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Quantization and Moment Maps

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Abstract

The main contribution of this paper is the relationship between quantum moment maps and states in deformation quantization. This allows for a Poincare-Birkhoff-Witt Theorem to be stated and proved in deformation quantization. Moreover, moment maps allows for the construction of positive linear functionals and minimizing states on universal enveloping algebras. Finally, a nice Cauchy-Schwarz inequality is also stated and proved.

1 Introduction

This note, a contribution to physical theory, presents various aspects of moment mappings in classical mechanics and quantum mechanics, the latter in the so-called deformation quantization setting. In Hamiltonian systems with symmetry, one obtains a symplectic reduction theorem and a classification theorem for symplectic manifolds via coadjoint orbits. This last result shows the deep connection between symplectic geometry and the representation theory of Lie groups.

In deformation quantization, quantum moment maps are the the quantum analogues of classical moment maps. Quantum moment maps are maps from $\mathcal{U}(\mathfrak{g})[[\lambda]]$ into the space of derivations of $C^\infty(M)[[\lambda]]$, where $\mathfrak{g}$ is the Lie algebra of a Lie group of symplectomorphisms of M. By its very definition, quantum moment maps provide representations of $\mathfrak{g}$. Concrete computations of quantum moment maps for Lie groups will show how representations of the Lie groups are obtained. We will also give the quantum analogue of the Poincare-Birkhoff-Witt theorem where the quantized universal algebra is realized as a space of differential operators.

In quantum mechanics, a fundamental role is played by the Heisenberg uncertainty principle. To state it in deformation theory requires a proper notion of states in associative algebras. This has been done by Waldmann and coworkers [2] and we will briefly present their construction here. Their theory is derived from the Gelfand theory of C^*-algebras. The uncertainty principle is stated by Przanowski and coworkers [10], which we will also briefly recall here. The case where the uncertainty principle is an equality will be presented here. In this case, it is the quantum moment map and an invariant due to Hamachi [4] that detects the equality. We will present a proof here of this result.

2 Hamiltonian Mechanics

2.1 Symplectic manifolds

Let a particle move in 3-space $\mathbb{R}^3$ under the influence of a potential $\nabla V(q)$. Let $L(q^i, \dot{q}^i) = \frac{m}{2}||\dot{q}||^2 - V(q)$ be the Langrangian of this system. Then the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0$$

will imply Newton's equation of motion $F = m\mathbf{a}$. Introducing the variable $p_i = mq_i$ and writing

$$H(q,p) = \frac{||p||^2}{2m} + V(q)$$

it is straightforward to verify that the Euler-Lagrange equation is equivalent to Hamilton's equations $\dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q_i}$ and $\dot{q}_i = \{q_i, H\} = \frac{\partial H}{\partial p_i}$. Now, one may arrive at more general forms of Hamilton's equations, even for infinite-dimensional systems, by exploiting the underlying geometric picture whose methods and tools fall under what is known as symplectic geometry. In the fairly simple situation under consideration, the basic objects are the motion space or phase space of points $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ and the symplectic form $\omega = \sum dq_i \wedge dp_i$. Then, for any observable $f(q,p,t)$, its time evolution is given by

$$\frac{df_t}{dt} = -\{H, f_t\} := -\omega(X_H, X_f),$$

where $\{f,g\} = \sum \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}$ is called the Poisson bracket of f and g. Hamilton's equations now take the form $\dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q_i}$ and $\dot{q}_i = \{q_i, H\}$. For $f = H$ itself, we get $\frac{dH}{dt} = \{H, H\} = 0$, also called the conservation of energy. One sees that the change in point of view gives a few important results almost for free.

Definition 1. *A differentiable manifold M with a nondegenerate closed 2-form ω is called a symplectic manifold. The form ω provides a skew-symmetric bilinear form $\omega_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ on tangent spaces T_pM for each $p \in M$.*

We immediately shift our focus, and instead of considering M as basic we consider the set of "observables" $\mathcal{A} = C^\infty(M) = \{\text{smooth functions} f : M \to \mathbb{R}\}$ as basic.

