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The Λ_H-Symmetric Matrices

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Abstract

Let $\mathcal{H}_n$ be the set of all nonsingular Hermitian matrices in $M_n(\mathbb{C})$. Let $H \in \mathcal{H}_n$ be given. Define $\Lambda_H : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $\Lambda_H(A) = H^{-1}A^*H$. We say that an $A \in M_n(\mathbb{C})$ is Λ_H*-symmetric* if $\Lambda_H(A) = A$. Let $A = SJS^{-1}$ be such that J is the Jordan Canonical Form of A. We show that there exists a nonsingular $P \in M_n(\mathbb{C})$ such that P^*HP has the same block structure as J. Let

$$\mathcal{G}_A = \{H \in \mathcal{H}_n : A \text{ is } \Lambda_H - \text{symmetric}\},$$

and let $G \in \mathcal{G}_A$ be given. We also determine the possible inertia of G.

Key words: Λ_H-symmetric matrices

1 Introduction

Let $M_{n,m}(\mathbb{F})$ be the set of all n-by-m matrices over $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$. When $m = n$, we write $M_n(\mathbb{F}) \equiv M_{n,n}(\mathbb{F})$. Let $\mathcal{H}_n$ be the set of all nonsingular Hermitian matrices in $M_n(\mathbb{C})$. Let $H \in \mathcal{H}_n$ be given. Define $\Lambda_H : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $\Lambda_H(A) = H^{-1}A^*H$. One checks that for $A, B \in M_n(\mathbb{C})$, we have $\Lambda_H(AB) = \Lambda_H(B)\Lambda_H(A)$ and that $\Lambda_H(\Lambda_H(A)) = A$. We say that an $A \in M_n(\mathbb{C})$ is Λ_H*-symmetric* if $\Lambda_H(A) = A$.

When $H = I$, the Λ_H-symmetric matrices are the Hermitian matrices. We note that Λ_H-symmetric matrices are also called H*-self adjoint matrices* [2]. They are used in identifying spectral properties of matrix polynomials with Hermitian coefficients and in solving continuous algebraic Ricatti equations.

We let $J_n(\lambda) \in M_n(\mathbb{C})$ be the (upper triangular) Jordan block corresponding to the eigenvalue λ. Let $A \in M_n(\mathbb{C})$ be given. There exists an $H \in \mathcal{H}_n$ such that A is Λ_H-symmetric if and only if A is similar to a real matrix [4, Theorem 4.1.7], so that the Jordan Canonical Form of A, say J, contains only blocks of the form (i) $J_l(\alpha)$ with $\alpha \in \mathbb{R}$, or (ii) $J_k(\lambda) \oplus J_k(\overline{\lambda})$ with $\lambda \notin \mathbb{R}$. It is known that H is $*$congruent to a block diagonal matrix that is conformal to J [1, Theorem 2.1]. This result is attributed to Weierstrass and Kronecker and the proof uses matrix pencils. We present an elementary approach.

Let $H \in M_n(\mathbb{C})$ be Hermitian. The inertia of H is the ordered triple of nonnegative integers $i(H) \equiv (i_+(H), i_-(H), i_0(H))$where $i_+(H), i_-(H)$ and $i_0(H)$ are respectively the number of positive, negative and zero eigenvalues of H, counting multiplicities. If $i(H) = (p, q, r)$, then $i(-H) = (q, p, r)$. Suppose that $A, B \in M_n(\mathbb{C})$ are both Hermitian. Then $A \oplus B$ is also Hermitian. Moreover, if $i(A) = (p_1, q_1, r_1)$ and if $i(B) = (p_2, q_2, r_2)$, then $i(A \oplus B) = (p_1 + p_2, q_1 + q_2, r_1 + r_2)$. For Hermitian $H, G \in M_n(\mathbb{C})$, there exists a nonsingular $X \in M_n(\mathbb{C})$ such that $H = X^*GX$ if and only if $i(H) = i(G)$ [4, Theorem 4.5.8].

We are also interested in the following problem. Let $A \in M_n(\mathbb{C})$ be given. Suppose that there exists an $H \in \mathcal{H}_n$ such that A is Λ_H-symmetric. Let $\mathcal{G}_A = \{H \in \mathcal{H}_n : A \text{ is } \Lambda_H - \text{symmetric}\}$, and let $G \in \mathcal{G}_A$ be given. We wish to determine the possible inertia of G.

Let $H \in \mathcal{H}_n$ be given. Let $X \in M_n(\mathbb{C})$ be nonsingular. Then $X^*HX \in \mathcal{H}_n$. Let $A \in M_n(\mathbb{C})$ be given. One checks that $A = H^{-1}A^*H$ if and only if $X^{-1}AX = (X^*HX)^{-1}\left(X^{-1}AX\right)^*(X^*HX)$, that is, A is Λ_H-symmetric if and only if $X^{-1}AX$ is $\Lambda_{(X^*HX)}$-symmetric.

1.1 Notation

We denote the set of positive integers by $\mathbb{N}$. Let $n \in \mathbb{N}$ be given. For $A \in M_n(\mathbb{C})$, we denote the spectrum of A by $\sigma(A)$. We define the n-by-n backward identity matrix $B_n = [b_{ij}] \in M_n(\mathbb{C})$ with

$$b_{ij} = \begin{cases} 1 & \text{if } i + j = n + 1 \\ 0 & \text{otherwise} \end{cases} .$$

Notice that $B_n \in \mathcal{H}_n$ and that $B_n^{-1} = B_n$. The eigenvalues of B_n are 1 and -1. If $n = 2k$ is even, then $\operatorname{tr}(B_n) = 0$ and $i(B_n) = (k, k, 0)$. If $n = 2k + 1$ is odd, then $\operatorname{tr}(B_n) = 1$ and $i(B_n) = (k + 1, k, 0)$.

Let $A = [a_{ij}] \in M_n(\mathbb{C})$ be given. Then $B_nAB_n = [a_{n+1-i,n+1-j}]$. In particular, we have $B_nJ_n(\lambda)B_n = (J_n(\lambda))^T$, so that for every nonnegative integer k, we have

$$\left((J_n(\lambda))^T\right)^k B_n = B_n (J_n(\lambda))^k . \tag{1}$$

We use the convention that when $k = 0$, we have $(J_n(\lambda))^k = I_n$. Now, notice that $B_nJ_n(0)$ is a matrix with 1 in the diagonal just below the nonzero diagonal of B_n and 0 elsewhere. In general, $B_nJ_n^r(0)$ is a matrix with 1 in the r^{th} diagonal $(1 \le r \le n-1)$ below the nonzero diagonal of B_n and 0 elsewhere. Moreover, $J_n^r(0)B_n$ is a matrix with 1 in the r^{th} diagonal $(1 \le r \le n - 1)$ above the nonzero diagonal of B_n and 0 elsewhere.

