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On the Distribution of the Sums, Products and Quotient of Singh-Maddala Distributed Random Variables Based on FGM Copula

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Abstract

In this article, a Singh-Maddala distribution (Burr Type XII) based on Farlie-Gumbel-Morgenstern copula is introduced. Derivations of exact distribution $R = X + Y$ for $\gamma = 1$, $V = XY$, $W = X/Y$ and $Z = X/(X+Y)$ are obtained in closed form. Corresponding moment properties of these distributions are also derived. The expressions turn out to involve known special functions.

1 Introduction

Copula from the Latin word *copulare* means to connect or to join. Essentially, copulas' are functions that join or "couple" multivariate distributions to their one-dimensional marginal distribution functions [17]. Its sole purpose is to describe the interdependence of several random variables [20]. A copula is a joint distribution function of the uniform marginals [17]. When marginals are uniform, they are independent. It implies a flat probability density function, and any deviation will indicate dependency. To date, there has been growing interest in copula owing to its usefulness and popularity though not exempt of criticism [11]. A listing of copula can be found in [3].

In this study, a Farlie-Gumbel-Morgenstern (FGM) copula is considered in constructing a bivariate pdf that accounts dependence between two random variables. Let $f_X(x)$ and $f_Y(y)$ be the pdf's of random variable X and Y with their corresponding distribution functions $F_X(x)$ and $F_Y(y)$ respectively, and $\theta \in [-1, 1]$ the dependence parameter of X and Y. Then the joint probability density function (pdf) of X and Y is given by

$$f_{X,Y}(x, y) = f_X(x) f_Y(y) \cdot c(F_X(x), F_Y(y)) \tag{1}$$

where

$$c\left(F_X(x), F_Y(y)\right) = 1 + \theta\left(2F_X(x) - 1\right)\left(2F_Y(y) - 1\right) \tag{2}$$

is called the copula density function or FGM copula.

The FGM copula was first introduced by Morgenstern [12] in 1956. Note that when θ in (1) equals zero, the joint pdf collapses to a definition of independence of X and Y. By Sklar's theorem, any copula is unique (and hence for FGM copula) whenever the marginals were both continuous; otherwise a copula can be uniquely determined by the range of the product of its marginals [3, pp. 34-35]. Furthermore, any multivariate distribution can be specified via two independent components, (1) marginal distribution functions, and (2) a copula function that provides the dependence structure.

Our choice for FGM copula was driven largely by its applications in biological science, actuarial science and finances [9] most notably in hydrology [8]. The FGM copula is by far the most simple and tractable [18]. In our case, it provides closed form formula for our exact distributions given the Singh-Maddala as marginals. We are mindful however that it is restrictive in the sense that dependency of two marginals could only be modest in magnitude [8]. In fact, several dependence measures, e.g. Pearson's coefficient ρ and Kendall's τ for FGM copula can never exceed 0.333 and 0.222, respectively. This restriction motivates several authors in extending the FGM copula.

Our choice for Singh-Maddala distribution as marginals comes for two reasons. First, Singh-Maddala's wide applicability is well regarded, especially on income distribution [10]. Second, Singh-Maddala includes log-logistics, Pareto and Weibull distributions in which these distributions has been widely used for applications in hydrology [13, 15]. Singh-Maddala also has a beautiful property being the reciprocal of Dagum distribution that has a better fit when it comes to income distribution data [7]. Extensive applications on FGM with varying marginals can be found in [3].

Below are some examples on sums, products and quotient of random variables (see [13], [15]).

1. In hydrology, the stream flow from two bodies of water say, rivers X and Y, may combine to provide water supply to a single region or it would also be possible that the combined flow can create a flood hazards instead. Thus, the occurrence of these simultaneous flows, $Z = X + Y$, would be of interest. Furthermore, the inter-arrival time of droughts can be expressed as the sum of drought duration and non-drought duration.

2. If X and Y denote extreme rainfall at station A and B then $X/X + Y$ represents relative extremity of rainfall at station A compared to B. Similarly, the proportion of droughts can be thought of drought duration over the sum of drought duration and non-drought duration.

3. If X denotes the rainfall intensity or drought severity while Y the rainfall duration or drought duration respectively, then XY is the amount of rainfall or the magnitude of drought, respectively.

Throughout the years, exact distributions of the sum, product, and quotient of random variables (correlated or independent) has been extensively studied by many researchers. See for instance, [13], [14], and [15]. A comprehensive list on exact distributions according to family of distributions can be found in [2]. Definitions of a sum, product, and quotient of two independent random variables can be found in Rohatgi and Saleh [19].

All of the above-mentioned studies used almost classical families of bivariate distributions. The aim of this paper is to give a different approach on modeling dependency aside from *correlated* random variables applied on the sum, product and quotient. Traditionally, correlation is used to describe dependence between random variables, but recent studies have ascertained the superiority of copulas to model dependence, as they offer much more flexibility than the correlation approach [5]. Consider the linear correlation coefficient, though widely used as a test for dependence, is not a measure of general, but only of linear dependence. If the random variables do not follow a normal distribution, the use of the linear correlation coefficient as a measure of dependence may induce misleading conclusions [1]. Copula modeling approach, however, allows us to look beyond the usual correlation technique.

Furthermore, it allows us to model and estimate the univariate margins and the dependence structure independently. For example, we can decide to take exponential distributions for the margins and Gaussian copula for the dependence structure. We can also make different margins for each random variable. This approach is in contrast to the classical technique of using bivariate distribution in which one have to fix the margins, for example, bivariate normal which margins must also be normal- a restriction that is not always realistic [6]. In fact, empirical evidence of a non-linear form of dependence and non-normality of data is widely unquestionable [4]. In essence, copulas allow us to study the dependence structure independently of the marginal behaviour. Therefore, the copula-based approach should be considered an alternative methods for capturing co-dependency.

Domma and Giordano [4] laid the ground work on using copula in its evaluation of probability $P(X < Y)$ and so far no works has been done using copula applied on sums, product and quotient of two random variables. We fill the gap by studying the Singh-Maddala constructed from FGM copula.

