbb{F}\langle\mathcal{X}\rangle \to \Delta$ on (41) in order to obtain

$$\frac{[BAC]}{(q+q^{-1})(q-q^{-1})^2} = B^2 - A^2 + \frac{A\alpha - B\beta}{q+q^{-1}} \in \Delta_1^c. \tag{58}$$

Apply $-\rho^2$ to both sides of (58). We get

$$\frac{[CAB]}{(q+q^{-1})(q-q^{-1})^2} = C^2 - A^2 + \frac{A\alpha - C\gamma}{q+q^{-1}} \in \Delta_1^c. \quad \square$$

Proposition 24. *For nonzero $i,j,k \in \mathbb{N}$ with $i \geq 2$, the following hold in Δ.*

$$\left[BAB^jC^k\right] - (-1)^{j+k} q^{-j} (q^{2j}-1)^k (q-q^{-1})^{j+1} B^jC^{k-1}\Omega \in \Delta_{j+k+1}, \tag{59}$$
$$\left[CA^iB^j\right] - (-1)^{i+j} q^{1-i} (q^{2(i-1)}-1)^j (q-q^{-1})^i A^{i-1}B^{j-1}\Omega \in \Delta_{i+j}, \tag{60}$$
$$\left[CA^iBC^k\right] - (-1)^{i+1} q^{(1-i)(1+2k)} (q^{2(i-1)}-1)^{k+1} (q-q^{-1})^i A^{i-1}C^k\Omega \in \Delta_{i+k+1}. \tag{61}$$

Proof: We show (59) holds. We first consider the case $k=1$. By setting $i=1$ in (55), we get

$$\left[BAB^j\right] - (-1)^j q^{j+1} (q-q^{-1})^{j+1} AB^{j+1} \in \Delta_{j+1}. \tag{62}$$

Apply $-\mathrm{ad}\, C$ to the element in (62). We get

$$\left[BAB^jC\right] - (-1)^j q^{j+1} (q-q^{-1})^{j+1} \left[AB^{j+1}, C\right] \in \Delta_{j+2}. \tag{63}$$

In (54) set i, j, k to $1, j+1, 0$, respectively, and combine with (63). We have

$$\left[BAB^jC\right] - (-1)^{j+1}q^{j+1}(q^{2j}-1)(q-q^{-1})^{j+1}AB^{j+1}C \in \Delta_{j+2}. \tag{64}$$

Using (13), it is routine to show that

$$AB^n - q^{-2n}B^nA \in \Delta_n, \tag{65}$$

for $n \in \mathbb{N}$. Set $n = j+1$ in (65) and multiply the element by C from the right. We get

$$AB^{j+1}C - q^{-2(j+1)}B^{j+1}AC \in \Delta_{j+2}. \tag{66}$$

From (64) and (66), we get

$$\left[BAB^jC\right] - (-1)^{j+1}q^{-(j+1)}(q^{2j}-1)(q-q^{-1})^{j+1}B^{j+1}AC \in \Delta_{j+2}. \tag{67}$$

Using the fact that Ω is equal to (26), we have

$$B^{j+1}AC - qB^j\Omega \in \Delta_{j+2}. \tag{68}$$

From (67) and (68), we get

$$\left[BAB^jC\right] - (-1)^{j+1}q^{-j}(q^{2j}-1)(q-q^{-1})^{j+1}B^j\Omega \in \Delta_{j+2}, \tag{69}$$

from which we see that (59) holds for $k = 1$ and for nonzero $j \in \mathbb{N}$. Using (54),(69) and induction on k, we find that (59) holds for nonzero $j, k \in \mathbb{N}$. We now show (60) holds. Since $i \geq 2$, we can rewrite (59) changing the exponents j, k to $i-1, j$, respectively.

$$\left[BAB^{i-1}C^j\right] - (-1)^{i+j-1}q^{1-i}(q^{2(i-1)}-1)^j(q-q^{-1})^iB^{i-1}C^{j-1}\Omega \in \Delta_{i+j} \tag{70}$$

Denote the element in (70) by X. Since Δ_{i+j} is invariant under ρ, we have $-\rho^2(X) \in \Delta_{i+j}$, where $-\rho^2(X)$ is the element in (60). Thus, (60) holds for nonzero $i, j \in \mathbb{N}$. Finally, we show (61) holds. Set $j = 1$ in (60).

$$\left[CA^iB\right] - (-1)^{i+1}q^{1-i}(q^{2(i-1)}-1)(q-q^{-1})^iA^{i-1}\Omega \in \Delta_{i+1} \tag{71}$$

Since Ω is central, if we apply $-\text{ad}\ C$ to the element in (71), we get

$$\left[CA^iBC\right] - (-1)^{i+1}q^{1-i}(q^{2(i-1)}-1)(q-q^{-1})^i\left[A^{i-1}, C\right]\Omega \in \Delta_{i+2}. \tag{72}$$

From (54) we obtain

$$\left[A^{i-1}, C\right]\Omega - q^{-2(i-1)}(q^{2(i-1)}-1)A^{i-1}C\Omega \in \Delta_{i+2}. \tag{73}$$

From (71) and (73),

$$\left[CA^iBC\right] - (-1)^{i+1}q^{(1-i)\cdot 3}(q^{2(i-1)}-1)^2(q-q^{-1})^iA^{i-1}C\Omega \in \Delta_{i+2}, \tag{74}$$

from which we see that (61) holds for $k = 1$. Using (54),(74) and induction on k, we find that (61) holds for nonzero $i, k \in \mathbb{N}$ with $i \geq 2$. □

Proposition 25. *The following hold in* Δ.

$$\left[CA^2B\right] + (q-q^{-1})^3A\Omega \in \Delta_3, \tag{75}$$

$$\left[BABC\right] - (q-q^{-1})^3B\Omega \in \Delta_3, \tag{76}$$

$$\left[\left[CB\right], \left[CA\right]\right] - (q-q^{-1})^3C\Omega \in \Delta_3. \tag{77}$$

Proof: The relations (75),(76) follow from (60),(59), respectively. We show (77) holds. Let $V := -\left[BAC^2\right] + [CABC]$. By Proposition 23, we have $V \in \Delta_3$. Denote the element in (76) by X. Using the fact that Δ_3 is invariant under ρ, we have

$$\rho(X) = [CBCA] - (q - q^{-1})^3 C\Omega \in \Delta_3.$$

Using the Jacobi identity to express $[CBCA]$ in terms of standard Lie monomials, we further have

$$\rho(X) = [[CB], [CA]] - V - (q - q^{-1})^3 C\Omega \in \Delta_3,$$

and it follows that

$$[[CB], [CA]] - (q - q^{-1})^3 C\Omega = V + \rho(X) \in \Delta_3. \quad \square$$

6 The standard Lie monomials of L of length at most 4

Recall that the span of the standard Lie monomials of L of length at most n is L_n. Our goal in this section is to show that the standard Lie monomials of L of length at most 4 are linearly independent, and hence form a basis for L_4.