Definition 2. *A Poisson manifold is a differentiable manifold M for which there is a bilinear form $\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \longrightarrow \mathbb{R}$ on the space of smooth functions satisfying the following conditions: derivation property $\{uv, w\} = u\{v,w\} + \{u,w\}v$ and Jacobi identity $\{\{u,v\},w\} + \{\{v,w\},u\} + \{\{w,u\},v\} = 0$.*

As an example, symplectic manifolds are Poisson manifolds, where $\{f,g\} = \omega(df \wedge dg)$.

Examples

1. V a vector space, V^*, the dual of V. Then $V \oplus V^*$ with symplectic form ω,

$$\omega((u_1, v_1^*), (u_2, v_2^*)) = v_1^*(u_2) - v_2^*(u_1).$$

2. Q the configuration space of a mechanical space, $M = T^*Q$ (the phase space = space of positions and velocities) with $\omega = d(p_1 \wedge dq^1 + \cdots + p_n \wedge dq^n) = \sum_{i=1}^{n} dq^i \wedge dp_n$.

3. $\mathfrak{g}^*$, the dual of the Lie algebra of a Lie group is a Poisson manifold. This will be discussed in a subsequent section.

2.2 Hamiltonian systems with symmetry

Let G be a Lie group acting on a symplectic manifold (M, ω) by symplectic transformations, i.e., $g^*\omega(x) = \omega(g \cdot x) = \omega(x), \forall x \in M, \forall g \in G$. Then this group action induces a Lie algebra action $\mathfrak{g} \times \mathcal{A} \longrightarrow \mathcal{A}$ on $\mathcal{A} = C^\infty(M)$ given by

$$(\tilde{X}f)(x) = \frac{d}{dt} f(exp\, tX \cdot x)|_{t=0}.$$

The vector fields $\tilde{X}$ are called Hamiltonian in case $i(\tilde{X})\omega = -df$ for some function f. We write $\tilde{X} = \tilde{X}_f$ for Hamiltonian vector fields. The $f_{\tilde{X}}$ are called Hamiltonian functions. As Lie algebra actions, these vector fields satisfy

$$f_{\widetilde{[X,Y]}} = \{f_{\tilde{X}}, f_{\tilde{Y}}\}.$$

We could have defined such vector fields without the presence of group actions but we want to come quickly to the following definition.

Definition 3. *Let G act on M by symplectic transformations and call this action ϕ. The moment map J of ϕ is the mapping $J : M \longrightarrow \mathfrak{g}^*$ into the dual of the Lie algebra $\mathfrak{g}$ of G, defined by $J(x) = f_{\tilde{X}}(x)$, where $i(\tilde{X})\omega = -df_X$. The quadruple (M, ω, ϕ, J) is called a Hamiltonian G-space. Let us define a related mapping $\hat{J} : \mathfrak{g} \longrightarrow C^\infty(M)$, given by $\hat{J}(X) = f_X$.*

We then have the following canonical commutation relation in classical mechanics or the so-called correspondence principle, $\hat{J}([X,Y]) = \{\hat{J}(X), \hat{J}(Y)\}$. Equivalently, $< J(x), [X,Y] >= \{\hat{J}(X), \hat{J}(Y)\}$.

As examples, if $M = \mathbb{R}^n$ and $G = \mathbb{R}^n$ acts on by translations then J is the linear momentum given by $< J(x), \varepsilon >= x \cdot \varepsilon = \sum x_i \varepsilon^i, x \in M, \varepsilon \in \mathfrak{g} \cong \mathbb{R}^n$. On the other hand, for $M = \mathbb{R}^3$ and $G = SO(3)$ is the group of rotations of Euclidean 3-space about the origin, then J is angular momentum, $< J(x), \varepsilon >= x \times \varepsilon(\text{ cross product in } \mathbb{R}^3), x \in M, \varepsilon \in \mathfrak{g} \cong \mathbb{R}^3$. The following results make clear the major role played by the moment map in mechanics and geometry. Details can be found in the beautiful book by Kirillov [7].