This paper is organized as follows. Section 2 is devoted to derivations of explicit expressions for the pdfs of $R = X + Y$, $V = XY$ and $Z = X/(X + Y)$, respectively, while section 3 is devoted in the derivation of raw moments of all pdfs obtained in section 2.

The following results which can be found in [14] are needed in the subsequent discussions.

Lemma 1. *For any $\rho > \alpha > 0$,*

$$\int_0^\infty \frac{s^{\alpha-1}}{(s+z)^\rho} ds = z^{\alpha-\rho} B(\alpha, \rho - \alpha), \quad z \in \mathbb{R}, \tag{3}$$

where

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx$$

for $a > 0$ and $b > 0$ is the beta function.

Lemma 2. *For $0 < \alpha < \rho + \lambda$,*

$$\begin{aligned} &\int_0^\infty x^{\alpha-1}(x+y)^{-\rho}(x+z)^{-\lambda} dx \\ &\qquad = z^{-\lambda} y^{\alpha-\rho} B(\alpha, \rho+\lambda-\alpha) {}_2F_1\left(\alpha, \lambda; \rho+\lambda; 1 - \frac{y}{z}\right). \end{aligned} \tag{4}$$

where

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

is the Gauss Hypergeometric function and $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial.

Lemma 3. *For $p > 0$ and $q > 0$,*

$$\int_a^b (x-a)^{p-1}(b-x)^{q-1}(cx+d)^r dx = (b-a)^{p+q-1}(ac+d)^r B(p,q)_2F_1\left(p,-r;p+q;\frac{c(a-b)}{ac+d}\right). \quad (5)$$

2 PDFs

Let X and Y be two independent Singh-Maddala distributed random variables with probability density functions (pdf) for all positive parameters α, γ, and τ are given by

$$f_X(x;\tau,\alpha,\gamma) = \frac{\alpha\gamma(x/\tau)^\gamma}{x[1+(x/\tau)^\gamma]^{\alpha+1}}, \quad x > 0 \quad (6)$$

and

$$f_Y(y;\tau,\alpha,\gamma) = \frac{\alpha\gamma(y/\tau)^\gamma}{y[1+(y/\tau)^\gamma]^{\alpha+1}}, \quad y > 0 \quad (7)$$

respectively.

The corresponding cumulative distribution functions (cdf) of X and Y are known to be

$$F_X(x;\tau,\alpha,\gamma) = 1-\left(\frac{1}{1+(x/\tau)^\gamma}\right)^\alpha, \quad x > 0 \quad (8)$$

and

$$F_Y(y;\tau,\alpha,\gamma) = 1-\left(\frac{1}{1+(y/\tau)^\gamma}\right)^\alpha, \quad y > 0 \quad (9)$$

respectively.

The the joint density function of X and Y that follows Singh-Maddala distribution is given by

$$f_{X,Y}(x,y;\tau,\alpha,\gamma;\rho) = \frac{(\alpha\gamma)^2\left(xy/\tau^2\right)^\gamma}{xy\left[(1+(x/\tau)^\gamma)(1+(y/\tau)^\gamma)\right]^{\alpha+1}} \left[1+\rho\left[2\left(1+(x/\tau)^\gamma\right)^{-\alpha}-1\right]\left[2\left(1+(y/\tau)^\gamma\right)^{-\alpha}-1\right]\right] \quad (10)$$

where $x,\ y,\ \alpha,\ \tau,\ \gamma$ are all positive and $|\rho| \leq 1$.

The following figure illustrates the pdf in (10) for specific values: $\alpha = 1.5$, $\gamma = 4.5$ $\tau = 2$, $\rho = 0.5$.

Theorems (4)–(7) derive the pdfs of $R = X+Y$, $V = XY$ and $W = X/(X+Y)$ when X and Y are distributed according to (10). In the subsequent, we assume that $\alpha,\ \tau,\ \gamma$ are positive real numbers and $\rho \in [-1,1]$.

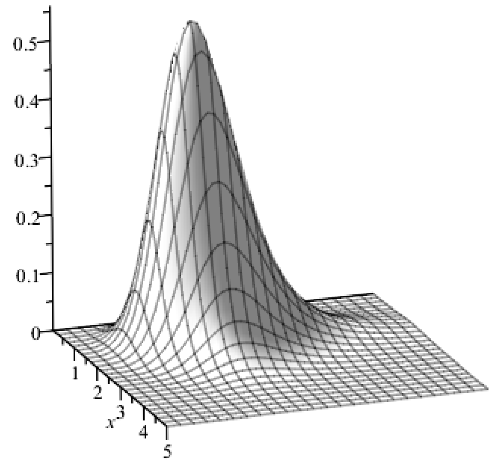

Figure 1: Graph of the pdf in (10)

Theorem 4. *If X and Y are jointly distributed according to (10), then the density function of $V = XY$ is*

$$f_V\left(x,y;\tau,\alpha,\gamma;\rho\right) = (\alpha\gamma)^2 v^{\gamma-1}\Bigg[\frac{(1+\rho)\tau^{2\gamma\alpha}v^{-\gamma(\alpha+1)}}{\gamma}B(\alpha+1,\alpha+1)\; {}_2F_1\left(\alpha+1,\alpha+1;2\alpha+2;1-\left(\frac{\tau^2}{v}\right)^{\gamma}\right) + \frac{4\rho\tau^{4\gamma\alpha}v^{-\gamma(2\alpha+1)}}{\gamma}B(2\alpha+1,2\alpha+1)\; {}_2F_1\left(2\alpha+1,2\alpha+1;4\alpha+2;1-\left(\frac{\tau^2}{v}\right)^{\gamma}\right) - \frac{2\rho\tau^{2\gamma\alpha}v^{-\gamma(\alpha+1)}}{\alpha}B(\alpha+1,2\alpha+1)\; {}_2F_1\left(\alpha+1,\alpha+1;3\alpha+2;1-\left(\frac{\tau^2}{v}\right)^{\gamma}\right) - \frac{2\rho\tau^{4\gamma\alpha}v^{-\gamma(2\alpha+1)}}{\gamma}B(2\alpha+1,\alpha+1)\; {}_2F_1\left(2\alpha+1,2\alpha+1;3\alpha+2;1-\left(\frac{\tau^2}{v}\right)^{\gamma}\right)\Bigg] \tag{11}$$

for $0 < v < \infty$.