Proposition 26. *For nonzero $j, k \in \mathbb{N}$, the following hold in Δ.*

$$\left[BAB^j\right] - (-1)^j q^{(j+1)}(q - q^{-1})^{j+1} AB^{j+1} \in \Delta_{j+1}, \tag{78}$$
$$\left[CAC^k\right] + q^{-(k+2)}(q - q^{-1})^{k+1} AC^{k+1} \in \Delta_{k+1}, \tag{79}$$
$$\left[CBC^k\right] - (-1)^k q^{(k+1)}(q - q^{-1})^{k+1} BC^{k+1} \in \Delta_{k+1}, \tag{80}$$
$$\left[BA^2B^j\right] - (-1)^j q^{2(j+1)}(q + q^{-1})^j (q - q^{-1})^{j+2} A^2B^{j+1} \in \Delta_{j+2}, \tag{81}$$
$$\left[CA^2C^k\right] - q^{-2(k+1)}(q + q^{-1})^k (q - q^{-1})^{k+1} A^2C^{k+1} \in \Delta_{k+2}, \tag{82}$$
$$\left[CB^2C^k\right] - (-1)^k q^{2(k+1)}(q + q^{-1})^k (q - q^{-1})^{k+2} B^2C^{k+1} \in \Delta_{k+2}. \tag{83}$$

Proof: Set $i = 1, 2$ in (55) to get (78),(81). Do similarly to (56) and (57) in order to show the other relations. $\square$

Lemma 27. *Fix a nonzero $n \in \mathbb{N}$. The following vectors are linearly independent in Δ for any $i, j, k \in \mathbb{N}$ such that $1 \leq i, j, k \leq n$.*

$$1, A, B, C, \tag{84}$$
$$[CAB], [BAC], \tag{85}$$
$$\left[CA^2B\right], [BABC], [[CB], [CA]], \tag{86}$$
$$\left[BA^i\right], \left[BAB^j\right], \left[BA^2B^j\right], \tag{87}$$
$$\left[CA^i\right], \left[CAC^k\right], \left[CA^2C^k\right], \tag{88}$$
$$\left[CB^j\right], \left[CBC^k\right], \left[CB^2C^k\right]. \tag{89}$$

Proof: Fix $n \in \mathbb{N}$. It suffices to show that there exists an upper triangular transition matrix from the above vectors to a subset of the basis of Δ consisting of the vectors in (27):

$$A^iB^jC^k\Omega^l\alpha^r\beta^s\gamma^t, \quad i, j, k, l, r, s, t \in \mathbb{N}, \quad ijk = 0.$$

Let $i, j, k \in \mathbb{N}$ such that $1 \leq i, j, k \leq n$. From Propositions 20, 23, 25, 26 we have the following data:

$$[CAB] - c_1C^2 - d_1A^2 - d_2C\gamma - d_3A\alpha \in \Delta_0, \tag{90}$$

$$
\begin{aligned}
[BAC] - c_2B^2 - d_4A^2 - d_5B\beta - d_6A\alpha &\in \Delta_0, & (91)\\
\left[CA^2B\right] - c_3A\Omega &\in \Delta_3, & (92)\\
[BABC] - c_4B\Omega &\in \Delta_3, & (93)\\
[[CB],[CA]] - c_5C\Omega &\in \Delta_3, & (94)\\
\left[BA^i\right] - e_iA^iB &\in \Delta_i, & (95)\\
\left[CA^i\right] - f_iA^iC &\in \Delta_i, & (96)\\
\left[CB^j\right] - g_jB^jC &\in \Delta_j, & (97)\\
\left[BAB^j\right] - e'_jAB^{j+1} &\in \Delta_{j+1}, & (98)\\
\left[CAC^k\right] - f'_kAC^{k+1} &\in \Delta_{k+1}, & (99)\\
\left[CBC^k\right] - g'_kBC^{k+1} &\in \Delta_{k+1}, & (100)\\
\left[BA^2B^j\right] - e''_jA^2B^{j+1} &\in \Delta_{j+2}, & (101)\\
\left[CA^2C^k\right] - f''_kA^2C^{k+1} &\in \Delta_{k+2}, & (102)\\
\left[CB^2C^k\right] - g''_kB^2C^{k+1} &\in \Delta_{k+2}, & (103)
\end{aligned}
$$

where the small letters (other than i, j, k) denote scalars. Each of (92) to (103) is of the form $M - aV \in \Delta_m$, where M is a Lie monomial, $a \in \mathbb{F}$, and V is an element of the basis of Δ consisting of the vectors in (27), and $V \notin \Delta_m$. Call V the *leading term* of M. For (90),(91), define the leading terms of $[CAB]$, $[BAC]$ by C^2, B^2, respectively. Observe that no two distinct Lie monomials found in (90) to (103) have the same leading terms. This yields a transition matrix from the vectors (84) to (89) to some of the vectors in (27) such that all entries below the main diagonal are zero, and that the diagonal entries are

$$c_1, \dots, c_5, e_i, f_i, g_j, e'_j, f'_k, g'_k, e''_j, f''_k, g''_k.$$

By Propositions 20, 23, 25 and 26, all such scalars are nonzero. Hence, the transition matrix is upper triangular. □

Notation 28. *Let $\mathcal{I}_n$ denote the set consisting of all the linearly independent vectors in Lemma 27.*

Lemma 29. *Fix nonzero $m, n \in \mathbb{N}$. The vectors $X\alpha^r\beta^s\gamma^t$ are linearly independent in Δ for any $X \in \mathcal{I}_n$ and any $r, s, t \in \mathbb{N}$ such that $r + s + t \leq m$.*

Proof: The proof is similar to that of Lemma 27, but with (90) to (103) modified as follows. For each of (90) to (103), multiply the element by $\alpha^r\beta^s\gamma^t$ and add $r + s + t$ to the index of the filtration subspace. Based on these new data, an upper triangular transition matrix can be constructed. □

Notation 30. *Let $\mathcal{I}_n^m$ denote the set consisting of all the linearly independent vectors in Lemma 29. Observe that the vectors*

$$[CB]\,\gamma,\ [BA]\,\beta,\ [CA]\,\alpha,\ [CB]\,\beta,\ [CA]\,\gamma,\ [BA]\,\alpha, \tag{104}$$

are in $\mathcal{I}_3^1$. Let $\mathcal{I}^*$ denote the set obtained from $\mathcal{I}_3^1$ by replacing the vectors in (104) *by the vectors*

$$\left[CAB^2\right],\ \left[BA^2C\right],\ [CABC],\ \left[BAC^2\right],\ [[CA],[BA]],\ [[CB],[BA]]. \tag{105}$$