Definition 1. *Let* $\overrightarrow{a} = [a_1 \ \cdots \ a_n]^T \in \mathbb{C}^n$ *be given. Then*

1. $T_U(\overrightarrow{a}) \equiv \sum_{i=1}^n a_i J_n^{i-1}(0)$ *is called upper Toeplitz.*

2. $T_L(\overrightarrow{a}) \equiv B_nT_U(\overrightarrow{a})B_n = \sum_{i=1}^n a_i \left(J_n^{i-1}(0)\right)^T$ *is called lower Toeplitz.*

3. $H_L(\overrightarrow{a}) \equiv B_n T_U(\overrightarrow{a}) = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 0 & 0 & \cdots & a_1 & a_2 \\ \vdots & \vdots & \swarrow & a_2 & \vdots \\ 0 & a_1 & \swarrow & \swarrow & a_{n-1} \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix} \in M_n(\mathbb{C})$ *is called lower Hankel.*

4. $H_U(\overrightarrow{a}) \equiv T_U(\overrightarrow{a}) B_n$ *is called upper Hankel.*

Let $A, B \in M_n(\mathbb{C})$ be given. We say that A is *congruent* to B if there exist nonsingular $X, Y \in M_n(\mathbb{C})$ such that $A = XBY$. One checks that congruence is an equivalence relation in $M_n(\mathbb{C})$. Moreover, if A and B are congruent, then either both are singular or both are nonsingular. In fact, A is congruent to B if and only if A and B have the same rank.

For each $\overrightarrow{a} \in \mathbb{C}^n$, notice that $T_U(\overrightarrow{a})$, $T_L(\overrightarrow{a})$, $H_L(\overrightarrow{a})$, and $H_U(\overrightarrow{a})$ are all congruent. If $\overrightarrow{a} = [a_1 \cdots a_n]^T$, then $T_U(\overrightarrow{a})$ is nonsingular if and only if $a_1 \neq 0$. Notice that a lower or an upper Hankel matrix is symmetric, and hence, is Hermitian if and only if $\overrightarrow{a} \in \mathbb{R}^n$. The set of all upper Toeplitz matrices is a subspace of $M_n(\mathbb{C})$. Moreover, a product of upper Toeplitz matrices is an upper Toeplitz matrix. Each respective set (that is, the set of lower Toeplitz, the set of lower Hankel, and the set of upper Hankel) is a subspace of $M_n(\mathbb{C})$. However, the product of two lower Hankel matrices is not necessarily lower Hankel.

Let $\overrightarrow{a}, \overrightarrow{b} \in \mathbb{C}^n$ be given. Then $T_U(\overrightarrow{a})H_U(\overrightarrow{b}) = T_U(\overrightarrow{a})T_U(\overrightarrow{b})B_n$ is upper Hankel. The following table gives the type of product achieved when a type of factor on the first column is multiplied to a type of factor on the top row. In [3], a similar table for products of triangular matrices is given.

Proposition 2. *Let H_L be a lower Hankel matrix, let H_U be an upper Hankel matrix, let T_L be a lower Toeplitz matrix, and let T_U be an upper Toeplitz matrix. Then we have the following multiplication table for the form of products of matrices. The column on the left represents the left factor of the product, while the top row represents the right factor of the product.*

$\bullet$	H_L	H_U	T_L	T_U
H_L		T_L		H_L
H_U	T_U		H_U	
T_L	H_L		T_L	
T_U		H_U		T_U

Let $A \in M_n(\mathbb{C})$ be upper Toeplitz and let $B \in M_n(\mathbb{C})$ be lower Hankel. Then A^T is lower Toeplitz, BA is lower Hankel, and $A^T BA$ is lower Hankel.

Let n and m be given integers with $n \geq m$. Let $T \in M_m(\mathbb{C})$ be lower Toeplitz. Set $A \equiv [T\ 0] \in M_{m,n}(\mathbb{C})$. Let k be an integer with $k \leq m$. Write $A = \begin{bmatrix} T_1 & 0 \\ T_2 & T_3 \end{bmatrix}$, with $T_3 \in M_k(\mathbb{C})$, and notice that T_3 is also lower Toeplitz.

Lemma 3. *Let n, m_1, and m_2 be given integers with $n \geq m_1 \geq m_2$. Let $T_1 \in M_{m_1}(\mathbb{C})$ be lower Toeplitz and let $H_1 \in M_{m_2}(\mathbb{C})$ be lower Hankel. Let $T = [T_1\ 0] \in M_{m_1,n}(\mathbb{C})$ and let $H = \begin{bmatrix} 0 \\ H_1 \end{bmatrix} \in M_{n,m_2}(\mathbb{C})$. Then $TH = \begin{bmatrix} 0 \\ H_2 \end{bmatrix} \in M_{m_1,m_2}(\mathbb{C})$ and $H_2 \in M_{m_2}(\mathbb{C})$ is lower Hankel.*

Proof: Write $T \equiv \begin{bmatrix} T_2 & 0 \\ T_3 & T_4 \end{bmatrix}$, with $T_4 \in M_{m_2}(\mathbb{C})$. Then T_4 is lower Toeplitz and $TH = \begin{bmatrix} 0 \\ T_4 H_1 \end{bmatrix}$. Proposition 2 guarantees that $H_2 \equiv T_4 H_1$ is lower Hankel.

Let $n \geq m$ be given integers. Let $H_1 \in M_m(\mathbb{C})$ be lower Hankel, and set $H \equiv \begin{bmatrix} 0 \\ H_1 \end{bmatrix} \in M_{n,m}(\mathbb{C})$. Write $B_n = \begin{bmatrix} 0 & B_m \\ B_{n-m} & 0 \end{bmatrix}$. Then, $B_n H = \begin{bmatrix} B_m H_1 \\ 0 \end{bmatrix}$ and that $B_m H_1$ is upper Toeplitz. Conversely, let $T_1 \in M_m(\mathbb{C})$ be upper Toeplitz, and set $T \equiv \begin{bmatrix} T_1 \\ 0 \end{bmatrix} \in M_{n,m}(\mathbb{C})$. Write $B_n = \begin{bmatrix} 0 & B_{n-m} \\ B_m & 0 \end{bmatrix}$. Then, $B_n T = \begin{bmatrix} 0 \\ B_m T_1 \end{bmatrix}$ and that $B_m T_1$ is lower Hankel. Suppose $n \leq m$. Let $H_1 \in M_n(\mathbb{C})$ be lower Hankel and let $H = [0\ H_1] \in M_{n,m}(\mathbb{C})$. Then $B_n H = [0\ B_n H_1]$, and that $B_n H_1$ is upper Toeplitz. Conversely, if $T_1 \in M_n(\mathbb{C})$ is upper Toeplitz and if $T = [0\ T_1] \in M_{n,m}(\mathbb{C})$, then $B_n T = [0\ B_n T_1]$ and that $B_n T_1$ is lower Hankel.