Proof: From (10), the joint pdf of $(X,Y)=\left(X,\frac{V}{X}\right)$ can be expressed as

$$f_{X,Y}\left(x,\frac{v}{x};\tau,\alpha,\gamma;\rho\right)=\frac{(\alpha\gamma)^2v^{\gamma-1}}{\tau^{2\gamma}}\left[\frac{1+\rho}{\left[(1+(x/\tau)^\gamma)\left(1+(v/x\tau)^\gamma\right)\right]^{\alpha+1}}\right.$$
$$+\frac{4\rho}{\left[(1+(x/\tau)^\gamma)\left(1+(v/x\tau)^\gamma\right)\right]^{2\alpha+1}}$$
$$-\frac{2\rho}{(1+(x/\tau)^\gamma)^{2\alpha+1}\left(1+(v/x\tau)^\gamma\right)^{\alpha+1}}$$
$$\left.-\frac{2\rho}{(1+(x/\tau)^\gamma)^{2\alpha+1}\left(1+(x/\tau)^\gamma\right)^{\alpha+1}}\right].$$

By Rohatgi's well-known result (Theorem 3, p. 139), the pdf of $V=XY$ becomes

$$f_V\left(v;\tau,\alpha,\gamma;\rho\right)=\frac{(\alpha\gamma)^2v^{\gamma-1}}{\tau^{2\gamma}}\Big[(1+\rho)G(1,1)+4\rho G(2,2)-2\rho G(2,1)-2\rho G(1,2)\Big] \tag{12}$$

where

$$G(h,k)=\int_0^\infty x^{-1}\left(1+\left(\frac{x}{\tau}\right)^\gamma\right)^{-(h\alpha+1)}\left(1+\left(\frac{v}{x\tau}\right)^\gamma\right)^{-(k\alpha+1)}dx$$
$$=\tau^{\gamma h\alpha+\gamma}\int_0^\infty x^{\gamma k\alpha+\gamma-1}\left(x^\gamma+\tau^\gamma\right)\left(x^\gamma+\left(\frac{v}{\tau}\right)^\gamma\right)^{-(k\alpha+1)}\,dx,\quad h,k\in\{1,2\}.$$

Using Lemma (2) we obtain

$$G(h,k)=\frac{\tau^{2\gamma k\alpha+2\gamma}v^{-\gamma(k\alpha+1)}}{\gamma}B(k\alpha+1,h\alpha+1)\,{}_2F_1\left(k\alpha+1,k\alpha+1;(h+k)\alpha+2;1-\left(\frac{\tau^2}{v}\right)^\gamma\right). \tag{13}$$

Applying (13) to the equation (12) will result to (11).

Figure 2 illustrate the shape of the pdf in (11) for $\tau=2,4$. Each plot contains three curves corresponding to selected values of α and γ. The effect of the parameters is evident.

Theorem 5. *If X and Y are jointly distributed according to (10), then the distribution of $W=\frac{X}{Y}$ is*

$$\begin{aligned}f_W(w;\tau,\alpha,\gamma;\rho)=\frac{\alpha^2\gamma}{w^{\gamma+1}}\Bigg[&(1+\rho)B(2,2\alpha)\,{}_2F_1\left(2,\alpha+1;2\alpha+2;1-\left(\frac{\tau^2}{w}\right)^\gamma\right)\\&+4\rho B(2,4\alpha)\,{}_2F_1\left(2,2\alpha+1;4\alpha+2;1-\left(\frac{\tau^2}{w}\right)^\gamma\right)\\&-2\rho B(2,3\alpha)\,{}_2F_1\left(2,\alpha+1;3\alpha+2;1-\left(\frac{\tau^2}{w}\right)^\gamma\right)\\&-2\rho B(2,3\alpha)\,{}_2F_1\left(2,2\alpha+1;3\alpha+2;1-\left(\frac{\tau^2}{w}\right)^\gamma\right)\Bigg]\end{aligned} \tag{14}$$

for $0<w<\infty$.