Proposition 31. *The following hold in Δ.*

$$\frac{[BA]\,\alpha}{(q+q^{-1})^2} = \frac{[[CB],[BA]]-[BABC]}{(q+q^{-1})^2(q-q^{-1})^2}+\frac{[BA^2]}{q-q^{-1}}-2\,[CA]\,, \tag{106}$$

$$\frac{[CB]\,\beta}{(q+q^{-1})^2} = \frac{[BAC^2]-[CABC]}{(q+q^{-1})^2(q-q^{-1})^2}+\frac{[CB^2]}{q-q^{-1}}+2\,[BA]\,, \tag{107}$$

$$\frac{[CA]\,\gamma}{(q+q^{-1})^2} = -\frac{[[CA],[BA]]+[CA^2B]}{(q+q^{-1})^2(q-q^{-1})^2}+\frac{[CAC]}{q-q^{-1}}-2\,[CB]\,, \tag{108}$$

$$\frac{[BA]\,\beta}{(q+q^{-1})^2} = \frac{[[CA],[BA]]-[BA^2C]}{(q+q^{-1})^2(q-q^{-1})^2}-\frac{[BAB]}{q-q^{-1}}+2\,[CB]\,, \tag{109}$$

$$\frac{[CB]\,\gamma}{(q+q^{-1})^2} = \frac{-[[CB],[BA]]+[BABC]-[CAB^2]}{(q+q^{-1})^2(q-q^{-1})^2}-\frac{[CBC]}{q-q^{-1}}-2\,[CA]\,, \tag{110}$$

$$\frac{[CA]\,\alpha}{(q+q^{-1})^2} = \frac{-[CABC]}{(q+q^{-1})^2(q-q^{-1})^2}-\frac{[CA^2]}{q-q^{-1}}-2\,[BA]\,. \tag{111}$$

Proof: Apply the canonical map $\mathbb{F}\langle\mathcal{X}\rangle \to \Delta$ to both sides of (44) to get (106). Apply ρ, ρ^2 to both sides of (106) to get (107),(108), respectively. To get (109), apply σ to both sides of (106). Apply ρ, ρ^2 to both sides of (109) to get (110),(111), respectively. □

Lemma 32. *The vectors in $\mathcal{I}^*$ are linearly independent in Δ.*

Proof: We use (106) to (111) to construct a transition matrix from the elements of $\mathcal{I}_3^1$ to the elements of $\mathcal{I}^*$. Denote such transition matrix by T. Order the rows of T such that the last 17 correspond to

$$[BA]\,,\ [CA]\,,\ [CB]\,,\ [BA^2]\,,\ [CA^2]\,,\ [BAB]\,, \tag{112}$$

$$[CB^2]\,,\ [CAC]\,,\ [CBC]\,,\ [CA^2B]\,,\ [BABC]\,, \tag{113}$$

$$[CAB^2]\,,\ [BA^2C]\,,\ [CABC]\,,\ [BAC^2]\,,\ [[CA],[BA]]\,,\ [[CB],[BA]]\,, \tag{114}$$

in that order, while order the columns of T such that the last 17 correspond to the vectors in (112),(113) together with

$$[CB]\,\gamma,\ [BA]\,\beta, [CA]\,\alpha,\ [CB]\,\beta,\ [CA]\,\gamma,\ [BA]\,\alpha. \tag{115}$$

Observe that all the vectors in $\mathcal{I}_3^1$ to be replaced to form $\mathcal{I}^*$ are in (115), all the replacements are in (114), and all the other vectors that appear in (106) to (111) (which we use to construct the transition matrix) appear in (112),(113). Then T is of the form

$$T = \left[\begin{array}{c|c} I & M \\ \hline 0 & T' \end{array}\right]$$

where I is an identity matrix, M is some matrix with 6 columns, and T' is a 6×6 matrix which has the following properties. All diagonal entries of T' are nonzero. Denote the ij-entry of T' by T'_{ij}. All entries of T' below the main diagonal and all entries in the first two columns are zero except the ones that appear below:

$$\frac{1}{(q-q^{-1})^2} = -T'_{11} = -T'_{61} = -T'_{22} = T'_{52} \neq 0.$$

By these observations about T', we find that T is invertible. This implies that the vectors in $\mathcal{I}^*$ are linearly independent. □

Theorem 33. *The standard Lie monomials of L of length at most 4 form a basis for L_4.*

Proof: All such vectors are in $\mathcal{I}^*$. Use Lemma 32 and the fact that the vectors in the statement span L_4. □

7 The standard Lie monomials of L of length at most 5

In this section, we show the Lie algebra relations that hold in L_5. We also exhibit a basis for L if q is not a sixth root of unity.

Lemma 34. *The following hold in Δ.*

$$[CA]\,\beta+[BA]\,\gamma = \frac{-\left[\left[BA^2\right],[CA]\right]+\left[\left[CA^2\right],[BA]\right]}{(q-q^{-1})^3}-\frac{\left[CA^2C\right]+\left[BA^2B\right]}{(q-q^{-1})^2} -\frac{(q+q^{-1})^2\,([BAC]-[CAB])}{q-q^{-1}} \tag{116}$$

$$[CB]\,\alpha-[BA]\,\gamma = \frac{-[[BAB],[CB]]+\left[\left[CB^2\right],[BA]\right]}{(q-q^{-1})^3}-\frac{\left[CB^2C\right]-\left[BA^2B\right]}{(q-q^{-1})^2} -\frac{(q+q^{-1})^2\,[CAB]}{q-q^{-1}} \tag{117}$$

Proof: In view of Remark 10, write each of the left and right sides of (116) as a linear combination of irreducible Δ-words. This yields the same linear combination of the basis vectors (16) of Δ. Apply ρ to both sides of (116) to get (117). □

Theorem 35. *The following relations hold in L.*

$$\frac{\left[BA^2BC\right]}{q-q^{-1}} = -\frac{(2q^2+1)(q^2+2)\left(\left[\left[BA^2\right],[CB]\right]+[[BAB],[CA]]\right)}{2q^2(q+q^{-1})^2(q-q^{-1})} -\frac{(q^4+3q^2+1)\,([[BAC],[BA]]-2\,[[CAB],[BA]])}{2q^2(q+q^{-1})^2(q-q^{-1})} -\left[BAC^2\right]+2\,[CABC]-\left[BAB^2\right]+\left[BA^3\right], \tag{118}$$

$$\frac{\left[\left[CB^2\right],[CA]\right]}{2(q+q^{-1})^2(q-q^{-1})} = \frac{-(3q^4+5q^2+3)\,[[BAC],[CB]]}{2q^2(q+q^{-1})^2(q-q^{-1})} +\frac{(q^4+3q^2+1)\,[[CAB],[CB]]}{2q^2(q+q^{-1})^2(q-q^{-1})} +\frac{(2q^4+3q^2+2)\,[[CBC],[BA]]}{2q^2(q+q^{-1})^2(q-q^{-1})} -\frac{\left[BABC^2\right]-\left[CAB^2C\right]}{q-q^{-1}} +\left[CBC^2\right]+\left[BA^2C\right]-\left[CB^3\right]+\left[CA^2B\right], \tag{119}$$