Lemma 4. *Let n and m be given integers with $n \geq m$. Let $H_1 \in M_m(\mathbb{C})$ be lower Hankel and let $H = \begin{bmatrix} 0 \\ H_1 \end{bmatrix} \in M_{n,m}(\mathbb{C})$. There exists an upper Toeplitz $T_1 \in M_m(\mathbb{C})$ such that $T = \begin{bmatrix} T_1 \\ 0 \end{bmatrix} \in M_{n,m}(\mathbb{C})$ and $B_n T + H = 0$. Conversely, let $T_1 \in M_m(\mathbb{C})$ be upper Toeplitz and let $T = \begin{bmatrix} T_1 \\ 0 \end{bmatrix} \in M_{n,m}(\mathbb{C})$. There exists a lower Hankel $H_1 \in M_m(\mathbb{C})$ such that $H = \begin{bmatrix} 0 \\ H_1 \end{bmatrix} \in M_{n,m}(\mathbb{C})$ and $B_n T + H = 0$.*

Let $(n_1, \ldots, n_k) \in \mathbb{N}^k$ be given, and let $n = \sum_{i=1}^k n_i$. Let $\lambda_1, ..., \lambda_k \in \mathbb{C}$ be given, and let $J = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$. Suppose that

$$JX = XJ. \tag{2}$$

Write $X = [X_{st}]$ conformal to J. Then, $J_{n_s}(\lambda_s) X_{st} = X_{st} J_{n_t}(\lambda_t)$. If $\lambda_s \neq \lambda_t$, then $X_{st} = 0$. If $\lambda_s = \lambda_t$, then we have $J_{n_s}(0) X_{st} = X_{st} J_{n_t}(0)$. (i) If $n_s = n_t$, then $X_{st} = T_U(\overrightarrow{a})$ for some $\overrightarrow{a} \in \mathbb{C}^n$; (ii) if $n_s > n_t$, then $X_{st} = \begin{bmatrix} T \\ 0 \end{bmatrix}$ for some upper Toeplitz $T \in M_{n_t}(\mathbb{C})$; and (iii) if $n_s < n_t$, then $X_{st} = [0\ T]$ for some upper Toeplitz $T \in M_{n_s}(\mathbb{C})$.

Definition 5. *Let $\alpha = (n_1, \ldots, n_k) \in \mathbb{N}^k$ and $n = \sum_{i=1}^k n_i$. Let $1 \leq s, t \leq k$ be integers, let $H_{st} \in M_{n_s, n_t}(\mathbb{C})$ be given, and let $H = [H_{st}] \in M_n(\mathbb{C})$. Then H is said to be $\alpha-$blockwise upper Toeplitz if for every s and t, there exists an upper Toeplitz $T_{st} \in M_p(\mathbb{C})$ with $p = min\{n_s, n_t\}$ such that H_{st} has one of the following forms:*

1. T_{st} *if* $n_s = n_t$
2. $\begin{bmatrix} T_{st} \\ 0 \end{bmatrix}$ *if* $n_s > n_t$
3. $[0\ T_{st}]$ *if* $n_s < n_t$.

The set of $\alpha-$blockwise upper Toeplitz matrices is a subspace of $M_n(\mathbb{C})$. We are interested in two particular types of Jordan blocks: (i) for a given $\lambda \in \mathbb{R}$,

$$L_1(\lambda) = J_{n_1}(\lambda) \oplus \cdots \oplus J_{n_p}(\lambda) \tag{3}$$

with $n_1 \geq \cdots \geq n_p$, and (ii) for $\beta \in \mathbb{C} \setminus \mathbb{R}$,

$$L_2(\beta) = \left(J_{k_1}(\beta) \oplus J_{k_1}\left(\overline{\beta}\right)\right) \oplus \cdots \oplus \left(J_{k_q}(\beta) \oplus J_{k_q}\left(\overline{\beta}\right)\right) \tag{4}$$

with $k_1 \geq \cdots \geq k_q$. In equation (2), suppose that $\lambda_1 = \cdots = \lambda_k$. Then X is a solution to equation (2) if and only if X is $\alpha-$blockwise upper Toeplitz. Suppose that J has the form of $L_2(\beta)$ in equation (4), and let $\alpha = (k_1, k_1, ..., k_q, k_q)$. Then $X = [X_{st}]$ (conformal to J) is a solution to equation (2), if and only if X is $\alpha-$blockwise upper Toeplitz and $X_{st} = 0$ whenever $s + t$ is odd.

For $\alpha = (n_1, \ldots, n_k) \in \mathbb{N}^k$, we assume that $n_1 \geq \cdots \geq n_k$.

We look at solutions to the equation

$$J^* Y = Y J. \tag{5}$$

Write $Y = [Y_{st}]$ conformal to J. Then, we have $(J_{n_s}(\lambda_s))^* Y_{st} = Y_{st} J_{n_t}(\lambda_t)$. If $\lambda_t \neq \overline{\lambda_s}$, then $Y_{st} = 0$. If $\lambda_t = \overline{\lambda_s}$, then $J_{n_s}^T(0) Y_{st} = Y_{st} J_{n_t}(0)$. Set $Z = B_{n_s} Y_{st}$. Then, we have $Z J_{n_t}(0) = B_{n_s} Y_{st} J_{n_t}(0) = B_{n_s} J_{n_s}^T(0) Y_{st} = J_{n_s}(0) B_{n_s} Y_{st} = J_{n_s}(0) Z$ so that $Z J_{n_t}(0) = J_{n_s}(0) Z$. Hence, (i) if $n_s = n_t$, then $Y_{st} = H_L(\overrightarrow{a})$ for some $\overrightarrow{a} \in \mathbb{C}^n$; (ii) if $n_s > n_t$, then $Y_{st} = \begin{bmatrix} 0 \\ H \end{bmatrix}$ for some lower Hankel $H \in M_m(\mathbb{C})$; and (iii) if $n_s < n_t$, then $Y_{st} = [0\ H]$ for some lower Hankel $H \in M_n(\mathbb{C})$.

Definition 6. *Let $\alpha = (n_1, \ldots, n_k) \in \mathbb{N}^k$ and $n = \sum_{i=1}^k n_i$. Let $1 \leq s, t \leq k$ be integers, let $H_{st} \in M_{n_s, n_t}(\mathbb{C})$ be given, and let $H = [H_{st}] \in M_n(\mathbb{C})$. Then H is said to be $\alpha-$blockwise lower Hankel if for every s and t, there exists a lower Hankel $G_{st} \in M_p(\mathbb{C})$ with $p = min\{n_s, n_t\}$ such that H_{st} has one of the following forms:*

1. G_{st} *if* $n_s = n_t$
2. $\begin{bmatrix} 0 \\ G_{st} \end{bmatrix}$ *if* $n_s > n_t$
3. $[0\ G_{st}]$ *if* $n_s < n_t$.