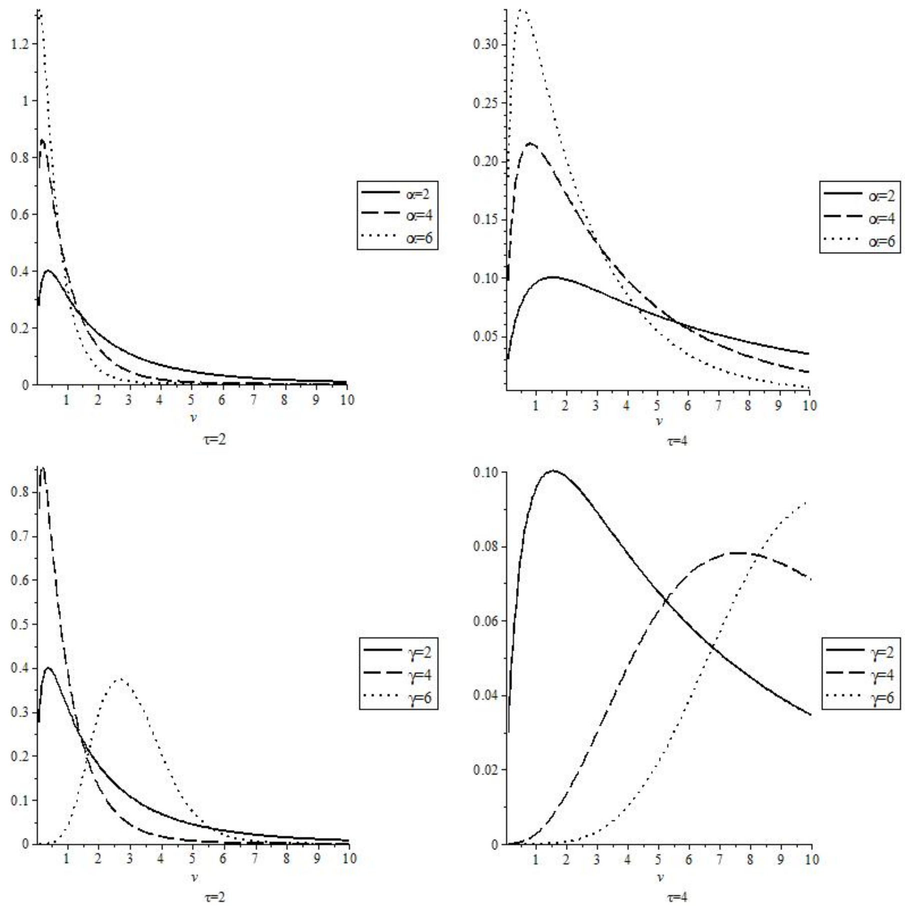

Figure 2: Graph of the pdf in (11) with selected values of θ and α.

Proof: From (10), the joint pdf of $(X, Y) = \left(X, \frac{X}{W}\right)$ can be expressed as

$$
\begin{aligned}
f_{X,W}(xw, x; \tau, \alpha, \gamma; \rho) = \left(\frac{\alpha\gamma}{\tau^{\gamma}}\right)^2 w^{\gamma-1} \Bigg[& \frac{(1+\rho)x^{2\gamma-2}}{\left[(1+(xw/\tau)^{\gamma})\,(1+(x/\tau)^{\gamma})\right]^{\alpha+1}} \\
& + \frac{4\rho x^{2\gamma-2}}{\left[(1+(xw/\tau)^{\gamma})\,(1+(x/\tau)^{\gamma})\right]^{2\alpha+1}} \\
& - \frac{2\rho x^{2\gamma-2}}{(1+(xw/\tau)^{\gamma})^{2\alpha+1}\,(1+(x/\tau)^{\gamma})^{\alpha+1}} \\
& - \frac{2\rho x^{2\gamma-2}}{(1+(xw/\tau)^{\gamma})^{\alpha+1}\,(1+(x/\tau)^{\gamma})^{2\alpha+1}} \Bigg].
\end{aligned}
$$

By Rohatgi's result, the pdf of $W = \frac{X}{Y}$ can be expressed as

$$
f_W(w; \tau, \alpha, \gamma; \rho) = \left(\frac{\alpha\gamma}{\tau^{\gamma}}\right)^2 w^{\gamma-1} \left[(1+\rho)P(1,1) + 4\rho P(2,2) - 2\rho P(2,1) - 2\rho P(1,2)\right] \quad (15)
$$

where

$$P(h,k) = \frac{\tau^{\gamma(h\alpha+k\alpha+2)}}{w^{\gamma(h\alpha+1)}} \int_0^\infty x^{2\gamma-1} \left(x^\gamma + (\tau/w)^\gamma\right)^{-(h\alpha+1)} \left(x^\gamma + \tau^\gamma\right)^{-(k\alpha+1)} dx \tag{16}$$

for $h, k \in \{1, 2\}$.

Using Lemma (2) one can get

$$P(h,k) = \frac{\tau^{2\gamma}}{\gamma w^{2\gamma}} B(2, (h+k)\alpha)\, {}_2F_1\left(2, k\alpha+1; (h+k)\alpha+2; 1 - (\tau^2/w)^\gamma\right). \tag{17}$$

By (17), the following terms in (15) are obvious.

(1) $(1+\rho)P(1,1) = (1+\rho)\dfrac{\tau^{2\gamma}}{\gamma w^{2\gamma}} B(2, 2\alpha)\, {}_2F_1\left(2, \alpha+1; 2\alpha+2; 1 - (\tau^2/w)^\gamma\right);$

(2) $4\rho P(2,2) = 4\rho\dfrac{\tau^{2\gamma}}{\gamma w^{2\gamma}} B(2, 4\alpha)\, {}_2F_1\left(2, 2\alpha+1; 4\alpha+2; 1 - (\tau^2/w)^\gamma\right);$

(3) $-2\rho P(2,1) = 2\rho\dfrac{\tau^{2\gamma}}{\gamma w^{2\gamma}} B(2, 3\alpha)\, {}_2F_1\left(2, \alpha+1; 3\alpha+2; 1 - (\tau^2/w)^\gamma\right);$

(4) $-2\rho P(1,2) = 2\rho\dfrac{\tau^{2\gamma}}{\gamma w^{2\gamma}} B(2, 3\alpha)\, {}_2F_1\left(2, 2\alpha+1; 3\alpha+2; 1 - (\tau^2/w)^\gamma\right).$

The result follows by using items (1)–(4) in (15).

The next figure illustrates the pdf in (14) for $\tau = 2, 4$, for specific values $\alpha = 2, 4, 6$ and $\gamma = 2, 4, 6$, respectively.

Theorem 6. *If X and Y are jointly distributed according to (10), then the distribution of $Z = \frac{X}{X+Y}$ is*

$$\begin{aligned} f_Z(z; \tau, \alpha, \gamma; \rho) = \frac{\alpha^2 \gamma (1-z)^{\gamma-1}}{z^{\gamma+1}} \Bigg[& (1+\rho) B(2, 2\alpha)\, {}_2F_1\left(2, \alpha+1; 2\alpha+2; 1 - \left(\frac{1-z}{z}\right)^\gamma\right) \\ & + 4\rho B(2, 4\alpha)\, {}_2F_1\left(2, 2\alpha+1; 4\alpha+2; 1 - \left(\frac{1-z}{z}\right)^\gamma\right) \\ & - 2\rho B(2, 3\alpha)\, {}_2F_1\left(2, \alpha+1; 3\alpha+2; 1 - \left(\frac{1-z}{z}\right)^\gamma\right) \\ & - 2\rho B(2, 3\alpha)\, {}_2F_1\left(2, 2\alpha+1; 3\alpha+2; 1 - \left(\frac{1-z}{z}\right)^\gamma\right) \Bigg] \end{aligned} \tag{18}$$

for $0 < z < 1$.