Theorem (Fundamental Conservation Law). Let $\phi : G \times M$ be a symplectic action on M with moment map J. Suppose $H : M \longrightarrow \mathbb{R}$ is invariant under the action: $H(m) = H(\phi(m))$. Then J is an integral for X_H : this means that if φ_t is the flow of X_H, then

$$J(\varphi_t(m)) = J(m).$$

That is, J is constant along the trajectory of X_H. In the case of the $\mathbb{R}$-action on M by $t \mapsto \varphi_t$ (flow of some fixed vector field), then $J = H$ and the conservation law is again the law of conservation of energy.

Theorem (**Symplectic Reduction**). Let J be a moment map and let $\lambda \in \mathfrak{g}^*$ be a regular value of J. Then $M_\lambda := J^{-1}(\lambda)/G_\lambda$ is a symplectic manifold, called the reduced symplectic manifold.

A classification result (Kostant-Kirillov-Souriau). Let M be a symplectic manifold for which (G, M, ϕ, J) is a homogeneous Hamiltonian $G-$space. Then M is homeomorphic to the inverse image (or a central covering of it) $J^{-1}(\mathcal{O})$, where $\mathcal{O}$ is an orbit of the coadjoint action of G on $\mathfrak{g}^*$.

This last result is very important in representation theory of Lie groups and the beginning of the circle of ideas leading to the Borel-Weil-Bott Theorem. The results above indicate the power of the formalism provided by symplectic geometry and moment maps on Poisson and symplectic manifolds.

3 Quantization

If Q is the configuration space (position space) of a mechanical system, then the cotangent bundle T^*Q is its phase space (motion space consisting of positions and momenta). Let $(q_1, ..., q_n, p_1, ..., p_n)$ be the coordinates on M and consider the Liouville 1-form $\theta = \sum p_i dq_i$. Then (M, ω) is a symplectic manifold, where $\omega = d\theta = \sum dp_i \wedge dq_i$. In the Hamiltonian formulation of classical mechanics, the elements of $\mathcal{A} = C^\infty(M)$ are the observables. The evolution of an observable $f_t \in N$ is governed by the equation

$$\frac{df_t}{dt} = -\{H, f_t\}.$$

For example Hamilton's equations are given by $\dot{p}_i = \{p_i, H\} = -\dfrac{\partial H}{\partial q_i}$ and $\dot{q}_i = \{q_i, H\} = \dfrac{\partial H}{\partial p_i}$.

Quantum mechanics, on the other hand, assigns to a quantum mechanical system a Hilbert space $\mathcal{H}$, whose elements are called states, and a set of self-adjoint linear operators A on $\mathcal{H}$ as the quantum observables. The time evolution of an observable A_t is governed by

$$\frac{dA_t}{dt} = i\hbar[H, A_t],$$

where the right hand side is the commutator $H \circ A_t - A_t \circ H$ of operators and $\hbar$ is Planck's constant.

Quantization, then, abstracts the similarity between the classical and quantum, and is defined as a linear mapping from a space of classical observables to a space of quantum observables

$$\mathcal{Q} : \mathcal{A} \longrightarrow Op(\mathcal{H}),$$

satisfying the following conditions.

1. $\mathcal{Q}(1) = Id_{\mathcal{H}}$
2. $\mathcal{Q}\{f, g\} = \frac{i}{\hbar}[\mathcal{Q}(f), \mathcal{Q}(g)]$
3. $\mathcal{Q}$ acts on a certain subspace of observables irreducibly.

An example is the Weyl quantization, one of the more important quantization schemes. For functions $a(q,p)$ of Schwartz class, it assigns the operator $\hat{a} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ given by

$$(Au)(q) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} exp\left(\frac{i}{h}p(q-q')\right) a\left(\frac{q+q'}{2}, p\right) u(q')dq'dp.$$

The function a is called the symbol of the operator A. For instance, one gets the usual operator representations of positions and momenta, $\hat{q}_i =$ multiplication by q_i and $\hat{p}_i = \frac{\partial}{\partial q_i}$.

4 Deformation Quantization

4.1 Basic Definitions and Example

In the early 1970s, Bayen and coworkers [1] suggested that quantization be viewed as a deformation of the algebra of classical observables and not as a radical change in the nature of the observables.