A solution to equation (5) is $\alpha-$blockwise lower Hankel. The set of $\alpha-$blockwise lower Hankel matrices is a subspace of $M_n(\mathbb{C})$.

Suppose that $J = L_2(\beta)$ in equation (4). Let $\alpha = (k_1, k_1, ..., k_p, k_p)$. If Y satisfies equation (5), then Y is $\alpha-$blockwise lower Hankel. Moreover, if $Y = [Y_{st}]$ is conformal to J, then $Y_{st} = 0$ whenever $s + t$ is even.

Definition 7. *Let $H = [H_{st}] \in M_n(\mathbb{C})$ be given. We say that H is checkered if $H_{st} = 0$ whenever $s + t$ is even, that is, whenever s and t are both odd or both even.*

2 Λ_H-Symmetric Matrices

Let $H \in \mathcal{H}_n$ be given. Let $A \in M_n(\mathbb{C})$ be Λ_H- symmetric, so that $H^{-1} A^* H = A$. Then A is similar to a real matrix. There exist a nonsingular $S \in M_n(\mathbb{C})$, scalars $\lambda_1, ..., \lambda_k \in \mathbb{R}$ and $\beta_1, ..., \beta_t \in \mathbb{C} \setminus \mathbb{R}$; Jordan matrices $J_{1i}(\lambda_i)$ that have the form $L_1(\lambda_i)$ in equation (3) for $i = 1, ..., k$, Jordan matrices $J_{2j}(\beta_j)$ that have the form $L_2(\beta_j)$ in equation (4) for $j = 1, ..., t$ such that $K_1 = J_{11}(\lambda_1) \oplus \cdots \oplus J_{1k}(\lambda_k)$, $K_2 = J_{21}(\beta_1) \oplus \cdots \oplus J_{2t}(\beta_t)$, $J = K_1 \oplus K_2$, and $A = SJS^{-1}$ is the Jordan Canonical Form of A.

Let $C = S^*HS$, so that $C \in \mathcal{H}_n$. Then, $J^*C = CJ$, so that $C = C_1 \oplus C_2$ with C_i having the same size as K_i for $i = 1, 2$. Moreover, $C_1 = C_{11} \oplus \cdots \oplus C_{1k}$, with C_{1i} having the same size as $J_{1i}(\lambda_i)$ for each $i = 1, ..., k$. We also have $C_2 = C_{21} \oplus \cdots \oplus C_{2t}$, with C_{2j} having the same size as $J_{2j}(\beta_j)$ for each $j = 1, ..., t$. We show that for each $i = 1, ..., k$, there exist a diagonal D_{1i} having the same block sizes as $J_{1i}(\lambda_i)$ and a nonsingular V_i that commutes with $J_{1i}(\lambda_i)$ such that $C_{1i} = V_i^* D_{1i} V_i$. We also show that for each $j = 1, ..., t$, there exist a diagonal D_{2j} having the same block sizes as $J_{2j}(\beta_j)$ and a nonsingular W_j such that $C_{2j} = W_j^* D_{2j} W_j$. Notice that W_j does not commute with $J_{2j}(\beta_j)$, otherwise, D_{2j} is checkered and is 0.

Let $\alpha_{1i} = (n_{i1}, \ldots, n_{ip})$ and let $\alpha_{2j} = (n_{j1}, n_{j1}, \ldots, n_{jq}, n_{jq})$. For each i and j, we have that C_{ij} is Hermitian, nonsingular, and $\alpha_{ij}-$blockwise lower Hankel. We look at a single block nonsingular lower Hankel matrix.

Let $\overrightarrow{a} = [a_1 \cdots\ a_n]^T \in \mathbb{C}^n$. Let $2 \le k \le n$ be an integer and let $A_k = I + \alpha_k J_n^{k-1}(0)$. Notice that A_k commutes with $T_U(\overrightarrow{a})$. Equation (1) guarantees that $A_k^T B_n = B_n A_k$. Since $\mathcal{H}_L(\overrightarrow{a}) = B_n T_U(\overrightarrow{a})$, we have

$$\mathcal{H}_L\left(\overrightarrow{b}\right) \equiv A_k^T \mathcal{H}_L(\overrightarrow{a}) A_k = \mathcal{H}_L(\overrightarrow{a})\left(I + 2\alpha_k J_n^{k-1}(0) + \alpha_k^2 J_n^{2k-2}(0)\right). \tag{6}$$

Hence $b_1 = a_1$ and $b_k = a_k + 2\alpha_k a_1$. Suppose that $a_1 \ne 0$. If $a_2 \ne 0$, then take $k = 2$ and choose $\alpha_2 = -\frac{a_2}{2a_1}$ so that $b_2 = 0$. If $b_3 \ne 0$, then take $k = 3$ and consider $\mathcal{H}_L(\overrightarrow{c}) \equiv A_3^T \mathcal{H}_L\left(\overrightarrow{b}\right) A_3$. Choose $\alpha_3 = -\frac{b_3}{2b_1}$ so that $c_3 = c_2 = 0$. Notice that A_2A_3 is nonsingular and upper Toeplitz. Continuing in this manner, there exists a nonsingular upper Toeplitz $P \in M_n(\mathbb{C})$ such that $P^T \mathcal{H}_L(\overrightarrow{a}) P = a_1 B_n$. If $\overrightarrow{a} \in \mathbb{R}^n$, then each α_k is real, and P is also real.

Lemma 8. *Let $\overrightarrow{a} = [a_1 \cdots\ a_n]^T \in \mathbb{C}^n$ be given with $a_1 \ne 0$. There exists a nonsingular upper Toeplitz $P \in M_n(\mathbb{C})$ such that $P^T \mathcal{H}_L(\overrightarrow{a}) P = a_1 B_n$. If $\overrightarrow{a} \in \mathbb{R}^n$, then P may be chosen to be real.*

We look at the blocks C_{1i}. Suppose that $J = L_1(\lambda)$ as in equation (3). Let $\alpha = (n_1, \ldots, n_k)$, let $n = \sum_{i=1}^k n_i$, and let $\delta = (n_2, ..., n_k)$. Suppose that $H = [H_{st}] \in \mathcal{H}_n$ (conformal to J) satisfies $J^*H = HJ$ so that H is $\alpha-$blockwise lower Hankel. Notice that each H_{ii} is lower Hankel and Hermitian, so that each H_{ii} is also real. Write $H = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, with $M_{11} = H_{11}$. Then M_{22} is Hermitian and $\delta-$blockwise lower Hankel.

If H_{1k} has the same size as H_{1j}, then adding a multiple of column k to column j (and putting it in column j) results in a matrix that is still $\alpha-$blockwise lower Hankel. This is the same as multiplying H by $P = [P_{st}]$ on the right, where $P_{ss} = I_{n_s}$, $P_{kj} = aI_{n_j}$, and the other P_{st} are 0. Similarly, if H_{j1} has the same size as H_{k1}, then adding a multiple of row j to row k (and putting it in row k) results in a matrix that is still $\alpha-$blockwise lower Hankel. This is the same as multiplying H by $Q = [Q_{st}]$ on the left, where $Q_{ss} = I_{n_s}$, $Q_{kj} = aI_{n_k}$, and the other Q_{st} are 0. Both P and Q commute with J.