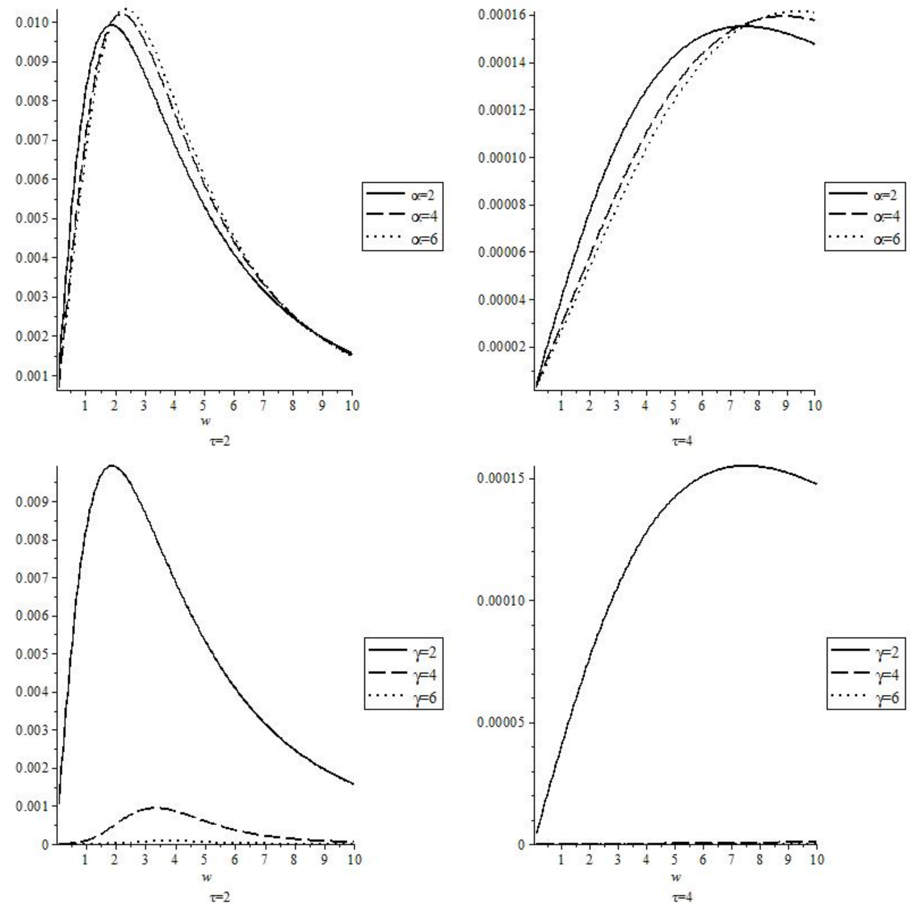

Figure 3: Graph of the pdf in (14).

Proof: Consider the transformation: $(X,Y) \longrightarrow (R,Z) = \left(X+Y, \frac{X}{X+Y}\right)$ so that

$$f_{R,Z}(r,z;\tau,\alpha,\gamma;\rho) = \frac{(\alpha\gamma)^2[z(1-z)]^{\gamma-1}}{\tau^{2\gamma}} \left[\frac{(1+\rho)r^{2\gamma-2}}{\left[\left(1+\left(\frac{rz}{\tau}\right)^{\gamma}\right)\left(1+\left(\frac{r-rz}{\tau}\right)^{\gamma}\right)\right]^{\alpha+1}} + \frac{4\rho r^{2\gamma-2}}{\left[\left(1+\left(\frac{rz}{\tau}\right)^{\gamma}\right)\left(1+\left(\frac{r-rz}{\tau}\right)^{\gamma}\right)\right]^{2\alpha+1}} - \frac{2\rho r^{2\gamma-2}}{\left(1+\left(\frac{rz}{\tau}\right)^{\gamma}\right)^{2\alpha+1}\left(1+\left(\frac{r-rz}{\tau}\right)^{\gamma}\right)^{\alpha+1}} - \frac{2\rho r^{2\gamma-2}}{\left(1+\left(\frac{rz}{\tau}\right)^{\gamma}\right)^{\alpha+1}\left(1+\left(\frac{r-rz}{\tau}\right)^{\gamma}\right)^{2\alpha+1}} \right]$$

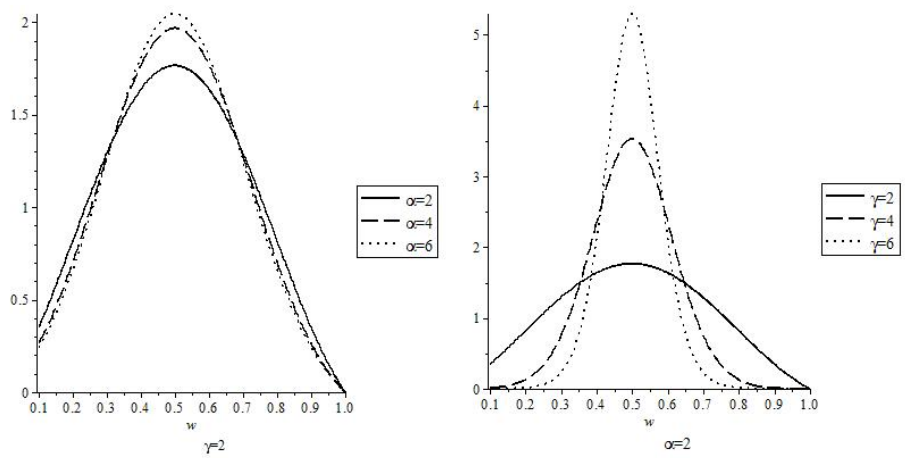

Figure 4: Graph of the pdf in (18).