$$\frac{\left[\left[CA^2\right],[CB]\right]}{2(q+q^{-1})^2(q-q^{-1})} = \frac{\left(q^4+3q^2+1\right)(-2\,[[BAC],[CA]]+[[CAB],[CA]])}{2q^2(q+q^{-1})^2(q-q^{-1})} +\frac{(2q^2+1)(q^2+2)\,[[CAC],[BA]]}{2q^2(q+q^{-1})^2(q-q^{-1})}$$

$$
\begin{aligned}
& +\frac{\left[CA^2BC\right]}{q-q^{-1}} - 2\left[[CB],[BA]\right] - \left[CAC^2\right] \\
& +2\left[BABC\right] - \left[CAB^2\right] + \left[CA^3\right], \qquad (120)\\
[[CAC],[CB]] \;=\; & -[[BAB],[CB]] + [[CBC],[CA]] - \left[\left[BA^2\right],[CA]\right] \\
& +\left[\left[CB^2\right],[BA]\right] + \left[\left[CA^2\right],[BA]\right]. \qquad (121)
\end{aligned}
$$

Before we prove Theorem 35, we first discuss a manner of computations using the relations of Δ that will be useful in the proof. Using (9),(10),(11), we have

$$AB - \gamma = \frac{[BA]}{q(q-q^{-1})} - (q+q^{-1})C, \qquad (122)$$

$$AC - \beta = -\frac{q\,[CA]}{(q-q^{-1})} - (q+q^{-1})B, \qquad (123)$$

$$BC - \alpha = \frac{[CB]}{q(q-q^{-1})} - (q+q^{-1})A. \qquad (124)$$

Consider (122). Apply $-\text{ad}\ A$ on both sides. The resulting equation has a right side that includes $\left[BA^2\right]$ in the linear combination and a left side that is $ABA - A^2B$. Simplify $ABA - A^2B$ using (13). The resulting equation has a left side which is a linear combination of the basis (28) of Δ_n for $n = 2$, while the right side is a linear combination of standard Lie monomials. We use this procedure for each of (122),(123),(124) until we obtain one equation for each standard Lie monomial of length 3. The results are:

$$A^2B = \frac{\left[BA^2\right]}{q^2(q-q^{-1})^2} + (q+q^{-1})\,[CA] + A\gamma + (q+q^{-1})^2B - (q+q^{-1})\beta, \qquad (125)$$

$$A^2C = \frac{q^2\left[CA^2\right]}{(q-q^{-1})^2} + (q+q^{-1})\,[BA] + A\beta + (q+q^{-1})^2C - (q+q^{-1})\gamma, \qquad (126)$$

$$AB^2 = \frac{-\,[BAB]}{q^2(q-q^{-1})^2} - (q+q^{-1})\,[CB] + B\gamma + (q+q^{-1})^2A - (q+q^{-1})\alpha, \qquad (127)$$

$$C^2 - A^2 = \frac{[CAB]}{(q+q^{-1})(q-q^{-1})^2} + \frac{C\gamma}{q+q^{-1}} - \frac{A\alpha}{q+q^{-1}}, \qquad (128)$$

$$B^2C = \frac{\left[CB^2\right]}{q^2(q-q^{-1})^2} - (q+q^{-1})\,[BA] + B\alpha + (q+q^{-1})^2C - (q+q^{-1})\gamma, \qquad (129)$$

$$B^2 - A^2 = \frac{[BAC]}{(q+q^{-1})(q-q^{-1})^2} + \frac{B\beta}{q+q^{-1}} - \frac{A\alpha}{q+q^{-1}}, \qquad (130)$$

$$AC^2 = \frac{-q^2\,[CAC]}{(q-q^{-1})^2} + (q+q^{-1})\,[CB] + C\beta + (q+q^{-1})^2A - (q+q^{-1})\alpha, \qquad (131)$$

$$BC^2 = \frac{-\,[CBC]}{q^2(q-q^{-1})^2} + (q+q^{-1})\,[CA] + C\alpha + (q+q^{-1})^2B - (q+q^{-1})\beta. \qquad (132)$$

Observe that the left sides are expressed in terms of the basis (28) of Δ_n for $n = 3$. The right sides are expressed in terms of elements of $\mathcal{I}_3^1$. We can continue this process of obtaining equations of the said form such that there is one equation for each standard Lie monomial of lengths 4 and 5. We simply apply the appropriate map $-\text{ad}\ X$ where $X \in \{A, B, C\}$ to both sides of each equation from (125) to (132). Use the relations (13) to (15) and those which arise from the equality of each of (21) to (26) to Ω, with the goal of writing the left side as a linear combination of elements of some filtration subspace, and the right side as a linear combination of elements of $\mathcal{I}_n^m$ for some $m, n \in \mathbb{N}$. These steps will yield the equations for length 4. If we do the procedures again to all equations obtained thus far, we get those equations for length 5. We will not show all such equations here, only the ones needed in order to prove (133). Denote these equations by $X_i = Y_i$ for $1 \leq i \leq 3$ where the left sides are

$$\begin{aligned} X_1 &= A^3B - A^2\gamma, \\ X_2 &= A^2B^2 + \frac{(q^4+1)\Omega}{q^2} - (q^4+1)C^2 - 2B^2 - \frac{(q^4+1)A^2}{q^4}, \\ X_3 &= AB^3 - B^2\gamma, \end{aligned}$$

and the right sides for $1 \leq i \leq 3$ are

$$\begin{aligned} Y_1 &= \frac{\left[BA^3\right]}{q^3(q-q^{-1})^3} - \frac{(q^6+1)\left[CA^2\right]}{q^3(q-q^{-1})^2} - \frac{(q^2-2)(q+q^{-1})^2\left[BA\right]}{q(q-q^{-1})} \\ &\quad -(q+q^{-1})^3C - (q+q^{-1})A\beta + (q+q^{-1})^2\gamma, \\ Y_2 &= \frac{-\left[BA^2B\right]}{q^4(q+q^{-1})(q-q^{-1})^3} - \frac{\left[CAB\right]}{q^2(q-q^{-1})} + \frac{(q^2+2)\left[BA\right]\gamma}{q^2(q+q^{-1})(q-q^{-1})} \\ &\quad -\frac{(q^4+q^2+2)C\gamma}{q} - (q+q^{-1})B\beta - \frac{(q^4+1)A\alpha}{q^3} + \gamma^2, \\ Y_3 &= \frac{\left[BAB^2\right]}{q^3(q-q^{-1})^3} - \frac{(q^6+1)\left[CB^2\right]}{q^3(q-q^{-1})^2} - \frac{(q^2-2)(q+q^{-1})^2\left[BA\right]}{q(q-q^{-1})} \\ &\quad -(q+q^{-1})^3C - (q+q^{-1})B\alpha + (q+q^{-1})^2\gamma. \end{aligned}$$

We are now ready to prove Theorem 35.