Let $\overrightarrow{h} = [h_1 \cdots h_{n_1}]^T \in \mathbb{R}^{n_1}$ and let $H_{11} = \mathcal{H}_L\left(\overrightarrow{h}\right)$. We show that without loss of generality, we may assume that H_{11} is nonsingular.

If $h_1 \ne 0$, then H_{11} is nonsingular. Suppose that $h_1 = 0$. There exist a $k \in \{2, \ldots, n\}$ such that $n_1 = n_2 = \cdots = n_k$, a $\overrightarrow{b} = [b_1 \cdots b_{n_1}]^T \in \mathbb{C}^{n_1}$ with $b_1 \ne 0$, and $H_{1k} = \mathcal{H}_L\left(\overrightarrow{b}\right)$; otherwise, the first row of H is 0 and H is singular. Let $\overrightarrow{c} = [c_1 \cdots\ c_{n_1}]^T \in \mathbb{R}^{n_1}$ be given and let $H_{kk} = \mathcal{H}_L(\overrightarrow{c})$. Let $P_1 = [P_{st}] \in M_n(\mathbb{C})$ be such that $P_{ss} = I_{n_s}$ for all s,

$P_{k1} = aI_{n_1}$, and the other blocks are 0. Then P_1 commutes with J and $P_1^* H P_1 \in \mathcal{H}_n$ is $\alpha-$blockwise lower Hankel. The $(1,1)$ block of $P_1^* H P_1$ is $G_1 = H_{11} + aH_{1k} + \overline{a} H_{1k}^* + |a|^2 H_{kk}$. Let $\overrightarrow{d} = [d_1 \cdots d_{n_1}]^T \in \mathbb{R}^{n_1}$ and let $G_1 = \mathcal{H}_L(\overrightarrow{d})$. Then, $d_1 = ab_1 + \overline{ab_1} + |a|^2 c_1$. If $c_1 = 0$, then choose $a = \overline{b_1}$, otherwise, choose a so that $\mathrm{Re}(ab_1)$ (the real part of ab_1) has the same sign as c_1. Hence, G_1 may be assumed to be nonsingular.

Let $H = [H_{st}] \in \mathcal{H}_n$ be $\alpha-$blockwise lower Hankel and suppose that H_{11} is nonsingular. Lemma 8 guarantees that there exists a nonsingular upper Toeplitz $T \in M_{n_1}(\mathbb{R})$ such that $T^* H_{11} T = h_1 B_{n_1}$. Set $P_2 = T \oplus I_{n-n_1}$. Then P_2 commutes with J and $P_2^* H P_2 = \begin{bmatrix} h_1 B_{n_1} & G_4 \\ G_4^* & G_3 \end{bmatrix}$ where $G_4 = [(T^* H_{12}) \ \cdots \ (T^* H_{1k})]$ and $G_3 = [H_{(s+1)(t+1)}]$ for $s, t \in \{1, \ldots, k-1\}$ is $\delta-$blockwise lower Hankel. We systematically reduce the block components of G_4. Because T^* is lower Toeplitz, Lemma 3 ensures that for each $i = 2, ..., k$, we have $T^* H_{1i} = \begin{bmatrix} 0 \\ F_i \end{bmatrix}$, where $F_i \in M_n(\mathbb{C})$ is lower Hankel. Lemma 4 guarantees that there exists $T_2 \in M_{n_1, n_2}$ such that $T_2 = \begin{bmatrix} S_2 \\ 0 \end{bmatrix}$ with $S_2 \in M_{n_2}(\mathbb{C})$ an upper Toeplitz, and $h_1 B_{n_1} T_2 + T^* H_{12} = 0$. Set $N_2 = [T_2 \ \ 0 \ \cdots \ 0] \in M_{n_1, n-n_1}$ and set $Q_2 = \begin{bmatrix} I_{n_1} & N_2 \\ 0 & I_{n-n_1} \end{bmatrix}$. Then Q_2 is $\alpha-$blockwise upper Toeplitz and commutes with J. Moreover, $Q_2^*(P_2^* H P_2) Q_2 = \begin{bmatrix} h_1 B_{n_1} & G_5 \\ G_5^* & G_6 \end{bmatrix}$, where $G_5 = [0 \ (T^* H_{13}) \ \cdots \ (T^* H_{1k})]$ and $G_6 = \begin{bmatrix} C_1 & C_2 \\ C_2^* & C_3 \end{bmatrix}$, with $C_1 = -T_2^* T^* H_{12} + H_{12}^* T T_2 + H_{22}$ (a lower Hankel), $C_2 = [(H_{23} + T_2^* T^* H_{13}) \ \cdots \ (H_{2k} + T_2^* T^* H_{1k})]$, and $C_3 = [H_{(s+2)(t+2)}]$ for $s, t \in \{1, \ldots, k-2\}$. One checks that G_6 is a $\delta-$blockwise lower Hankel. We apply a similar process to zero-out the entries of G_5 while maintaining that G_6 is a $\delta-$blockwise lower Hankel matrix. At each step, the matrix we use in the reduction commutes with J.

Conversely, suppose that there exists a nonsingular $P \in M_n(\mathbb{C})$ such that $JP = PJ$ and $P^* H P = x B_{n_1} \oplus G$, with x real and $G \in \mathcal{H}_{n-n_1}$ is $\delta-$blockwise lower Hankel. Then $H \in \mathcal{H}_n$ and $P^* H P$ is $\alpha-$blockwise lower Hankel so that $J^* P^* H P = P^* H P J$, that is, $J^* H = H J$. Hence, H is $\alpha-$blockwise lower Hankel.

Lemma 9. *Let $\lambda \in \mathbb{R}$ and $\alpha = (n_1, \ldots, n_k) \in \mathbb{N}^k$ be given. Set $J = J_{n_1}(\lambda) \oplus \cdots \oplus J_{n_k}(\lambda)$, set $n = \sum_{i=1}^k n_i$, and set $\delta = (n_2, \ldots, n_k)$. Then $H \in \mathcal{H}_n$ is $\alpha-$blockwise lower Hankel if and only if there exist a nonsingular $P \in M_n(\mathbb{C})$, a nonzero $x \in \mathbb{R}$ and a $\delta-$blockwise lower Hankel $G \in \mathcal{H}_{n-n_1}$ such that $JP = PJ$ and $P^* H P = x B_{n_1} \oplus G$.*

If $x > 0$, then we may choose P such that $JP = PJ$ and $P^* H P = B_{n_1} \oplus G$. If $x < 0$, then we may choose P such that $JP = PJ$ and $P^* H P = -B_{n_1} \oplus G$.