Note that the jacobian of transformation is r, thus

$$f_Z(z;\tau,\alpha,\gamma;\rho) = \frac{(\alpha\gamma)^2[z(1-z)]^{\gamma-1}}{\tau^{2\gamma}}\left[(1+\rho)\Phi(1,1)+4\rho\Phi(2,2)-2\rho\Phi(2,1)-2\rho\Phi(1,2)\right] \tag{19}$$

where

$$\Phi(h,k) = \int_0^\infty r^{2\gamma-1}\left(1+\left(\frac{rz}{\tau}\right)^\gamma\right)^{-(h\alpha+1)}\left(1+\left(\frac{r-rz}{\tau}\right)^\gamma\right)^{-(k\alpha+1)} dr \tag{20}$$

for $h,k\in\{1,2\}$.
Let $u=r^\gamma$. Then $\dfrac{du}{\gamma}=r^{\gamma-1}dr$. One can obtain $\Phi(h,k)$ as follows

$$\Phi(h,k) = \frac{1}{\gamma}\int_0^\infty u\left(1+\left(\frac{z}{\tau}\right)^\gamma u\right)^{-(h\alpha+1)}\left(1+\left(\frac{1-z}{\tau}\right)^\gamma u\right)^{-(k\alpha+1)} du. \tag{21}$$

Using Lemma (2), we have

$$\Phi(h,k) = \frac{1}{\gamma}\frac{\tau^{2\gamma}}{z^{2\gamma}}B(2,(h+k)\alpha)\;{}_2F_1\left(2,k\alpha+1;(h+k)\alpha+2;1-\left(\frac{1-z}{z}\right)^\gamma\right). \tag{22}$$

Combining (22) and (19) the result in (18) follows.

The next figure illustrates the pdf in (18) for specific values: $\rho=0.5$, with $\gamma=2$ for $\alpha=2,4,6$; $\alpha=2$ for $\gamma=2,4,6$. Note that $\frac{X}{X+Y}$ is between 0 and 1. The graph shows the domain on $[0,1]$.

Theorem 7. *If X and Y are jointly distributed according to (10) with $\gamma=1$, then the* *density function of $R = X + Y$ is given by*

$$f_R(r;\alpha,\theta;\rho) = r(\alpha\theta^{\alpha})^2 \Bigg\{ (1+\rho)\,\theta^{-(\alpha+1)}(r+\theta)^{-(\alpha+1)} \sum_{j=1}^{\infty}\left[\binom{\alpha+j-1}{j-1}\left(-\frac{r}{\theta}\right)^{j-1} j^{-1}A\right]$$
$$+ 4\rho\theta^{2\alpha}\theta^{-(2\alpha+1)}(r+\theta)^{-(2\alpha+1)} \sum_{j=1}^{\infty}\left[\binom{2\alpha+j-1}{j-1}\left(-\frac{r}{\theta}\right)^{j-1} j^{-1}B\right]$$
$$- 2\rho\theta^{\alpha}\theta^{-(\alpha+1)}(r+\theta)^{-(\alpha+1)} \sum_{j=1}^{\infty}\left[\binom{2\alpha+j-1}{j-1}\left(-\frac{r}{\theta}\right)^{j-1} j^{-1}A\right]$$
$$- 2\rho\theta^{\alpha}\theta^{-(2\alpha+1)}(r+\theta)^{-(2\alpha+1)} \sum_{j=1}^{\infty}\left[\binom{\alpha+j-1}{j-1}\left(-\frac{r}{\theta}\right)^{j-1} j^{-1}B\right]\Bigg\} \tag{23}$$

where $A = {}_2F_1\left(j,\alpha+1;j+1;\frac{r}{r+\theta}\right)$ *and* $B = {}_2F_1\left(j,2\alpha+1;j+1;\frac{r}{r+\theta}\right)$ *for* $0 < r < \infty$.

Proof: Consider the transformation: $(X,Y) \longrightarrow (R,Z) = \left(X+Y, \frac{X}{X+Y}\right)$ so that

$$f_{R,Z}(r,z;\tau,\alpha,\gamma;\rho) = \frac{(\alpha\gamma)^2[z(1-z)]^{\gamma-1}}{\tau^{2\gamma}}\Bigg[\frac{(1+\rho)r^{2\gamma-2}}{\left[\left(1+\left(\frac{rz}{\tau}\right)^{\gamma}\right)\left(1+\left(\frac{r-rz}{\tau}\right)^{\gamma}\right)\right]^{\alpha+1}}$$
$$+ \frac{4\rho r^{2\gamma-2}}{\left[\left(1+\left(\frac{rz}{\tau}\right)^{\gamma}\right)\left(1+\left(\frac{r-rz}{\tau}\right)^{\gamma}\right)\right]^{2\alpha+1}}$$
$$- \frac{2\rho r^{2\gamma-2}}{\left(1+\left(\frac{rz}{\tau}\right)^{\gamma}\right)^{2\alpha+1}\left(1+\left(\frac{r-rz}{\tau}\right)^{\gamma}\right)^{\alpha+1}}$$
$$- \frac{2\rho r^{2\gamma-2}}{\left(1+\left(\frac{rz}{\tau}\right)^{\gamma}\right)^{\alpha+1}\left(1+\left(\frac{r-rz}{\tau}\right)^{\gamma}\right)^{2\alpha+1}}\Bigg]$$

The jacobian of transformation is r, thus

$$f_R(r;\tau,\alpha,\gamma;\rho) = \frac{\alpha\gamma^2}{\tau^{\gamma}} r^{2\gamma-2}\left[(1+\rho)G(1,1) + 4\rho G(2,2) - 2\rho G(2,1) - 2\rho G(1,2)\right] \tag{24}$$

where

$$G(h,k) = \int_0^1 [z(1-z)]^{\gamma-1}\left(1+(rz/\tau)^{\gamma}\right)^{-(h\alpha+1)}\left(1+\left(\frac{r-rz}{\tau}\right)^{\gamma}\right)^{-(k\alpha+1)} dz \tag{25}$$

for $h,k \in \{1,2\}$.
The above integral is mathematically challenging. Thus, we set $\gamma = 1$. Then we have the following equivalent equation.

