MATIMYÁS MATEMATIKA
ISSN 0115-6926

Journal of the Mathematical Society of the Philippines
Vol. 38 Nos. 1-2 (2015) pp. 35-42

Subgroups as dominating sets for a Cayley graph of the dicyclic group

JOSE MARIA P. BALMACEDA
Institute of Mathematics
University of the Philippines Diliman
Quezon City, Philippines
joey@math.upd.edu.ph

JORIS N. BULORON
Department of Mathematics
Cebu Normal University
Cebu City, Philippines
jorisbuloron@yahoo.com

CARMELITA M. LOQUIAS
Department of Mathematics
University of San Carlos
Cebu City, Philippines
lilithloquias@yahoo.com

Abstract

A dominating set for a graph $\Gamma(V, E)$ with vertex set V and edge set E, is a subset D of V such that for every $v \in V \backslash D$ there exists $w \in D$ with $\{v, w\} \in E$. We determine all subgroups of the dicyclic group $Dic_n = \langle a, x : a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1} \rangle$ of order $4n, n > 1$, that form dominating sets for the Cayley graph of Dic_n with respect to a minimal symmetric generating set. We also give some results on efficient domination in the graphs considered.

Mathematics Subject Classification: 05C25, 05C69
Keywords: Cayley graph, dicyclic group, dominating set, efficient domination.

1 Introduction

A dominating set of vertices for a graph $\Gamma(V, E)$ with vertex set V and edge set E, is a subset D of V such that for every $v \in V \backslash D$ there exists $w \in D$ with $\{v, w\} \in E$. An efficient dominating set is a dominating set D with the property that no two vertices of D are adjacent and every vertex in the graph is adjacent to exactly one element in D. Dominating sets are studied in graph theory for various reasons, notably in relation to network connection theory. Efficient dominating sets are also studied in coding theory.

Dominating sets of Cayley graphs of $\mathbb{Z}_n$ were studied by Chelvam and Rani [2]. They computed the domination number of $Cay(\mathbb{Z}_n, A)$ and a minimal dominating set whenever $|A|$ is even. Chelvam and Mutharasu investigated subgroups of $\mathbb{Z}_n$ that form efficient dominating sets for some of its Cayley graphs [1]. A paper of Ma focused on the isomorphism of Cayley digraphs of the dicyclic group [6]. These papers motivated this study.

We recall the definition of a Cayley graph of a group. A subset X of group G is called a generating subset if $G = \langle X \rangle$. We say X is inverse-symmetric whenever $g \in X$ implies $g^{-1} \in X$. A Cayley graph $Cay(G, X)$ of a group G with respect to subset X is a graph with vertex set G and edge set $\{\{a, b\} \,|\, b = ax \text{ for some } x \in X\}$ [5]. To obtain a simple connected undirected Cayley graph, we use a symmetric generating subset X not containing the identity. A Cayley graph encodes the algebraic structure of the group.

We consider in this paper the Cayley graph of the group $Dic_n = \langle a, x : a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1} \rangle$. This group is the *dicyclic group* of degree n, sometimes called a *generalized quaternion group* [7]. It can be proven that $|Dic_n| = 4n$. Our main result is the classification of all subgroups of Dic_n that form dominating sets for $Cay(G, S)$, for some minimal inverse-symmetric generating subset S not containing the identity.

The findings of this paper are dependent on the classification of subgroups of the dicyclic group, as may be found in [3]. We state the subgroup classification below.

Proposition 1. *Let $n \geq 2$ and $Dic_n = \langle a, x : a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1} \rangle$ be the dicyclic group of order $4n$. Then every subgroup of Dic_n is one of the following:*

i. cyclic group $\langle a^{\frac{2n}{r}} \rangle$, of order r, where r is a divisor of $2n$;

ii. cyclic group $\langle a^i x \rangle$ of order 4, where $i \in \{1, ..., n\}$;

iii. dicyclic groups $\langle a^{\frac{n}{r}}, a^i x \rangle$ of order $4r$, where r is a divisor of n, $i \in \left\{1, ..., \frac{n}{r}\right\}$.