A repeated application of Lemma 9 shows the following.

Theorem 10. *Let $\lambda \in \mathbb{R}$ and $\alpha = (n_1, \ldots, n_k) \in \mathbb{N}^k$ be given. Set $J = J_{n_1}(\lambda) \oplus \cdots \oplus J_{n_k}(\lambda)$ and set $n = \sum_{i=1}^k n_i$. Then $H \in \mathcal{H}_n$ is $\alpha-$blockwise lower Hankel if and only if there exist a nonsingular $P \in M_n(\mathbb{C})$ and nonzero $x_1, ...x_k \in \mathbb{R}$ such that $JP = PJ$ and $P^* H P = x_1 B_{n_1} \oplus \cdots \oplus x_k B_{n_k}$.*

We now look at the blocks C_{2j}. Suppose that $J = L_2(\beta)$ as in equation (4). Let $\alpha = (k_1, k_1, \ldots, k_p, k_p)$, let $n = 2 \sum_{i=1}^p k_i$, and let $\delta = (k_2, k_2, ..., k_p, k_p)$. Suppose that $H = [H_{st}] \in \mathcal{H}_n$ (conformal to J) satisfies $J^* H = H J$ so that H is $\alpha-$blockwise lower Hankel and checkered.

We follow the discussion after Lemma 8. Let $\overrightarrow{h} = [h_1 \cdots h_{k_1}]^T \in \mathbb{C}^{k_1}$ and let $H_{12} = \mathcal{H}_L\left(\overrightarrow{h}\right)$. First, we show that it is without loss of generality to assume that H_{12} is nonsingular. If H_{12} is singular and if $k_1 > k_2$, then H is singular. Hence, there exists an integer t such that $k_1 = \cdots = k_t$ and $H_{1,2t}$ is nonsingular. Let Q be the matrix such that HQ adds column $2t$ of H to column 2 (and places it in column 2). Then $G = Q^*HQ = [G_{st}]$ is $\alpha-$blockwise lower Hankel, Hermitian, and is checkered. Moreover G_{12} is nonsingular.

Suppose that H_{12} is nonsingular so that $H_2 \equiv \begin{bmatrix} 0 & H_{12} \\ H_{12}^* & 0 \end{bmatrix}$ is also nonsingular. Lemma 8 guarantees that there exists an upper Toeplitz $P \in M_{k_1}(\mathbb{C})$ such that $P^T H_{12} P = h_1 B_{k_1}$. Set $P_1 = \overline{P} \oplus P$ and notice that $P_1^* H_2 P_1 = \begin{bmatrix} 0 & h_1 B_{k_1} \\ \overline{h_1} B_{k_1} & 0 \end{bmatrix}$. Set $P_2 = P_1 \oplus I_{n-2k_1}$. Then P_2 commutes with J and $G = P_2^* H P_2 = [G_{st}]$ is $\alpha-$blockwise lower Hankel, is Hermitian, and is checkered. Moreover, $G_{12} = h_1 B_{k_1}$.

We now use G_{12} to systematically zero out the first row of G. Next, we systematicaly zero out the second row using $G_{21} = G_{12}^*$. Suppose that Q_1 is the matrix that does the job. Then Q_1 commutes with J and $Q_1^* G Q_1$ is $\alpha-$blockwise lower Hankel, is Hermitian, is checkered, and the first two rows are zero except for G_{12} and G_{21}. Hence, the first and second columns of G except for G_{12} and G_{21}have also been zeroed out, that is, $Q_1^* G Q_1 = P_1^* H_2 P_1 \oplus A_2$, with A_2 $\alpha-$blockwise lower Hankel, Hermitian, and checkered.

Conversely, suppose that there exist a nonsingular $P \in M_n(\mathbb{C})$, a nonzero $x \in \mathbb{C}$, and a $\delta-$blockwise lower Hankel and checkered $G \in \mathcal{H}_{n-2n_1}$ such that $H_1 = \begin{bmatrix} 0 & x B_{n_1} \\ \overline{x} B_{n_1} & 0 \end{bmatrix}$, $JP = PJ$ and $P^*HP = H_1 \oplus G$. Then $H \in \mathcal{H}_n$ and P^*HP is $\alpha-$blockwise lower Hankel and checkered so that $J^*P^*HP = P^*HPJ$, that is, $J^*H = HJ$. Hence, H is $\delta-$blockwise lower Hankel and checkered.

Lemma 11. *Let $\beta \in \mathbb{C} \setminus \mathbb{R}$ and $\alpha = (n_1, n_1, \ldots, n_k, n_k) \in \mathbb{N}^{2k}$ be given. Set $J = \left(J_{n_1}(\beta) \oplus J_{n_1}\left(\overline{\beta}\right)\right) \oplus \cdots \oplus \left(J_{n_k}(\beta) \oplus J_{n_k}\left(\overline{\beta}\right)\right)$, set $n = 2\sum_{i=1}^k n_i$, and set $\delta = (n_2, n_2, \ldots, n_k, n_k)$. Then $H \in \mathcal{H}_n$ is $\alpha-$blockwise lower Hankel and checkered if and only if there exist a nonsingular $P \in M_n(\mathbb{C})$, a nonzero $x \in \mathbb{C}$, and a $\delta-$blockwise lower Hankel and checkered $G \in \mathcal{H}_{n-2n_1}$ such that $H_1 = \begin{bmatrix} 0 & x B_{n_1} \\ \overline{x} B_{n_1} & 0 \end{bmatrix}$, $JP = PJ$, and $P^*HP = H_1 \oplus G$.*

A repeated application of Lemma 11 shows the following.

Theorem 12. *Let $\beta \in \mathbb{C} \setminus \mathbb{R}$ and $\alpha = (n_1, n_1, \ldots, n_k, n_k) \in \mathbb{N}^{2k}$ be given. Set $J = \left(J_{n_1}(\beta) \oplus J_{n_1}\left(\overline{\beta}\right)\right) \oplus \cdots \oplus \left(J_{n_k}(\beta) \oplus J_{n_k}\left(\overline{\beta}\right)\right)$ and set $n = 2\sum_{i=1}^k n_i$. Then $H \in \mathcal{H}_n$ is $\alpha-$blockwise lower Hankel and checkered if and only if there exist a nonsingular $P \in M_n(\mathbb{C})$ and nonzero $x_1, ..., x_k \in \mathbb{C}$ such that for each $i = 1, ..., k$ we have $H_i = \begin{bmatrix} 0 & x_i B_{n_i} \\ \overline{x_i} B_{n_i} & 0 \end{bmatrix}$, $JP = PJ$, and $P^*HP = H_1 \oplus \cdots \oplus H_k$.*

We now use Theorems 10 and 12 to show the following.