Let (M,ω) be symplectic manifold. Write $\mathcal{A} = C^\infty(M)$ for the algebra of smooth functions on M.

Definition 4. *A deformation quantization of M is an associative algebra structure $*_\lambda$ on the space of formal power series $\mathcal{A}[[\lambda]]$ depending on the formal parameter λ, and satisfies the following properties:*

1. $f *_\lambda g = f \cdot g +$ *higher order terms in* λ
2. $f *_\lambda g - g *_\lambda f = -i\lambda\{f,g\} +$ *higher order terms in* λ
3. $f *_\lambda 1 = f = 1 *_\lambda f$
4. *for* $f, g \in \mathcal{A}, f *_\lambda g = \sum_{r=0}^{\infty} \lambda^r C_r(f,g)$, *where each C_r is a bidifferential operator on $\mathcal{A}$.*

Remark 5. *Item 1. means that the $*_\lambda$-product is a deformation of the commutative pointwise product of functions in $\mathcal{A}$. By defining the Lie bracket*

$$[a,b]_{*_\lambda} = \frac{1}{2\lambda}(a *_\lambda b - b *_\lambda a),$$

Item 3. means that the $*_\lambda$ generates a deformation of the Poisson bracket $\{\, , \}$ on $\mathcal{A}$. Item 4. means that the $*_\lambda$-product is local.

Example. The basic example is that of the symplectic manifold $(\mathbb{R}^{2n}, \omega = \sum dp^i \wedge dq_i)$, which has for its deformation quantization the algebra
$(C^\infty(\mathbb{R}^{2n})[[ih/2]], *_M)$ where,

$$\begin{aligned} u *_M v &= \exp(\frac{ih}{2}\omega^{-1}\partial_r u\, \partial_r v) \\ &= \sum \left(\frac{ih}{2}\right)^r \frac{1}{r!}\omega^{i_1 j_1} \cdots \omega^{i_r j_r} \frac{\partial^r u}{\partial x^{i_1} ... \partial x^{i_r}}, \\ &= u \cdot v + \text{terms of higher order in } i\hbar \end{aligned}$$

$u, v \in C^\infty(\mathbb{R}^{2n}), (x^1, ..., x^{2n}) = (p^1, ..., p^n, q_1, ..., q_n)$, and
$(\omega^{ij}) = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. The Lie bracket between u and v is

$$\begin{aligned} [u, v]_{*_M} &= \frac{1}{2i} \sinh(\frac{i\hbar}{2} \omega^{-1} \partial_r u \, \partial_r v) \\ &= \{u, v\} + \text{terms of higher order in } (ih)^2, \end{aligned}$$

showing the deformation of the Poisson bracket.

It was discovered by Moyal [8] that the $*$-product on $\mathbb{R}^{2n}$ above is intimately connected with the Weyl quantization. Indeed, if a and b are two symbols with corresponding operators A, B, respectively, then the symbol c that corresponds to the operator AB is

$$c(x) = \sum \frac{1}{r!} \left(-\frac{ih}{2} \right)^r \omega^{i_1 j_1} \cdots \omega^{i_r j_r} \frac{\partial^r a}{\partial x^{i_1} \cdots \partial x^{i_r}} \frac{\partial^r b}{\partial x^{j_1} \cdots \partial x^{j_r}}.$$

4.2 The Gutt $*-$product

We now introduce the Gutt $*$-product on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$. Let G be a finite-dimensional Lie group whose Lie algebra is $\mathfrak{g}$. Let $\mathfrak{g}^*$ be the dual space to $\mathfrak{g}$. If $X_1, \ldots, X_n$ form a basis of $\mathfrak{g}$, whose commutation relations satisfy

$$[X_i, X_j] = \sum_k c_{ij}^k X_k,$$

then a Poisson bracket $\{\cdot, \cdot\}$ maybe defined on $C^\infty(\mathfrak{g}^*)$. Let $x_i, \ldots, x_n$ be linear coordinates of $\mathfrak{g}^*$. Below, we will choose the x_i to be the functions satisfying $x_i(X_j) = \delta_{ij}$. Then the Poisson bracket of two functions $f, g \in C^\infty(\mathfrak{g}^*)$ is defined as