We note that $\langle a^i x \rangle = \left\{a^i x, a^n, a^{n+i} x, 1\right\}$ for any choice of $n \geq 2$. We now state the main result of this paper.

Theorem 2. *Let $n \geq 2$ and $Dic_n = \langle a, x : a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1} \rangle$ be the dicyclic group of order $4n$. Let $S_{\delta,i} = \{a, a^{2n-(-1)^\delta 2i} x, a^{2n-1}, a^{n-(-1)^\delta 2i} x\}$ where $\delta \in \{0, 1\}$ and $i \in \{0, 1, 2, ..., 2n-1\}$, and $Cay(Dic_n, S_{\delta,i})$ be the Cayley graph with respect to $S_{\delta,i}$. If K is a proper subgroup of Dic_n, then K is a dominating set of $Cay(Dic_n, S_{\delta,i})$ if and only if K is one of the following:*

i. $\langle a \rangle$;

ii. $\langle a^2 \rangle$ where n is odd;

iii. $\langle a^{(-1)^\delta (t-2i)} x \rangle$ for $n = 2, 3$ and any $t \in \{0, 1, 2, ..., 2n-1\}$;

iv. $\langle a^{(-1)^\delta (2-2i)} x \rangle$ for $n = 4$;

v. $\langle a^2, a^{(-1)^\delta (t-2i)} x \rangle$ where n is even and $t \in \{1, 2\}$;

vi. $\langle a^3, a^{(-1)^\delta (t-2i)} x \rangle$ where 3 divides n and $t \in \{1, 2, 3\}$;

vii. $\langle a^4, a^{(-1)^\delta (2-2i)} x \rangle$ where 4 divides n.

2 Proof of the Main Theorem

The proof of the main theorem will be presented after a series of lemmas. We will start by classifying all proper subgroups of Dic_n which are dominating sets for the Cayley graph with respect to the subset $X = \{a, x, a^{2n-1}, a^n x\}$.

Lemma 3. *Let $n \geq 2$ and $Dic_n = \langle a, x : a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1} \rangle$ be the dicyclic group of order $4n$. Let $H = \langle a \rangle$. Then H is a dominating set for the Cayley graph $Cay(Dic_n, X)$. Moreover, any right coset of H in Dic_n is also a dominating set for $Cay(Dic_n, X)$.*

Proof: If $v \in Dic_n \backslash H$ then $v = a^i x$ for some $i \in \{0, 1, 2, ..., 2n-1\}$. Now,

$$va^n x = a^i x a^n x = a^i a^n x^2 = a^i \in H.$$

Hence, $\{v, va^n x\} \in E(Cay(Dic_n, X))$. Now, suppose $v \in Dic_n \backslash Hx$. It follows that $v = a^i$ for some $i \in \{0, 1, 2, ..., 2n-1\}$ and $vx = a^i x \in Hx$. □

Lemma 4. *Let $n \geq 2$ be odd and $H = \langle a^2 \rangle$. Then H is a dominating set for the graph $Cay(Dic_n, X)$. Moreover, any right coset of H in Dic_n is also a dominating set for $Cay(Dic_n, X)$.*

Proof: The proof follows a similar computation as in the proof of Lemma 3. □

Lemma 5. *Let Dic_n be the dicyclic group of order $4n$ and $H = \langle a^i x \rangle$ for $i \in \{0, 1, \ldots, 2n-1\}$. If $n \in \{2, 3\}$ then H is a dominating set for the graph $Cay(Dic_n, X)$. If $n = 4$ then only the subgroup $\langle a^2 x \rangle$ forms a dominating set for $Cay(Dic_n, X)$.*