Proof of Theorem 35. First, we claim that the following relation holds in Δ.

$$\begin{aligned} \frac{\left[BA^2BC\right]}{(q+q^{-1})^2(q-q^{-1})^2} &= \frac{-\left[BAC^2\right] + 2\left[CABC\right]}{q-q^{-1}} \\ &\quad + \frac{(q^4+1)\left(\left[BAB^2\right] - \left[BA^3\right]\right)}{q^2(q-q^{-1})} \\ &\quad - \frac{2(q^6-1)\left(\left[CB^2\right] - \left[CA^2\right]\right)}{q^3(q-q^{-1})} \\ &\quad + \frac{\left[BAC\right]\gamma}{(q+q^{-1})^2} + \left[BAB\right]\beta + \left[BA^2\right]\alpha. \end{aligned} \tag{133}$$

Once we have established our claim, we argue as follows. To show (118), apply $-\text{ad } A$, $-\text{ad } B$, $-\text{ad } B$, $-\text{ad } C$ to both sides of (106),(108),(109),(110), respectively. Write all Lie monomials in standard form. We get

$$\left[BA^2\right]\alpha = f_1, \tag{134}$$

$$\left[CA^2\right]\beta = f_2, \tag{135}$$

$$\left[CAB\right]\gamma = f_3, \tag{136}$$

$$\left[BAC\right]\gamma + k\left[CAB\right]\gamma = f_4, \tag{137}$$

for some $k \in \mathbb{F}$ and some $f_1, f_2, f_3, f_4 \in L$. Eliminate $\left[BA^2\right]\alpha, \left[CA^2\right]\beta, \left[BAC\right]\gamma$ in (133) using (134) to (137). The result is (118). Apply ρ, ρ^2 to both sides of (118) in order to obtain (119),(120), respectively. We now show (121) holds. Add (116) and (117). We get

$$[CB]\,\alpha + [CA]\,\beta = g_1 + g_2, \tag{138}$$

where g_1, g_2 are the right sides of (116),(117), respectively. Apply $-\rho$ to both sides of (117). We get

$$[CB]\,\alpha + [CA]\,\beta = g_3, \tag{139}$$

for some $g_3 \in L$ such that $[[CAC],[CB]]$ appears with nonzero coefficient in g_3. Eliminate $[CB]\,\alpha + [CA]\,\beta$ in (138),(139). We get (121) as desired. This completes the proof of the theorem. We now prove our claim. We start with the equation $X_2 = Y_2$. Use (13) to express A^2B^2 in X_2 as a linear combination that involves $ABAB$. The resulting linear combination in the left side involves ACB, which we eliminate using the fact that Ω is equal to (25). This yields a linear combination in the left side that involves $AB\gamma$. To eliminate $AB\gamma$ at this stage, use the equation which arises from multiplying both sides of (122) by γ. Apply $-\mathrm{ad}\, C$ on both sides of the resulting equation, and multiply both sides by q^2. At this point, the linear combination in the left side involves $ABABC, CABAB$. Consider $ABABC$. Since Ω is equal to (21), we can write $ABABC$ as a linear combination of

$$AB\Omega, ABA^2, AB^3, ABC^2, ABA\alpha, AB^2\beta, ABC\gamma.$$

Similarly, since Ω is equal to (23), we find that $CABAB$ is a linear combination of

$$AB\Omega, A^3B, B^2AB, C^2AB, A^2B\alpha, BAB\beta, CAB\gamma.$$

When we write $ABABC, CABAB$ in terms of such linear combinations, the coefficient of $AB\Omega$ vanishes. The new linear combination in the left side contains the following vectors.

$$ABA^2,\ ABA\alpha,\ B^2AB,\ BAB\beta, \tag{140}$$

$$ABC^2,\ C^2AB,\ (ABC - CAB)\gamma. \tag{141}$$

Use (13) to write each of (140) into a linear combination that involves

$$BA^3,\ A^2B\alpha,\ B^3A,\ AB^2\beta,$$

respectively. As for the vectors in (141), use the fact that Ω is equal to each of (21),(23) in order to eliminate $(ABC - CAB)\gamma$, and in order to write ABC^2 as a linear combination of $C\Omega, A^2C, B^2C, C^3, AC\alpha, BC\beta, C^2\gamma$, and in order to write C^2AB as a linear combination of $C\Omega, CA^2, CB^2, C^3, CA\alpha, CB\beta, C^2\gamma$. After all these steps, call the resulting equation $X_4 = Y_4$. Observe that the following appear in the linear combination for X_4.

$$A^3B,\ AB^3,\ A^2C,\ B^2C, \tag{142}$$

$$(B^2 - A^2)\gamma,\ A^2B\alpha,\ AB^2\beta,\ AC\alpha,\ BC\beta, \tag{143}$$

$$BA^3,\ B^3A, \tag{144}$$

$$CA^2,\ CB^2, \tag{145}$$

$$CA\alpha,\ CB\beta. \tag{146}$$

Consider A^3B. We use the equation $X_1 = Y_1$ to eliminate A^3B in X_7. The equations needed in order to eliminate the other vectors in (142) are $X_3 = Y_3$ and (126),(129). To eliminate $(B^2 - A^2)\gamma$ in X_4, use the equation which arises from multiplying both sides of (130) by

γ. We do similarly for the other vectors in (143), by using the appropriate equation from (125) to (132), which we multiply both sides by the appropriate $\delta \in \{\alpha, \beta, \gamma\}$. To eliminate BA^3 in X_4, use the equation which is the result of applying σ to both sides of $X_3 = Y_3$. We keep in mind that after the application of σ (or any other element of $PSL_2(\mathbb{Z})$ which we shall use in the succeeding steps), all Lie monomials are to be expressed in standard form. Do similarly to $X_1 = Y_1$, and use the result to eliminate B^3A. We do similarly for (145), but we use ρ^2 instead of σ to the equations (127),(131). Apply ρ^2 to both sides of each of (122),(123), and multiply both sides by α, β, respectively. This yields equations that we can use to eliminate (146) in X_4. Recall that in all these eliminations, all Lie monomials in the equations used must be written in standard form. When all the vectors in (142) to (146) have been eliminated, the result is (133). This completes the proof of the claim. □

At this point we have shown that each of

$$\left[BA^2BC\right], \left[\left[CB^2\right], [CA]\right], \left[\left[CA^2\right], [CB]\right], [[CAC], [CB]], \tag{147}$$

is linearly dependent on standard Lie monomials of length at most 5 that are not in (147). In what follows, we shall show that the standard Lie monomials of length at most 5 except (147) are linearly independent in Δ.