$$\{f, g\} = \sum x_k c_{ij}^k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Consider now the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. This is just the space of all linear combinations of elements $X_{i_1}^{n_1} \cdots X_{i_k}^{n_k}$ (Poincare-Birkhoff-Witt, see Section 5.3) where $X_{i_j} \in \mathfrak{g}, k, i_k \in \mathbb{N}$. Define a new bracket $[\cdot, \cdot]_\lambda$ on $\mathfrak{g}$ by $[X, Y]_\lambda = \lambda[X, Y]$ ($[X, Y]$ is old $\mathfrak{g}$-bracket). On the algebra of formal power series $\mathcal{U}(\mathfrak{g})[[\lambda]]$, $[\cdot, \cdot]_\lambda$ defines a Lie algebra structure by linear extension. Consider also the space $\mathcal{S}(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{g})/\sim$, where $X_i X_j \sim X_j X_i$. The symmetrizer map $S : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{S}(\mathfrak{g})$ given by $S(X_{i_1} \cdots X_{i_k}) = \dfrac{1}{k!} \sum X_{\sigma(i_1)} \cdots X_{\sigma(i_k)}, \sigma \in \mathfrak{S}_k$ provides the Gutt $*$-product on $\mathrm{Poly}(\mathfrak{g}^*) \cong \mathcal{S}(\mathfrak{g}) : f * g = \sigma(\sigma^{-1}(f) \circ \sigma^{-1}(g))$. One can show that $\dfrac{1}{\lambda}(f * g - g * f) = \{f, g\} + O(\lambda^2)$.

5 Quantum Moment Mappings

5.1 Basic Definitions

In [11] P. Xu defined quantum moment mappings which are the quantum analogs, in deformation quantization, of moment mappings in classical mechanics. Again, write $\mathcal{A} = C^\infty(M)$ and let $*$ be a star product on $\mathcal{A}$.

Let G be a Lie group of symplectic transformations on M such that $\phi_g u * \phi_g v = u * v$, for all $g \in G$. Recall that $X \in \mathfrak{g}$ induces a derivation $\tilde{X}$ of $\mathcal{A}$, hence also of $\mathcal{A}[[\lambda]]$.

A quantum moment mapping : $J_* : \mathcal{U}(\mathfrak{g}_\lambda) \longrightarrow \mathcal{A}[[\lambda]]$ is an associative algebra homomorphism satisfying

$$[J_* Z, u]_* := \frac{1}{\lambda}((J_* Z) * u - u * (J_* Z)) = \lambda \frac{d}{dt} u(exp(tZ))|_{t=0}, u \in \mathcal{A},$$

where both sides are viewed as derivations of $\mathcal{A}[[\lambda]]$.

Theorem 6. *A quantum moment mapping is a deformation of some classical moment mapping. This means that, in the notations above,* $J_*(f) = J(f) + O(\lambda)$.

Quantum moment mappings give us an invariant of star-products.

Definition 7. *Let G act on M transitively and by symplectic transformations. Let $*$ be a $G-$invariant star-product on M and J_* is a quantum moment mapping of $*$. Let $\mathfrak{Z}$ denote the center with respect to Gutt star-product of $\mathcal{U}(\mathfrak{g})[[\lambda]]$. Then*

$$c_*(X) := J_*(X) = constant \in \mathbb{C}[[\lambda]], \forall X \in \mathfrak{g}.$$

This invariant is due to Hamachi and several examples can be found in [4].

5.2 Example

In the paper [9], a simple construction of quantum moment mappings were given and was carried out in a couple of low-dimensional Lie groups to obtain unitary representations of these Lie groups. It realizes, at least in the examples given there, the program of applying deformation quantization to Lie group representations in an autonomous manner. To carry out this program, we define a mapping

$$ad_* : \mathcal{U}(\mathfrak{g}) \longrightarrow Der(\mathcal{A}[[\lambda]]),$$

into the space of derivations of $\mathcal{A}[[\lambda]]$ by

$$T \mapsto ad_* f_T, \text{ where } ad_* f_T(u) = \frac{1}{\lambda}(f_T * u - u * f_T).$$