Proof: It is a matter of computational verification to show that $H = \langle a^i x \rangle$ for every $i \in \{0, 1, \ldots, 2n-1\}$ becomes a dominating set for the Cayley graph $Cay(Dic_n, X)$ for $n = 2, 3$. We can also easily verify that only the case $i = 2$ satisfies the domination requirement for $n = 4$. □

Lemma 6. *Let $n \geq 2$ be an even integer and $H = \langle a^2, a^i x \rangle$ where $i \in \{1, 2\}$. Then H is a dominating set for the graph $Cay(Dic_n, X)$. Moreover, any right coset of H in Dic_n is also a dominating set for $Cay(Dic_n, X)$.*

Proof: The proof follows a similar computation as in the proof of Lemma 3. □

Lemma 7. *Let 3 divide n and Dic_n be the dicyclic group of order $4n$. Let $H = \langle a^3, a^i x \rangle$ where $i \in \{1, 2, 3\}$. Then H is a dominating set for the graph $Cay(Dic_n, X)$. Moreover, any right coset of H in Dic_n is also a dominating set for $Cay(Dic_n, X)$.*

Proof: Let $n = 3k$ where $k \in \mathbb{Z}^+$. Then $X = \{a, x, a^{6k-1}, a^{3k} x\}$. The rest of the proof is similar to the proof of Lemma 3. □

We note that the subgroup $\langle a^3, a^3 x \rangle$ above has the property that every element outside this subgroup is adjacent to exactly one element in the subgroup. These dominating sets are called *perfect dominating sets.*

Lemma 8. *Let 4 divide n and Dic_n be the dicyclic group of order $4n$. Let $H = \langle a^4, a^2x\rangle$. Then H is a dominating set for $Cay(Dic_n, X)$. Moreover, any right coset of H is also a dominating set for $Cay(Dic_n, X)$.*

Proof: Let $n = 4k$ where $k \in \mathbb{Z}^+$. Then $X = \{a, x, a^{8k-1}, a^{4k}x\}$. The rest of the proof is similar to the proof of Lemma 3. □

We now show that only the subgroups above can be dominating sets for $Cay(Dic_n, X)$ where $X = \{a, x, a^{2n-1}, a^n x\}$.

Lemma 9. *Let $n \geq 2$ be an even integer and Dic_n the dicyclic group of order $4n$. Then $\langle a^2\rangle$ is not a dominating set for $Cay(Dic_n, X)$.*

Proof: Consider $a^{2n-1}x \in Dic_n$. Then we have the following:

a. $(a^{2n-1}x)a = a^{2n-1}(xa) = a^{2n-1}a^{2n-1}x = a^{2n-2}x \notin \langle a^2\rangle$;

b. $(a^{2n-1}x)a^{2n-1} = a^{2n-1}(xa^{2n-1}) = a^{2n-1}ax = x \notin \langle a^2\rangle$;

c. $(a^{2n-1}x)a^n x = a^{2n-1}(xa^n)x = a^{2n-1}a^n x^2 = a^{2n-1} \notin \langle a^2\rangle$;

d. $(a^{2n-1}x)x = a^{2n-1}x^2 = a^{n-1} \notin \langle a^2\rangle$.

This proves the lemma. □

Lemma 10. *Let $n \geq 2$ and Dic_n the dicyclic group of order $4n$. If $i \in \mathbb{Z}$ such that $i \geq 3$ and i divides $2n$, then $\langle a^i\rangle$ is not a dominating set for $Cay(Dic_n, X)$.*

Proof: Consider $a^{2n-1}x \in Dic_n$. Then we have the following:

a. $(a^{2n-1}x)a = a^{2n-1}(xa) = a^{2n-1}a^{2n-1}x = a^{2n-2}x \notin \langle a^i\rangle$;

b. $(a^{2n-1}x)a^{2n-1} = a^{2n-1}(xa^{2n-1}) = a^{2n-1}ax = x \notin \langle a^i\rangle$;

c. $(a^{2n-1}x)a^n x = a^{2n-1}(xa^n)x = a^{2n-1}a^n x^2 = a^{2n-1} \notin \langle a^i\rangle$.