Proposition 36. *The following hold in* Δ.

$$\left[CA^3B\right] - (q+q^{-1})(q-q^{-1})^4A^2\Omega \in \Delta_4, \tag{148}$$
$$\left[CA^2B^2\right] - q(q-q^{-1})^4AB\Omega \in \Delta_4, \tag{149}$$
$$\left[CA^2BC\right] + q^{-1}(q-q^{-1})^4AC\Omega \in \Delta_4, \tag{150}$$
$$\left[BAB^2C\right] + (q+q^{-1})(q-q^{-1})^4B^2\Omega \in \Delta_4, \tag{151}$$
$$\left[BABC^2\right] + q(q-q^{-1})^4BC\Omega \in \Delta_4, \tag{152}$$
$$[[CBC], [CA]] + (q+q^{-1})(q-q^{-1})^4C^2\Omega \in \Delta_4. \tag{153}$$

Proof: Use Proposition 24 to show (148) to (152) hold. To show (153), set $i = 2$ in (49). We get

$$\left[CA^2\right] - q^{-2}(q-q^{-1})^2A^2C \in \Delta_2. \tag{154}$$

Applying $-\rho^2$ to the element in (154) and using the fact that Δ_2 is invariant under ρ, we have

$$[CBC] + q^{-2}(q-q^{-1})^2C^2B \in \Delta_2. \tag{155}$$

Set $i = 1$ in (49). We get

$$[CA] + q^{-1}(q-q^{-1})AC \in \Delta_1. \tag{156}$$

By taking the Lie bracket of the elements in (155) and (156), we have

$$[[CBC], [CA]] - q^{-3}(q-q^{-1})^3\left(C^2BAC - AC^3B\right) \in \Delta_4. \tag{157}$$

Using (14), it is routine to show that

$$C^2A - q^{-4}AC^2 \in \Delta_2,$$

from which we obtain

$$(q-q^{-1})^3\left(qC^2ACB - q^{-3}AC^3B\right) \in \Delta_4. \tag{158}$$

Combining (157) and (158), we obtain

$$[[CBC], [CA]] - (q - q^{-1})^3 C^2 \left(q^{-3} BAC - qACB\right) \in \Delta_4. \tag{159}$$

Using the fact that (25) and (26) are both equal to Ω, we have

$$(q + q^{-1})(q - q^{-1})\Omega + q^{-3} BAC - qACB \in \Delta_2. \tag{160}$$

Finally, we get (153) from (159) and (160). □

Proposition 37. *For nonzero $j, k \in \mathbb{N}$, the following hold in Δ.*

$$\left[BA^3B^j\right] - (-1)^j q^3 (q^6 - 1)^j (q - q^{-1})^3 A^3 B^{j+1} \in \Delta_{j+3}, \tag{161}$$

$$\left[CA^3C^k\right] + q^{-3(2k+1)} (q^6 - 1)^k (q - q^{-1})^3 A^3 C^{k+1} \in \Delta_{k+3}, \tag{162}$$

$$\left[CB^3C^k\right] - (-1)^k q^3 (q^6 - 1)^k (q - q^{-1})^3 B^3 C^{k+1} \in \Delta_{k+3}. \tag{163}$$

Proof: Use Proposition 22. □

Lemma 38. *Assume q is not a sixth root of unity. Fix a nonzero $n \in \mathbb{N}$. The following vectors are linearly independent in Δ for any $i, j, k \in \mathbb{N}$ such that $1 \leq i, j, k \leq n$.*

$$1, A, B, C, \tag{164}$$

$$[CAB], [BAC], \tag{165}$$

$$\left[CA^2B\right], [BABC], [[CB], [CA]], \tag{166}$$

$$\left[CA^3B\right], \left[CA^2B^2\right], \left[CA^2BC\right], \tag{167}$$

$$\left[BAB^2C\right], \left[BABC^2\right], [[CBC], [CA]], \tag{168}$$

$$\left[BA^i\right], \left[BAB^j\right], \left[BA^2B^j\right], \left[BA^3B^j\right], \tag{169}$$

$$\left[CA^i\right], \left[CAC^k\right], \left[CA^2C^k\right], \left[CA^3C^k\right], \tag{170}$$

$$\left[CB^j\right], \left[CBC^k\right], \left[CB^2C^k\right], \left[CB^3C^k\right]. \tag{171}$$

Proof: The proof is similar to that of Lemma 27. In order to construct the desired upper triangular transition matrix, we combine the data from (90) to (103) to that in (148) to (153), and (161) to (163). Recall that the transition matrix that can be constructed from the data is upper triangular if the scalar coefficients of the leading terms are nonzero. Those in (90) to (103) are nonzero as shown in the proof of Lemma 27. The scalar coefficients in (148) to (153) are nonzero by the manner q is defined. Finally, the scalar coefficients in (161) to (163) are nonzero since q is further assumed to be not a sixth root of unity. □

Notation 39. *Let $\mathcal{J}_n$ denote the set consisting of all the linearly independent vectors in Lemma 38.*

Lemma 40. *Assume q is not a sixth root of unity. Fix nonzero $m, n \in \mathbb{N}$. The vectors $Y\alpha^r\beta^s\gamma^t$ are linearly independent in Δ for any $Y \in \mathcal{J}_n$ and any $r, s, t \in \mathbb{N}$ such that $r + s + t \leq m$.*

Proof: The proof is similar to that of Lemma 27. For each of the data used in the proof of Lemma 38, which are (90) to (103), (148) to (153), and (161) to (163), multiply the element by $\alpha^r\beta^s\gamma^t$ and add $r + s + t$ to the index of the filtration subspace. Use these new data to construct a similar upper triangular transition matrix. □

Notation 41. *Let $\mathcal{J}_n^m$ denote the set consisting of all the linearly independent vectors in Lemma 40. Observe that the vectors*

$$[CB]\,\gamma,\ [BA]\,\beta,\ [CA]\,\alpha,\ [CB]\,\beta,\ [CA]\,\gamma,\ [BA]\,\alpha, \tag{172}$$

are in $\mathcal{J}_4^1$. Let $\mathcal{J}^*$ denote the set obtained from $\mathcal{J}_4^1$ by replacing the vectors in (172) *by the following vectors*

$$\left[CAB^2\right],\ \left[BA^2C\right],\ [CABC],\ \left[BAC^2\right],\ \ [[CA],[BA]],\ \ [[CB],[BA]]. \tag{173}$$

Lemma 42. *Assume q is not a sixth root of unity. The vectors in $\mathcal{J}^*$ are linearly independent in Δ.*

Proof: The proof is similar to that of Lemma 32. □

Proposition 43. *Assume q is not a sixth root of unity. If V is a subspace of* Span $\mathcal{J}^*$ *such that*

$$\text{Span } \mathcal{J}^* = V + \text{Span } \mathcal{J}_4^0, \qquad (\textit{direct sum})$$

then a basis for V is $\mathcal{J}^*\backslash\mathcal{J}_4^0$.