Corollary 13. *Let $H \in \mathcal{H}_n$ be given. Let $A \in M_n(\mathbb{C})$ be given. Then A is Λ_H-symmetric if and only if there exist $\lambda_1, ..., \lambda_k \in \mathbb{R}$, $\beta_1, ..., \beta_t \in \mathbb{C} \setminus \mathbb{R}$, nonsingular $P \in M_n(\mathbb{C})$, nonzero $x_1, ..., x_k \in \mathbb{R}$, and nonzero $y_1, ..., y_t \in \mathbb{C}$ such that*

1. *$K_1 = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$, for $j = 1, ..., t$, we have $J_{2,j}(\beta_j) = J_{m_j}(\beta_j) \oplus J_{m_j}\left(\overline{\beta_j}\right)$, $K_2 = J_{2,1}(\beta_1) \oplus \cdots \oplus J_{2,t}(\beta_t)$, $J = K_1 \oplus K_2$, and $P^{-1}AP = J$ is the Jordan Canonical Form of A, and*

2. $H_1 = x_1 B_{n_1} \oplus \cdots \oplus x_k B_{n_k}$, *for each* $j = 1, ..., t$ *we have* $H_{2,j} = \begin{bmatrix} 0 & y_j B_{m_j} \\ \overline{y_j} B_{m_j} & 0 \end{bmatrix}$, $H_2 = H_{2,1} \oplus \cdots \oplus H_{2,t}$, *and* $P^* H P = H_1 \oplus H_2$.

Let $a \in \mathbb{C}$ be nonzero. Then $C \equiv \begin{bmatrix} 0 & aB_n \\ \overline{a}B_n & 0 \end{bmatrix}$ is nonsingular. Set $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\overline{a}} I_n & \frac{1}{\overline{a}} I_n \\ I_n & -I_n \end{bmatrix}$. Then, $Q^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} \overline{a} I_n & I_n \\ \overline{a} I_n & -I_n \end{bmatrix}$. Moreover, $D \equiv Q^* C Q = \begin{bmatrix} B_n & 0 \\ 0 & -B_n \end{bmatrix}$. Hence, H_2 in Corollary 13 is $*$congruent to $(B_{m_1} \oplus -B_{m_1}) \oplus \cdots \oplus (B_{m_t} \oplus -B_{m_t})$.

Corollary 14. *Let* $H \in \mathcal{H}_n$ *be given. Let* $A \in M_n(\mathbb{C})$ *be* Λ_H-*symmetric. Let* $\lambda_1, ..., \lambda_k \in \mathbb{R}$ *and* $\beta_1, ..., \beta_t \in \mathbb{C} \setminus \mathbb{R}$ *be given. Suppose that* $K_1 = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$ *and that for* $j = 1, ..., t$, *we have* $J_{2,j}(\beta_j) = J_{m_j}(\beta_j) \oplus J_{m_j}(\overline{\beta_j})$ *and* $K_2 = J_{2,1}(\beta_1) \oplus \cdots \oplus J_{2,t}(\beta_t)$. *Suppose that* $J = K_1 \oplus K_2$, *that* $S \in M_n(\mathbb{C})$ *is nonsingular, and that* $A = SJS^{-1}$ *is the Jordan Canonical Form of* A. *There exist a nonsingular* $Q \in M_n(\mathbb{C})$ *and nonzero* $x_1, ..., x_k \in \mathbb{R}$ *such that* $H_1 = x_1 B_{n_1} \oplus \cdots \oplus x_k B_{n_k}$, $H_2 = (B_{m_1} \oplus -B_{m_1}) \oplus \cdots \oplus (B_{m_t} \oplus -B_{m_t})$ *and* $Q^* H Q = H_1 \oplus H_2$.

2.1 Inertia of H

Let $H \in \mathcal{H}_n$ be given. Suppose that $A \in M_n(\mathbb{C})$ is Λ_H-symmetric. Let $\mathcal{G}_A = \{H \in \mathcal{H}_n : A \text{ is } \Lambda_H - \text{symmetric}\}$, and let $G \in \mathcal{G}_A$ be given. Corollaries 13 and 14 allow for an easy calculation of the inertia of G. Let $\lambda \in \mathbb{R}$ be nonzero. Then the inertia of λB_n depends only on the parity of n and on the sign of λ. If $n = 2k$ is even, then $i(\lambda B_n) = (k, k, 0)$, no matter the sign of λ. If $n = 2k+1$ is odd and if λ is positive, then $i(\lambda B_n) = (k+1, k, 0)$. If $n = 2k+1$ is odd and if λ is negative, then $i(\lambda B_n) = (k, k+1, 0)$. Let $a \in \mathbb{C}$ be nonzero. Then the inertia of $\begin{bmatrix} 0 & aB_n \\ \overline{a}B_n & 0 \end{bmatrix}$ is $(n, n, 0)$.

Let z be the number of Jordan blocks of A corresponding to a real eigenvalue and of odd size. If $z = 0$, then there is a positive integer k such that $n = 2k$ so that for every $G \in \mathcal{G}_A$, we have $i(G) = i(H) = (k, k, 0)$. Suppose $z > 0$. Let $G \in \mathcal{G}_A$ be given and let $P \in M_n(\mathbb{C})$ be such that $P^* G P = G_1 \oplus G_2$ is as in Corollary 13. Choose G_1 so that $x_i > 0$ whenever n_i is odd. If $i(G) = (p, q, 0)$, then we have $p - q = z$ and $p + q = n$. The other possible inertias can be achieved by changing the sign of one or more x_i. If k such x_i were changed, then the new inertia $(p_1, q_1, 0)$ satisfies $p_1 - q_1 = z - 2k$. Here, $k = 0, ..., z$.

Theorem 15. *Let* $H \in \mathcal{H}_n$ *be given and let* $A \in M_n(\mathbb{C})$ *be* Λ_H-*symmetric. Let* z *be the number of Jordan blocks of* A *corresponding to a real eigenvalue and of odd size. There exists* $G \in \mathcal{G}_A$ *with inertia* $(p, q, 0)$, *where* $p - q = z$ *and* $p + q = n$. *If* $F \in \mathcal{G}_A$, *then the possible inertia of* F *is* $i(F) = (p_1, q_1, 0)$, *where* $p_1 + q_1 = n$ *and* $p_1 - q_1 = z - 2k$ *with* $k = 0, ..., z$. *Moreover, if* $(p_1, q_1, 0)$ *satisfies* $p_1 + q_1 = n$ *and* $p_1 - q_1 = z - 2k$ *with* $k = 0, ..., z$, *then there exists* $F \in \mathcal{G}_A$ *such that* $i(F) = (p_1, q_1, 0)$.

Let $A \in M_n(\mathbb{C})$ be Λ_H-symmetric. Then $G \in \mathcal{G}_A$ if and only if $-G \in \mathcal{G}_A$. Hence there exists $G \in \mathcal{G}_A$ such that $i(G) = (p, q, 0)$ if and only if there exists $F \in \mathcal{G}_A$ such that $i(F) = (q, p, 0)$. Hence, when we find all the possible inertia of $G \in \mathcal{G}_A$, we need only find those $(p, q, 0)$ for which $p \geq q$.