Suppose $(a^{2n-1}x)x = a^{2n-1}x^2 = a^{n-1} \in \langle a^i\rangle$. It follows that i divides $n-1$. There exists an integer m such that $n-1 = mi$ holds modulo $2n$. Now, $2n - mi = n+1$ and so i also divides $n+1$. Since $n+1 = mi+2$, then i divides 2. This is absurd. Hence, $a^{n-1} \notin \langle a^i\rangle$. □

Lemma 11. *Let $n \geq 5$ and Dic_n the dicyclic group of order $4n$. If $i \in \mathbb{Z}$ such that $i \geq 5$ and i divides n, then $\langle a^i, a^j x\rangle$ where $j \in \{1, 2, ..., i\}$ is not a dominating set for $Cay(Dic_n, X)$.*

Proof: For $j \neq 2$, we consider $a^2 \in Dic_n$. We have the following:

a. $a^2 a = a^3 \notin \langle a^i, a^j x\rangle$;

b. $a^2 a^{2n-1} = a \notin \langle a^i, a^j x\rangle$.

Assume $a^2(a^n x) = a^{n+2}x \in \langle a^i, a^j x\rangle$. Then there exists an integer $i_\circ$ such that $a^{n+2}x = a^{ii_\circ + j}x$ and so $n+2 \equiv ii_\circ + j (mod\ 2n)$. It means that $n+2-ii_\circ - j = 2nm$ for some integer m. This will imply that i divides $j-2$ which is absurd. If $a^2x \in \langle a^i, a^j x\rangle$ then $2 \equiv ii_\circ + j(mod\ 2n)$. As above, i will divide $j-2$ which is impossible.

For $j = 2$, we can also show that a^3 cannot be dominated by $\langle a^i, a^j x\rangle$. □

Lemma 12. *Let 4 divide n and Dic_n be the dicyclic group of order $4n$. Then $\langle a^4, a^j x\rangle$ is not a dominating set for $Cay(Dic_n, X)$ for $j \in \{1, 3, 4\}$.*

Proof: Let $n = 4k$. Since $j \neq 2$, we can consider $a^2 \in Dic_n$. We have the following:

a. $a^2 a = a^3 \notin \langle a^4, a^j x\rangle$;

b. $a^2 a^{2n-1} = a \notin \langle a^4, a^j x\rangle$.

If $a^2(a^n x) = a^{n+2}x \in \langle a^4, a^j x\rangle$, then there exists an integer i such that $4k + 2 \equiv 4i + j (mod\ 2n)$. Thus, there exists an integer m where $4k + 2 - 4i - j = 8km$. This means that 2 divides j. We only have one choice and that is $j = 4$. Now, $4k - 4i - 2 = 8km$ which is a contradiction and so $a^{n+2}x \notin \langle a^4, x\rangle$. If $a^2 x \in \langle a^4, a^j x\rangle$, then $2 \equiv 4i + j (mod\ 2n)$ for some integer i. Again, 2 divides j as above and we will get an inconsistency. Therefore, $a^2 x \notin \langle a^4, x\rangle$. □

We define the *closed neighborhood* of v in V by $N[v] = \{w \in V : \{v, w\} \in E\} \bigcup \{v\}$.

Lemma 13. *Let $n \geq 5$ and Dic_n be the dicyclic group of order $4n$. Then $\langle a^i x\rangle$ is not a dominating set for $Cay(Dic_n, X)$.*

Proof: It is a simple computation for $n = 5$. Assume $n > 5$. For $\langle a^i x\rangle$ to dominate $Cay(Dic_n, X)$, we must have $Dic_n = \bigcup_{j=1}^{4} N[(a^i x)^j]$. However, we have the following inequalities

$$|\bigcup_{j=1}^{4} N[(a^i x)^j]| \leq \sum_{j=1}^{4} |N[(a^i x)^j]| = 20 < 4(6) \leq 4n = |Dic_n|.$$

Thus, $\langle a^i x\rangle$ cannot be a dominating set for this Cayley graph. □

We now collect the properties to prove the following proposition.