Proof: This follows from the fact that $\mathcal{J}^*, \mathcal{J}_4^0$ are both linearly independent sets and that $\mathcal{J}_4^0 \subset \mathcal{J}^*$. □

Lemma 44. *The following hold in Δ.*

$$[CAC]\,\alpha + \frac{\left[CABC^2\right]}{(q-q^{-1})^2} \quad \in \quad \text{Span } \mathcal{J}_4^0, \tag{174}$$

$$[CBC]\,\beta - \frac{\left[BAC^3\right]}{(q-q^{-1})^2} + \frac{\left[CABC^2\right]}{(q-q^{-1})^2} \quad \in \quad \text{Span } \mathcal{J}_4^0, \tag{175}$$

$$\left[BA^2\right]\gamma - \frac{(q^6-1)\left[\left[BA^2\right],[BA]\right]}{q^3(q-q^{-1})^3} \quad \in \quad \text{Span } \mathcal{J}_4^0, \tag{176}$$

$$[BAB]\,\gamma + \frac{(q+q^{-1})^2\,[[BAB],[BA]]}{(q-q^{-1})^2} \quad \in \quad \text{Span } \mathcal{J}_4^0, \tag{177}$$

$$\left[CA^2\right]\beta - \frac{(q^6-1)\left[\left[CA^2\right],[CA]\right]}{q^3(q-q^{-1})^3} \quad \in \quad \text{Span } \mathcal{J}_4^0, \tag{178}$$

$$[CAC]\,\beta + \frac{(q+q^{-1})^2\,[[CAC],[CA]]}{(q-q^{-1})^2} \quad \in \quad \text{Span } \mathcal{J}_4^0, \tag{179}$$

$$\left[CB^2\right]\alpha - \frac{(q^6-1)\left[\left[CB^2\right],[CB]\right]}{q^3(q-q^{-1})^3} \quad \in \quad \text{Span } \mathcal{J}_4^0, \tag{180}$$

$$[CBC]\,\alpha + \frac{(q+q^{-1})^2\,[[CBC],[CB]]}{(q-q^{-1})^2} \quad \in \quad \text{Span } \mathcal{J}_4^0. \tag{181}$$

Proof: Apply $-$ad C to both sides of (111),(107) and write all Lie monomials in standard form in order to get (174),(175), respectively. To get (176),(177), we first show that the equation

$$\frac{\left[BA^2\right]\gamma}{2(q+q^{-1})^2} \quad = \quad \frac{(q^6-1)\left[\left[BA^2\right],[BA]\right]}{2q^3(q+q^{-1})^2(q-q^{-1})^3} - \frac{\left[BA^3B\right]}{2(q+q^{-1})^2(q-q^{-1})^2}$$

$$-\frac{[[CA],[BA]]+\left[BA^2C\right]}{2(q-q^{-1})}+[CAC]-[BAB] \tag{182}$$

holds in Δ. In view of Remark 10, write each of the left and right sides of (182) as a linear combination of irreducible Δ-words. This yields the same linear combination of the basis vectors (16) of Δ. This proves (182), from which (176) follows. To prove (177), apply σ to both sides of (182). We obtain

$$\begin{aligned}\frac{[BAB]\,\gamma}{2(q+q^{-1})^2} &= \frac{-[[BAB],[BA]]}{2(q-q^{-1})^2}-\frac{\left[BA^2B^2\right]}{2(q+q^{-1})^2(q-q^{-1})^2} \\ &\quad+\frac{[[CB],[BA]]+[BABC]}{2(q-q^{-1})}-[CBC]-\left[BA^2\right],\end{aligned} \tag{183}$$

from which (177) follows. Finally, to prove (178) to (181), apply ρ,ρ^2 to both sides of (182),(183). □

Notation 45. *Observe that the vectors*

$$[CAC]\,\alpha,\ [CBC]\,\beta,\ \left[BA^2\right]\gamma,\ [BAB]\,\gamma, \tag{184}$$

$$\left[CA^2\right]\beta,\ [CAC]\,\beta,\ \left[CB^2\right]\alpha,\ [CBC]\,\alpha, \tag{185}$$

are in $\mathcal{J}^*$. Let $\mathcal{K}_0$ denote the set obtained from $\mathcal{J}^*$ by replacing the vectors in (184),(185) *by the vectors*

$$\left[CABC^2\right],\ \left[BAC^3\right],\ \left[\left[BA^2\right],[BA]\right],\ [[BAB],[BA]], \tag{186}$$

$$\left[\left[CA^2\right],[CA]\right],\ [[CAC],[CA]],\ \left[\left[CB^2\right],[CB]\right],\ [[CBC],[CB]]. \tag{187}$$

Lemma 46. *Assume q is not a sixth root of unity. The vectors in $\mathcal{K}_0$ are linearly independent in Δ.*

Proof: We use Lemma 44 in order to obtain a transition matrix from the vectors in $\mathcal{J}^*$ into the vectors in $\mathcal{K}_0$. Denote each of (174) to (181) by

$$M_i+f_i\in \text{Span}\ \mathcal{J}_4^0,$$

where $1\le i\le 8$, the vector M_i is an element of $\mathcal{J}^*$ that is to be replaced in order to form $\mathcal{K}_0$, while the standard Lie monomials that appear in f_i are the replacements. Using the usual ordering of standard Lie monomials, define $\overline{M}_i$ as the largest standard Lie monomial in f_i. Let $j,k\in\mathbb{N}$, with $1\le j,k\le 8$. Observe that if $M_j\ne M_k$ then $\overline{M}_j\ne\overline{M}_k$. Observe also that $M_i\in\mathcal{J}^*\backslash\mathcal{J}_4^0$ for $1\le i\le 8$. By Proposition 43, the coefficient of M_i in f_i is -1 for all i. By these observations, it follows that there exists an upper triangular transition matrix from the vectors in $\mathcal{J}^*$ into the vectors in $\mathcal{K}_0$ with nonzero diagonal entries. These diagonal entries are precisely the scalar coefficients of $\overline{M}_i$ for all i. These coefficients are all nonzero since q is assumed to be not a sixth root of unity. Since the vectors in $\mathcal{J}^*$ are linearly independent, the existence of a transition matrix just described implies that the vectors in $\mathcal{K}_0$ are also linearly independent. □

Theorem 47. *Assume q is not a sixth root of unity. The standard Lie monomials of L of length at most* 5 *except the vectors from* (147) *form a basis for L_5.*