Suppose that $n = 2$. The possible Jordan blocks of A are (i) $a \oplus \overline{a}$ for some $a \in \mathbb{C} \setminus \mathbb{R}$, (ii) $b \oplus c$ for some $b, c \in \mathbb{R}$, and (iii) $J_2(d)$ for some $d \in \mathbb{R}$. In cases (i) and (iii), we have $i(G) = (1, 1, 0)$ for every $G \in \mathcal{G}_A$. In case (ii) we have $p = 2$, $q = 0$, and $z = 2$, so that $p_1 - q_1 = 2$ or 0. Hence, if $G \in \mathcal{G}_A$, then the possible inertia of G is $(2, 0, 0)$ or $(1, 1, 0)$.

Suppose that $n = 6$. The following two tables give the possible inertia of $G \in \mathcal{G}_A$, given the Jordan Canonical Form of A. Here, we have $\lambda_1, \ldots, \lambda_6 \in \mathbb{R}$ and $\mu_1, \mu_2, \mu_3 \in \mathbb{C} \setminus \mathbb{R}$. Let z be the number of Jordan blocks of A corresponding to a real eigenvalue and having an odd size, and let $i(G) = (p, q, 0)$. Then, we have $p + q = n$ and $p - q = z - 2k$, where k is an integer less than or equal to $\frac{1}{2}z$. The first table shows the possible Jordan blocks of A when the largest Jordan block corresponding to a real eigenvalue has size 6, 5, or 4.

Jordan Canonical Form of A	Possible $i(G)$
$J_6(\lambda_1)$	$(3,3,0)$
$J_5(\lambda_1) \oplus J_1(\lambda_2)$	$(4,2,0),\ (3,3,0)$
$J_4(\lambda_1) \oplus J_2(\lambda_2)$	$(3,3,0)$
$J_4(\lambda_1) \oplus J_1(\lambda_2) \oplus J_1(\lambda_3)$	$(4,2,0),\ (3,3,0)$
$J_4(\lambda_1) \oplus J_1(\mu_1) \oplus J_1(\overline{\mu_1})$	$(3,3,0)$

The following table shows the other possible Jordan Canonical Forms for A.

Jordan Canonical Form of A	Possible $i(G)$
$J_3(\lambda_1) \oplus J_3(\lambda_2)$	$(4,2,0),\ (3,3,0)$
$J_3(\mu_1) \oplus J_3(\overline{\mu_1})$	$(3,3,0)$
$J_3(\lambda_1) \oplus J_2(\lambda_2) \oplus J_1(\lambda_3)$	$(4,2,0),\ (3,3,0)$
$J_3(\lambda_1) \oplus J_1(\lambda_2) \oplus J_1(\lambda_3) \oplus J_1(\lambda_4)$	$(5,1,0),\ (4,2,0),\ (3,3,0)$
$J_3(\lambda_1) \oplus J_1(\lambda_2) \oplus J_1(\mu_1) \oplus J_1(\overline{\mu_1})$	$(4,2,0),\ (3,3,0)$
$J_2(\lambda_1) \oplus J_2(\lambda_2) \oplus J_2(\lambda_3)$	$(3,3,0)$
$J_2(\lambda_1) \oplus J_2(\mu_1) \oplus J_2(\overline{\mu_1})$	$(3,3,0)$
$J_2(\lambda_1) \oplus J_2(\lambda_2) \oplus J_1(\lambda_3) \oplus J_1(\lambda_4)$	$(4,2,0),\ (3,3,0)$
$J_2(\lambda_1) \oplus J_2(\lambda_2) \oplus J_1(\mu_1) \oplus J_1(\overline{\mu_1})$	$(3,3,0)$
$J_2(\mu_1) \oplus J_2(\overline{\mu_1}) \oplus J_1(\lambda_1) \oplus J_1(\lambda_2)$	$(4,2,0),\ (3,3,0)$
$J_2(\mu_1) \oplus J_2(\overline{\mu_1}) \oplus J_1(\mu_2) \oplus J_1(\overline{\mu_2})$	$(3,3,0)$
$J_2(\lambda_1) \oplus J_1(\lambda_2) \oplus J_1(\lambda_3) \oplus J_1(\lambda_4) \oplus J_1(\lambda_5)$	$(5,1,0),\ (4,2,0),\ (3,3,0)$
$J_2(\lambda_1) \oplus J_1(\lambda_2) \oplus J_1(\lambda_3) \oplus J_1(\mu_1) \oplus J_1(\overline{\mu_1})$	$(4,2,0),\ (3,3,0)$
$J_2(\lambda_1) \oplus J_1(\mu_1) \oplus J_1(\overline{\mu_1}) \oplus J_1(\mu_2) \oplus J_1(\overline{\mu_2})$	$(3,3,0)$
$J_1(\lambda_1) \oplus J_1(\lambda_2) \oplus J_1(\lambda_3) \oplus J_1(\lambda_4) \oplus J_1(\lambda_5) \oplus J_1(\lambda_6)$	$(6,0,0),\ (5,1,0),\ (4,2,0),\ (3,3,0)$
$J_1(\lambda_1) \oplus J_1(\lambda_2) \oplus J_1(\lambda_3) \oplus J_1(\lambda_4) \oplus J_1(\mu_1) \oplus J_1(\overline{\mu_1})$	$(5,1,0),\ (4,2,0),\ (3,3,0)$
$J_1(\lambda_1) \oplus J_1(\lambda_2) \oplus J_1(\mu_1) \oplus J_1(\overline{\mu_1}) \oplus J_1(\mu_2) \oplus J_1(\overline{\mu_2})$	$(4,2,0),\ (3,3,0)$
$J_1(\mu_1) \oplus J_1(\overline{\mu_1}) \oplus J_1(\mu_2) \oplus J_1(\overline{\mu_2}) \oplus J_1(\mu_3) \oplus J_1(\overline{\mu_3})$	$(3,3,0)$

References

[1] Y. Bolshakov, C. V. M. van der Mee, A. C. M. Ran, B. Reichstein, and L. Rodman, Polar decompositions in finite dimensional indefinite scalar product spaces: general theory, *Linear Algebra and Its Applications* **261** (1997) 91–141.

[2] I. Gohberg, P. Lancaster, and L. Rodman, Indefinite Linear Algebra and Applications, Birhäuser Verlag, Basel, Switzerland, 2005.

[3] E.V. Haynsworth and A. M. Ostrowski, On the inertia of some classes of partitioned matrices, *Linear Algebra and Its Applications* **1** (1968) 299–316.

[4] R. A. Horn and C. R. Johnson. Matrix Analysis, Cambridge University Press, New York, 1985.

[5] R. A. Horn and C. R. Johnson. Topics in Matrix Analysis, Cambridge University Press, New York, 1991.