Proposition 14. *Let $n \geq 2$ and $Dic_n = \langle a, x : a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1}\rangle$ be the dicyclic group of order $4n$. Let $Cay(Dic_n, X)$ be the Cayley graph with respect to $X = \{a, x, a^{2n-1}, a^n x\}$. If H is a proper subgroup of Dic_n, then H is a dominating set for $Cay(Dic_n, X)$ if and only if H is one of the following:*

i. $\langle a\rangle$;

ii. $\langle a^2\rangle$ *where n is odd;*

iii. $\langle a^i x\rangle$ *for $n = 2, 3$ and any $i \in \{0, 1, 2, ..., 2n - 1\}$;*

iv. $\langle a^2 x\rangle$ *for $n = 4$;*

v. $\langle a^2, a^i x\rangle$ *where n is even and $i \in \{1, 2\}$;*

vi. $\langle a^3, a^i x\rangle$ *where 3 divides n and $i \in \{1, 2, 3\}$;*

vii. $\langle a^4, a^2 x\rangle$ *where 4 divides n.*

Proof: Suppose H is a proper subgroup of Dic_n. Proposition 1 restricts the possibilities for H. Lemmas 3 through 8 show that if H is one of the subgroups listed in Proposition 14 then H is a dominating set for $Cay(Dic_n, X)$. On the other hand, Lemmas 9 through 13 prove that only the subgroups in the list can form dominating sets for their respective Cayley graphs. □

We now present a proof of the main result of this paper.

Proof of Theorem 2:

Let $f : G_1 \to G_2$ be an isomorphism of groups and Y be an inverse-symmetric generating subset of G_1 not containing the identity. It follows immediately that $f[Y]$ is also an inverse-symmetric generating subset for G_2 not containing the identity. We can show that if D is a dominating set for the Cayley graph $Cay(G_1, Y)$ then $f[D]$ must also be a dominating set for the Cayley graph $Cay(G_2, f[Y])$. In the same way we can verify that for a dominating set D' for $Cay(G_2, f[Y])$, the inverse image $f^{-1}[D']$ is a dominating set for $Cay(G_1, Y)$ where $f[f^{-1}[D']] = D'$.

Now, observe that $S_{\delta,i} = \{a, a^{2n-(-1)^{\delta}2i}x, a^{2n-1}, a^{n-(-1)^{\delta}2i}x\} = X^{a^i x^{\delta}}$ where $\delta \in \{0, 1\}$ and $i \in \{0, 1, 2, ..., 2n-1\}$. Note that conjugation of a group by any of its elements gives an automorphism of the group. Consequently, if D is a dominating set of $Cay(Dic_n, X)$ then $D^{a^i x^{\delta}}$ is a dominating set for $Cay(Dic_n, S_{\delta,i})$ and any dominating set of $Cay(Dic_n, S_{\delta,i})$ is a conjugate of a dominating set of $Cay(Dic_n, X)$ by $a^i x^{\delta}$. Using Proposition 14, some well-known facts on conjugation, and the following computations on the dicyclic group:

i. $(a^t)^{a^i x^{\delta}} = a^{(-1)^{\delta}t}$;

ii. $(a^t x)^{a^i x^{\delta}} = a^{(-1)^{\delta}(t-2i)}x$,

our main theorem is now proved. □

3 Remarks on Efficient Domination

Recall that an *efficient dominating set* of a graph $\Gamma(V, E)$ is a dominating set D such that for every $x \in V$, $|N[x] \cap D| = 1$. We conclude this paper with some findings on efficient domination in the Cayley graphs of the dicyclic group.