Proof: Let $\mathcal{K}$ denote the set obtained from $\mathcal{K}_0$ by replacing the vectors

$$[BAC]\,\gamma,\ [CAB]\,\gamma,\ [BAB]\,\beta,\ \left[BA^2\right]\alpha, \tag{188}$$

$$\left[BA^2\right]\beta,\ \left[CA^2\right]\gamma,\ [CA]\beta, \tag{189}$$
$$[BAC]\beta,\ [CAC]\gamma,\ \left[CA^2\right]\alpha,\ [CAB]\beta, \tag{190}$$
$$\left[CB^2\right]\gamma,\ [CB]\alpha,\ [BAB]\alpha, \tag{191}$$
$$[CAB]\alpha,\ \left[CB^2\right]\beta,\ [BAC]\alpha,\ [CBC]\gamma, \tag{192}$$

by the vectors

$$[[CAB],[BA]],[[BAC],[BA]],[[BAB],[CA]],\left[\left[BA^2\right],[CB]\right], \tag{193}$$
$$\left[BA^3C\right],\left[\left[CA^2\right],[BA]\right],\left[\left[BA^2\right],[CA]\right], \tag{194}$$
$$\left[BA^2C^2\right],[[CAC],[BA]],[[CAB],[CA]],[[BAC],[CA]], \tag{195}$$
$$\left[CAB^3\right],\left[\left[CB^2\right],[BA]\right],[[BAB],[CB]], \tag{196}$$
$$\left[CAB^2C\right],[[CBC],[BA]],[[CAB],[CB]],[[BAC],[CB]]. \tag{197}$$

We claim that $\mathcal{K}$ is linearly independent. Observe that all vectors mentioned in the statement of the theorem are in $\mathcal{K}$. By Theorem 35, the vectors in (147) are linearly dependent on these vectors. The result follows. We now prove our claim. We construct five more sets in a manner similar to the construction of $\mathcal{J}^*$ from $\mathcal{J}_4^1$ and to that of $\mathcal{K}_0$ from $\mathcal{J}^*$. The goal is that at each step, we prove that the constructed set is linearly independent. Let $\mathcal{K}_1$ denote the set obtained from $\mathcal{K}_0$ by replacing the vectors (188) in $\mathcal{K}_0$ by the vectors (193). Let $\mathcal{K}_2$ denote the set obtained from $\mathcal{K}_1$ in a similar manner until $\mathcal{K}_5$, which is obtained from $\mathcal{K}_4$ by replacing the vectors (192) in $\mathcal{K}_4$ by the vectors (197). Observe that $\mathcal{K}_5 = \mathcal{K}$. We show that each of $\mathcal{K}_1, \ldots, \mathcal{K}_5$ is a linearly independent set in Δ. Apply $-\mathrm{ad}\ A, -\mathrm{ad}\ B, -\mathrm{ad}\ B, -\mathrm{ad}\ A$ to both sides of (110),(108),(109),(106), respectively. Write all Lie monomials in standard form. We get

$$[BAC]\gamma + k[CAB]\gamma = f_1, \tag{198}$$
$$[CAB]\gamma = f_2, \tag{199}$$
$$[BAB]\beta = f_3, \tag{200}$$
$$\left[BA^2\right]\alpha = f_4, \tag{201}$$

for some $k \in \mathbb{F}$ and some $f_1, f_2, f_3, f_4 \in L$. Eliminate $\left[BA^2BC\right]$ in (198),(200),(201) using the relation (118). Solve the resulting system in order to obtain

$$[BAC]\gamma = g_1, \tag{202}$$
$$[CAB]\gamma = g_2, \tag{203}$$
$$[BAB]\beta = g_3, \tag{204}$$
$$\left[BA^2\right]\alpha = g_4, \tag{205}$$

where each of $g_1, \ldots, g_4 \in L$, is a linear combination of

$$[[CAB],[BA]],[[BAC],[BA]],[[BAB],[CA]],\left[\left[BA^2\right],[CB]\right],$$

together with the vectors in $\mathcal{K}_0$. We use the equations (202) to (205) to construct a transtion matrix T_1 from the vectors in $\mathcal{K}_0$ into those in $\mathcal{K}_1$. Index the columns of T_1 such that the last 4 correspond to (188), while index the rows such that the last 4 correspond to (193). We find that T_1 has the form

$$T_1 = \left[\begin{array}{c|c} I_1 & U_1 \\ \hline 0 & L_1 \end{array}\right]$$

where I_1 is an identity matrix, U_1 is some matrix with 4 columns, and L_1 is a 4×4 matrix, with

$$\det L_1 = -\frac{1}{2(q+q^{-1})^2(q-q^{-1})^8} \neq 0,$$

which implies that $\det T_1 \neq 0$. Thus, $\mathcal{K}_1$ is linearly independent. For $2 \leq i \leq 5$, we can also construct a transtion matrix T_i from the vectors in $\mathcal{K}_{i-1}$ into those in $\mathcal{K}_i$ in a similar manner. The equations that can be used to construct T_i are also derived from (106) to (111) with the application of the appropriate map ad X where $X \in \{A, B, C\}$. Furthermore, we find that T_i can be partitioned into four matrices similar to T_1, and that if we denote the bottom right partition as L_i, then

$$\begin{aligned} \frac{1}{(q-q^{-1})^7} &= -\det L_2 = \det L_4 \neq 0, \\ \frac{1}{(q-q^{-1})^8} &= -\det L_3 = -\det L_5 \neq 0, \end{aligned}$$

which imply that T_i is invertible for $2 \leq i \leq 5$. Therefore, $\mathcal{K}_5 = \mathcal{K}$ is a linearly independent set in Δ. □

8 Acknowledgements

This work was done while the author is a graduate student in De La Salle University, Manila, Philippines, and in part while he was an Honorary Fellow (Sept. to Dec. 2015) in the University of Wisconsin - Madison. He extends his thanks to his mentors, Prof. Arlene Pascasio and Prof. Paul Terwilliger. This research is supported by the Science Education Institute of the Department of Science and Techonology (DOST-SEI), Republic of the Philippines. The author extends his gratitude to the anonymous referees whose comments greatly improved this work.

References

[1] R. Alperin, $PSL_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$, *Amer. Math. Monthly* **100** (1993) 385-386.

[2] R. Askey, J. Wilson, Some basic hypergeometric polynomials that generalize Jacobi polynomials, *Mem. Amer. Math. Soc.* **54** (1985) 319.

[3] G. Bergman, The diamond lemma for ring theory, *Adv. Math.* **29** (1978) 178-218.

[4] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, Stratified Lie groups and potential theory for their sub-Laplacians, *Springer-Verlag*, 2007.

[5] M. Hall, A basis for free Lie rings and higher commutators in free groups. *Proc. Amer. Math. Soc.* **1** (1950) 575-581.

[6] C. Reutenauer, Free Lie algebras, *Oxford Univ. Press, New York,* 1993.

[7] P. Terwilliger, The universal Askey-Wilson algebra, *SIGMA* **7** (2011) 069, 24 pages, arXiv:1104.2813.

[8] A. Zhedanov, "Hidden symmetry" of the Askey-Wilson polynomials, *Theoret. and Math. Phys.* **89** (1991) 1146-1157.