By linearly extending this to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})[[\lambda]]$, we obtain a representation of the Gutt $*$-product in a space of derivations. Let $X_1, ..., X_n$ be basis elements of the the Lie algebra $\mathfrak{g}$ satisfying the commutation relations

$$[X_i, X_j] = \sum_k c_{ij}^k X_k,$$

which induces the Poisson tensor $\sum x_k c_{ij}^k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$. Then

$$J_*(X_i) = \sum_{j,k} x_i \left(c_{jk}^i \frac{\partial}{\partial x_j} - c_{kj}^i \frac{\partial}{\partial x_k} \right).$$

This is a quantum moment map because it is the classical moment map $J : \mathfrak{g} \to C^\infty(M)$ when restricted to $\mathfrak{g}$. In fact, the computation is very simple

$$[J_*(X_i), u]_* = \frac{1}{2\lambda}(f_{X_i} * u - u * f_{X_i}) = \{f_X, u\} = \frac{d}{dt}u(exp\, tX_i)|_{t=0},$$

for which details the reader can supply easily. For $X, Y \in \mathfrak{g}$, one has $([J_*(X), J_*(Y)]_* = \{f_X, f_Y\} = f_{[X,Y]} = J_*([X,Y])$, so that indeed one gets a representation of the Lie algebra $\mathfrak{g}$ in a space of derivations. In [9], one considers the Casimir element or else look at another set of coordinates for the representation space to obtain irreducible subspaces. In both cases, deformation quantization theory autonomously provides the irreducible representations.

5.3 A Poincare-Birkhoff-Witt Theorem

Let $\mathfrak{g}$ be a Lie algebra and $\mathcal{U}(\mathfrak{g})$ be its universal enveloping algebra. Then the identification map $\iota : \mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})$ is an injection. The Poincare-Birkhoff-Witt Theorem [6] says, if $X_i, ..., X_n$ is a basis of $\mathfrak{g}$, then the set

$$1,\ \iota(X_{i_1})\cdots\iota(X_{i_n}),\ n \in \mathbb{N},$$

forms a basis of the universal enveloping algebra. This allows one to look at $\mathcal{U}(\mathfrak{g})$ as the span of

$$1,\ \frac{\partial^{\alpha_1}}{\partial x_{i_1}^{\alpha_1}}\cdots\frac{\partial^{\alpha_k}}{\partial x_{i_k}^{\alpha_k}},\ \alpha_j \in \mathbb{N},$$

where, again, the x_j are the linear coordinates of $\mathfrak{g}^*$.

Now, because of the Hamachi invariant for quantum moment maps, one obtains a similar result for the algebra $\mathcal{U}(\mathfrak{g})[[\lambda]]$.

Theorem 8. $ad_* : \mathcal{U}(\mathfrak{g})[[\lambda]]/\mathfrak{Z} \longrightarrow Der(C^\infty(M)[[\lambda]])$ *is injective, so realizes the universal enveloping algebra as a space of differential operators.*

6 Uncertainty Inequalities in Deformed Algebras

6.1 Uncertainty Principle

In the 1930s, Heisenberg formulated his famous uncertainty principle [5], one of the most important results in Quantum Mechanics, viz., $\Delta p\, \Delta q \geq \frac{\hbar}{2}$, where Δq is change in position of a particle and Δp is the change in momentum. Thus, measurement of one quantity to a high degree of accuracy will entail a large error in the measurement of the conjugate quantity. For the conjugate variables E and t (energy and time), the uncertainty relation is $\Delta E\, \Delta t \geq \frac{\hbar}{2}$. In quantum mechanics, where the phase space is a Hilbert space $\mathcal{H}$ and the observables are self-adjoint operators $A : \mathcal{H} \to \mathcal{H}$, the uncertainty principle takes the form

$$||Au||\, ||Bu|| \geq |<[A, B]u, u>|,$$

where the vectors u belong to the domain of the commutator.