Proposition 15. *Let $n \geq 2$ and $Dic_n = \langle a, x : a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1}\rangle$ be the dicyclic group of order $4n$. Then $Cay(Dic_n, X)$, where $X = \{a, x, a^{2n-1}, a^n x\}$, has no efficient dominating set.*

Proof: The result is easily verified for $n \in \{2, 3\}$ by checking the corresponding graphs. Assume $n \geq 4$. Suppose D is an efficient dominating set of $Cay(Dic_n, X)$. Without loss of generality, assume $1 \in D$. It follows that $\{a, a^2, ax\} \cap D = \varnothing$. If $a^3 \in D$ then $\{a, a^2, x, ax, a^2x, a^3x\} \cap D = \varnothing$. Since D dominates the graph, then $\{a^{n+1}, a^{n+2}\} \subseteq D$. This implies that $|N[a^{n+1}] \cap D| \geq 2$ which contradicts the definition of efficient domination. If $a^3x \in D$ then $\{a, a^2, a^3, x, ax, a^2x\} \cap D = \varnothing$. This means that $\{a^{n+1}, a^{n+2}x\} \subseteq D$. Thus, $|N[a^{n+2}] \cap D| \geq 2$ which is absurd. Finally, if $a^2x \in D$ then $\{a^n, a^{n+1}, a^{n+2}, a^{n+1}x, ax\} \cap D = \varnothing$. This contradicts the fact that D is a dominating set. All other options for the element in D immediately following 1 will give either $\{a, a^2, x, ax, a^2x, a^3x\} \cap D = \varnothing$ or $\{a, a^2, a^3, x, ax, a^2x\} \cap D = \varnothing$. □

Corollary 16. *Let $n \geq 2$ and $Dic_n = \langle a, x : a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1} \rangle$ be the dicyclic group of order $4n$. Let $S_{\delta,i} = \{a, a^{2n-(-1)^{\delta}2i}x, a^{2n-1}, a^{n-(-1)^{\delta}2i}x\}$ where $\delta \in \{0,1\}$ and $i \in \{0,1,2,...,2n-1\}$. Then the graph $Cay(Dic_n, S_{\delta,i})$ does not have an efficient dominating set.*

Proof: Let $f : G_1 \to G_2$ be a group isomorphism and X an inverse-symmetric generating subset of G_1 not containing the identity. Recall from the proof of Theorem 2 that if D is a dominating set of $Cay(G_1, X)$, then so is $f[D]$ with respect to $Cay(G_2, f[X])$. We now show that if D is an efficient dominating set, then so is $f[D]$. Assume, on the contrary, that there exists $y \in G_2$ with $|N[y] \cap f[D]| \geq 2$. It follows that we can find distinct y_1 and y_2 in $N[y] \cap f[D]$. If $y \notin \{y_1, y_2\}$ then we have $x'_1, x'_2 \in f[X]$ where $y_j = yx'_j$ $(j \in \{1,2\})$. Also, there exist $d_1, d_2 \in D$ such that $y_j = f(d_j)$ $(j \in \{1,2\})$. By the isomorphism of f, we have $x \in G_1; x_j \in X$ where $y = f(x)$ and $x'_j = f(x_j)$ for $j \in \{1,2\}$. Now, $d_1 \neq d_2$ and $xx_j = d_j$ $(j \in \{1,2\})$. Hence, $|N[x] \cap D| \geq 2$ which is absurd. The case $y \in \{y_1, y_2\}$ is treated similarly.

As before, $S_{\delta,i} = \{a, a^{2n-(-1)^{\delta}2i}x, a^{2n-1}, a^{n-(-1)^{\delta}2i}x\} = X^{a^i x^{\delta}}$ and conjugation gives an automorphism of the group G. By Proposition 15, the graph $Cay(G, S_{\delta,i})$ cannot have an efficient dominating set. □

Acknowledgment: The second author would like to thank the University of San Carlos for the encouragement and support while the research was being undertaken.

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