$$G(h,k) = \int_0^1 (1+rz/\tau)^{-(h\alpha+1)}\left(1+\frac{r-rz}{\tau}\right)^{-(k\alpha+1)} dz$$

Using Lemma (3), one can obtain $G(h,k)$ as follows

$$\begin{aligned} G(h,k) &= \theta^{-(h\alpha+1)} \sum_{j=0}^{\infty} \left[\binom{h\alpha+j}{j} \left(-\frac{r}{\theta}\right)^{j} \int_0^1 z^j \left(-rz+r+\theta\right)^{-(k\alpha+1)} dz \right] \\ &= \theta^{-(h\alpha+1)} \sum_{j=1}^{\infty} \left[\binom{h\alpha+j-1}{j-1} \left(-\frac{r}{\theta}\right)^{j-1} \int_0^1 (z-0)^{j-1} (1-z)^{1-1} \right. \\ &\qquad \left. (-rz+r+\theta)^{-(k\alpha+1)} dz \right] \\ &= \theta^{-(h\alpha+1)} \sum_{j=1}^{\infty} \left[\binom{h\alpha+j-1}{j-1} \left(-\frac{r}{\theta}\right)^{j-1} (r+\theta)^{-(k\alpha+1)} B(j,1) \right. \\ &\qquad \left. {}_2F_1\left(j, k\alpha+1; j+1; \frac{r}{r+\theta}\right) \right] \\ &= \theta^{-(h\alpha+1)} (r+\theta)^{-(k\alpha+1)} \sum_{j=1}^{\infty} \left[\binom{h\alpha+j-1}{j-1} \left(-\frac{r}{\theta}\right)^{j-1} j^{-1} \right. \\ &\qquad \left. {}_2F_1\left(j, k\alpha+1; j+1; \frac{r}{r+\theta}\right) \right]. \end{aligned} \tag{26}$$

Combining (24) and (26), the result follows immediately.

3 Moments

Theorem 8. *Let X and Y be jointly distributed according to (10). Then the (a,b)-th product moment of bivariate Singh-Maddala density function denoted by ${\mu'}_{a,b;\rho}(X,Y)$ is given by*

$$\begin{aligned} \mu'_{a,b;\rho}(X,Y) =& \tau^{a+b}\Gamma(a/\gamma+1)\Gamma(b/\gamma+1) \\ & \left[\frac{\Gamma(\alpha-a/\gamma)\Gamma(\alpha-b/\gamma)}{\Gamma^2(\alpha)} + \rho \left(\frac{\Gamma(2\alpha-a/\gamma)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha-a/\alpha)}{\Gamma(\alpha)} \right) \right. \\ & \left. \left(\frac{\Gamma(2\alpha-b/\gamma)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha-b/\alpha)}{\Gamma(\alpha)} \right) \right] \end{aligned} \tag{27}$$

where $x, y, \alpha, \gamma, \tau,$ are all positive, $|\rho| \le 1$ and $\max\{a,b\} < \alpha$.

Proof: By definition, one can expressed the (a,b)-th moment of $f_{X,Y}(x,y;\alpha,\tau;\rho)$ as

$$\begin{aligned} \mu'_{a,b}(X,Y) = & \int_0^\infty \int_0^\infty x^a y^b \frac{(\alpha\gamma)^2 \left(xy/\tau^2\right)^\gamma}{xy\left[(1+(x/\tau)^\gamma)(1+(y/\tau)^\gamma)\right]^{\alpha+1}} dx dy \\ & + \rho \left[\int_0^\infty \left(2\left(1+(x/\tau)^\gamma\right)^{-\alpha} - 1 \right) x^a \frac{\alpha\gamma(x/\tau)^\gamma}{x[1+(x/\tau)^\gamma]^{\alpha+1}} dx \right. \\ & \left. \int_0^\infty \left(2\left(1+(y/\tau)^\gamma\right)^{-\alpha} - 1 \right) y^b \frac{\alpha\gamma(y/\tau)^\gamma}{y[1+(y/\tau)^\gamma]^{\alpha+1}} dy \right]. \end{aligned}$$

By Lemma 1, one can show the following integrals:

(1)
$$\int_0^\infty x^a \frac{\alpha\gamma(x/\tau)^\gamma}{x[1+(x/\tau)^\gamma]^{\alpha+1}}dx = \frac{\tau^a\Gamma(a/\gamma+1)\Gamma(\alpha-a/\gamma)}{\Gamma(\alpha)};$$

(2)
$$\int_0^\infty y^b \frac{\alpha\gamma(y/\tau)^\gamma}{y[1+(y/\tau)^\gamma]^{\alpha+1}}dy = \frac{\tau^b\Gamma(b/\gamma+1)\Gamma(\alpha-b/\gamma)}{\Gamma(\alpha)};$$

(3)
$$2\int_0^\infty x^a \frac{\alpha\gamma(x/\tau)^\gamma}{x[1+(x/\tau)^\gamma]^{2\alpha+1}}dx = \frac{\tau^a\Gamma(a/\gamma+1)\Gamma(2\alpha-a/\gamma)}{\Gamma(2\alpha)};$$

Finally,

(4)
$$2\int_0^\infty y^b \frac{\alpha\gamma(y/\tau)^\gamma}{y[1+(y/\tau)^\gamma]^{2\alpha+1}}dy = \frac{\tau^b\Gamma(b/\gamma+1)\Gamma(2\alpha-b/\gamma)}{\Gamma(2\alpha)}.$$

Then the result follows directly.