Robertson and Heisenberg (see [10] and references therein) subsequently generalized the inequality to any finite number of observables as follows: Let (c_{jk}) be a nonnegative definite Hermitian matrix. Writing $c_{jk} = a_{jk} + ib_{jk}$, then $det\,(a_{jk}) \geq det\,(b_{jk})$. It is this inequality that we will be concerned with in the next section. The idea of Pzranowski and Turrubiates in [10] is to extract a nonnegative definite matrix from images of observables under appropriate states. It turns out that these states have been already defined much as in the theory of C^*-algebras by Bordemann and Waldmann in [2].

Mathematical formulations of the uncertainty principle are important in the areas of Fourier series, signal and wavelet analysis. On the other hand, one test of the validity of a quantum theory, say, deformation quantization, is an appropriate expression of the uncertainty principle.

6.2 Positive Functionals and States

Recently, a quantum analog, in deformation quantization setting, of the classical Robertson-Heisenberg uncertainty relation were formulated by Przanowski and Turrubiates. The following discussion is mostly taken from their paper [10], which the reader may consult for the details.

The main ingredient is to define an appropriate concept of positive linear functionals and states in associative algebras $\mathcal{A}$ of formal power series due to Bordemann and Waldmann. In what follows $\mathbb{K}$ is the field $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{K}(\lambda)$ denotes the field of quotients of the polynomials in $\mathbb{K}[\lambda]$. If $\mathcal{A}$ is an associative algebra, $\mathcal{A}((\lambda))$ will denote the algebra of formal Laurent series

$$\sum_{n=-N}^{\infty} \lambda^r a_r$$

in λ. See [2] for a beautiful and more complete discussion.

Let $\mathcal{A}$ be a complex associative algebra with involution over a field $\mathbb{K}(i)$, where $\mathbb{K}$ is an ordered field. A positive linear functional is a linear map

$$\rho : \mathcal{A} \longrightarrow \mathbb{K}(i),$$

satisfying

$$\rho(f\overline{f}) \geq 0,\ \ \forall f \in \mathcal{A}.$$

If, moreover, $\rho(1) = 1$, ρ is called a state.

For $\mathcal{A} = C^\infty(M)((\lambda))$, it turns out that what we need in our formulation of Heisenberg uncertainty inequalities is the field of formal Laurent series

$$\mathbb{C}((\lambda)) = \{ \sum_{r=-N}^{\infty} \lambda^r z_r : z_r \in \mathbb{C}, N \in \mathbb{Z}\}$$

This is an ordered field under the following ordering. $\omega_1 = \sum \lambda^k z_r \geq \omega_2 = \sum \lambda^r w_r$ if and only if $\min(\operatorname{supp}\omega_1) \geq \min(\operatorname{supp}\omega_2)$ where $\operatorname{supp}\omega = \{r : z_r \neq 0\}$.

Let $\rho : C^\infty(M)((\lambda)) \longrightarrow \mathbb{C}((\lambda))$ be a state. Let $f_1, ..., f_n \in C^\infty(M)((\lambda))$ such that $\overline{f}_k = f_k$. Such f's are called observables. Define $\delta f_j := f_j - \rho(f_j)$, $\varphi(f, g) := \rho(\overline{f} * g)$ and put $f := \sum v_j \delta f_j, v_j \in \mathbb{C}((\lambda))$.

Then $\varphi(f, f) = \sum \overline{v}_j v_k \varphi_{jk}$,where $\varphi_{jk} := \varphi(\delta f_j, \delta f_k)$ and $[\varphi] = (\varphi_{jk})$ is an n x n hermitian nonnegative-definite matrix over $\mathbb{C}((\lambda))$. Thus, the Heisenberg-Robertson inequality holds:

Writing $\phi_{jk} = a_{jk} + ib_{jk}, A = (a_{jk}), B = (b_{jk})$, then $detA \geq detB$.

In more detail $detA \geq detB$ translates to

$$det\left(\frac{1}{2}\rho(\delta f_j * \delta f_k + \delta f_k * \delta f_j)\right) \geq det\left(\frac{\hbar}{2}\rho([f_j, f_k]_*)\right).$$

In particular, for two observables f_1 and f_2,

$$\Delta f_1 \Delta f_2 \geq \frac{1}{2}\sqrt{(\hbar[f_1, f_2]_*)^2 + [\rho(f_1 * f_2 + f_2 * f_1) - 2\rho(f_1)\rho(f_2)]^2},$$

where $(\Delta f)^2 := \rho(f * f) - \rho(f)^2$ is the variance.