Theorem 9. *If X and Y are jointly distributed according to 10, then the a-th raw moment of the random variable V is*

$$\mu'_{a;\rho}(V) = \tau^{2a}\Gamma^2(a/\gamma+1)\left[\left(\frac{\Gamma(\alpha-a/\gamma)}{\Gamma(\alpha)}\right)^2 + \rho\left(\frac{\Gamma(2\alpha-a/\gamma)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha-a/\gamma)}{\Gamma(\alpha)}\right)^2\right]. \tag{28}$$

Proof: Notice that

$$E(V^a) = E((X\cdot Y)^a) = E(X^a \cdot Y^a).$$

Putting $b = a$ in (27), the result follows.

We state the next result without proof since the proof is similar to that of Theorem 9.

Theorem 10. *If X and Y are jointly distributed according to (10), then a-th raw moment of $W = \frac{X}{Y}$ is*

$$\mu'_{a;\rho}(W) = \Gamma(a/\gamma+1)\Gamma(1-a/\gamma)\left[\frac{\Gamma(\alpha-a/\gamma)\Gamma(\alpha+a/\gamma)}{\Gamma^2(\alpha)} + \rho\left(\frac{\Gamma(2\alpha-a/\gamma)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha-a/\gamma)}{\Gamma(\alpha)}\right)\left(\frac{\Gamma(2\alpha+a/\gamma)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha+a/\gamma)}{\Gamma(\alpha)}\right)\right]. \tag{29}$$

Theorem 11. *If X and Y are jointly distributed according to (10), then the a-th raw moment of $Z = \frac{X}{X+Y}$ is*

$$\mu'_{a;\rho}(Z) = \sum_{k=0}^{\infty}\binom{a-1+k}{k}(-1)^k\Gamma(a/\gamma+1)\Gamma(1-a/\gamma)(\mathcal{C}_1+\rho\mathcal{C}_2\mathcal{C}_3) \tag{30}$$

where

(i) $\mathcal{C}_1 = \dfrac{\Gamma(\alpha-(a+k)/\gamma)\Gamma(\alpha+(a+k)/\gamma)}{\Gamma^2(\alpha)}$

(ii) $\mathcal{C}_2 = \left(\frac{\Gamma(2\alpha - (a+k)/\gamma)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha - (a+k)/\gamma)}{\Gamma(\alpha)} \right)$

(iii) $\mathcal{C}_3 = \left(\frac{\Gamma(2\alpha + (a+k)/\gamma)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha + (a+k)/\gamma)}{\Gamma(\alpha)} \right)$.

Proof: Notice that

$$\begin{aligned} E\left(Z^a\right) &= E\left(X^a \cdot (X+Y)^{-a}\right) = E\left(\left(\frac{X}{Y}\right)^a \left(\frac{X}{Y}+1\right)^{-a}\right) \\ &= E\left(\left(\frac{X}{Y}\right)^a \sum_{k=0}^{\infty} \binom{a-1+k}{k} (-1)^k \left(\frac{X}{Y}\right)^k\right) \\ &= E\left(\sum_{k=0}^{\infty} \binom{a-1+k}{k} (-1)^k \left(\frac{X}{Y}\right)^{a+k}\right) \\ &= \sum_{k=0}^{\infty} \binom{a-1+k}{k} (-1)^k E\left(W^{a+k}\right). \end{aligned}$$

Using Theorem 9, the result in (30) follows.

Theorem 12. *If X and Y are jointly distributed according to (10), then the a-th raw moment of $R = X+Y$ is*

$$\begin{aligned} \mu'_{a;\rho}(R) = \tau^a \sum_{i=0}^{a} \binom{a}{i} \Gamma\left(\frac{i}{\gamma}+1\right) \Gamma\left(\frac{a-i}{\gamma}+1\right) \\ \left[\frac{\Gamma\left(\alpha - \frac{i}{\gamma}\right)\Gamma\left(\alpha - \frac{a-i}{\gamma}\right)}{\Gamma^2(\alpha)} + \rho\left(\frac{\Gamma\left(2\alpha - \frac{i}{\gamma}\right)}{\Gamma(2\alpha)} - \frac{\Gamma\left(\alpha - \frac{i}{\gamma}\right)}{\Gamma(\alpha)}\right)\right. \\ \left.\left(\frac{\Gamma\left(2\alpha - \frac{a-i}{\gamma}\right)}{\Gamma(2\alpha)} - \frac{\Gamma\left(\alpha - \frac{a-i}{\gamma}\right)}{\Gamma(\alpha)}\right)\right]. \end{aligned} \tag{31}$$

Proof: Since $R^a = (X+Y)^a = \sum_{i=0}^{a} \binom{a}{i} X^i \cdot Y^{a-i}$, then

$$\mu'_{a;\rho}(R) = E\left(R^a\right) = \sum_{i=0}^{a} \binom{a}{i} E\left(X^i Y^{a-i}\right) = \sum_{i=0}^{a} \binom{a}{i} \mu'_{i,a-i;\rho}\left(X^i Y^{a-i}\right).$$

By putting $a = i$ and $b = a-i$ in (27), the result follows.

4 Conclusion

In this paper, we have derived the probability density functions of the sum, product, and quotient of two random variables both having Singh-Maddala distribution. We also derived each corresponding rth raw moment. These moments are useful in the estimation of the sum, products or quotients of X and Y. Regardless of the application setting of random variables, the results that are expressed in terms of beta and hypergeometric functions can be implemented readily in most common software as these are special built-in functions. It is worth to mention that using FGM copula for accommodating the association for two random variables X and Y applied to sum, product and quotient of X and Y is new. One setback though of using this copula is due to its weak dependence [8]. However, through FGM, we produced fairly simple and elegant results. We may point out that our choice of using FGM is due to mathematical convenience. Hence, we suggest that one can always consider a rather general copula that better capture the association of random variables. We remark that equation (23) is the marginal pdf of Lomax distributed random variables X and Y. It is a special case of Singh-Maddala distribution when $\gamma = 1$. Furthermore, it remains an open problem on how to compute equation (25).

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