Example

Let $G = SO(3)$ be the real rotation group, $\mathfrak{g} = so(3)$ its Lie algebra and $\mathfrak{g}^3 = so(3)^*$ the dual of $so(3)$. In representation theory of Lie groups, an important role is played by the coadjoint action and the coadjoint orbits. For G, the orbits M are the spheres with integral radii. These orbits are symplectic manifolds, so are all locally homeomorphic to $\mathbb{R}^2$ via the map $(p, q) \mapsto r\cos p \sin q\, E^* + r \sin p \sin q\, F^* + r\cos q\, H^*$.

Consider the state $\omega : C^\infty(M)((\lambda)) \longrightarrow \mathbb{C}((\lambda))$ of the form $\omega = \sum \lambda^k \omega_k, \omega_k \in \mathbb{C}$.

For the basis elements E, F, H of $\mathfrak{g}$ satisfying the commutation relations $[H, E] = E, [H, F] = -F, [E, F] = 2H$, we get canonical functions $X_1 = E^*, X_2 = F^*, X_3 = H^*$. The uncertainty inequality for these functions is an equality and reduces to 0=0.

A state $\rho : C^\infty(M)((\lambda)) \longrightarrow \mathbb{C}((\lambda))$ is called a **minimizing state** for the functions $X_1, ..., X_n$ if

$$det\left(\frac{1}{2}\rho(\delta X_j * \delta X_k + \delta X_k * \delta X_j)\right) = det\left(\frac{\hbar}{2}\rho([X_j, X_k]_*)\right).$$

We have seen above an example of minimizing states on functions on coadjoint orbits.

We now state some results on the existence of states and minimizing states on Gutt star-product $\mathcal{U}(\mathfrak{g}_\lambda)$. These results simply puts together the concepts of state in associative algebras, quantum moment mappings and Hamachi's invariant.

Theorem 9. *Let $\rho : C^\infty(M)((\lambda)) \longrightarrow \mathbb{C}((\lambda))$ be a positive linear functional.*

1. *If $J_* : \mathcal{U}(\mathfrak{g}_\lambda) \longrightarrow C^\infty(M)((\lambda))$ is a quantum moment mapping, then $\rho \circ J_* : \mathcal{U}(\mathfrak{g}_\lambda) \longrightarrow \mathbb{C}((\lambda))$ is a a positive linear functional.*

2. *Let the restriction $J_*|\mathfrak{Z} \neq 0$. Then the center $\mathfrak{Z}$ of $\mathcal{U}(\mathfrak{g}_\lambda)$ consists of functions for which $\rho \circ J_*$ is a minimizing state.*

3. *Let J_* be the moment map given by ad_*. Then for two observables $J_*(X) = f_X, J_*(Y) = f_Y$, we have a form of the Cauchy-Schwarz inequality:*

$$(\rho \circ J_*(X))(\rho \circ J_*(Y)) = \rho(f_X)\rho(f_Y) \geq 2\rho(\{f_X, f_Y\}).$$

Proof: For the first assertion, just check $\rho \circ J_*(\overline{f} * f) \geq 0 = \rho(J_*(\overline{f}) * J_*(f)) = \rho(\overline{J_*(f)} * J_*(f)) \geq 0$. Since J_* is constant on $\mathfrak{Z}$, we may normalize it and the second assertion follows immediately from the first. The proof of the last assertion is a direct computation as follows. From the inequality

$$det\left(\frac{1}{2}\rho(\delta f_j * \delta f_k + \delta f_k * \delta f_j)\right) \geq det\left(\frac{\hbar}{2}\rho([f_j, f_k]_*)\right), \; j, k = 1, 2,$$

then

$$a_{11}a_{22} - a_{12}a_{21} = \frac{1}{4}\rho(f_1)^2\rho(f_2)^2,$$

while

$$b_{11}b_{22} - b_{12}b_{21} = 0 - (\frac{1}{i})^2\rho(\{f_X, f_Y\})^2.$$

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