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Schlömilch-Type Formula for r-Whitney Numbers of the First Kind

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Abstract

In this paper, a modified approach in deriving Schlömilch-type formulas is employed for certain family of Stirling-type pairs. Specifically, Schlömilch-type formulas for r-Stirling and r-Whitney numbers of the first kind are established.

Keywords: r-Stirling numbers, r-Whitney numbers, Schlömilch-type formula, generating function.

1 Introduction

The exact value of the Stirling numbers of both kinds can be obtained using their corresponding explicit formula. The known explicit formula for the classical Stirling numbers of the first kind is called the *Schlömilch formula*. The process of deriving this formula is quite complicated compared to deriving the explicit formula for Stirling numbers of the second kind. More precisely, the Schlömilch formula is given by

$$s(n,k) = \sum_{r=0}^{n-k} \sum_{j=0}^{r} (-1)^{j+r} \binom{r}{j} \binom{n-1+r}{n-k+r} \binom{2n-k}{n-k-r} \frac{(r-j)^{r-k+n}}{r!}. \tag{1}$$

In deriving this formula, one can use the fact that if $C_{t^n}F :=$ coefficient of t^n in $F(t)$ and $\{f(t), g(t)\}$ is a pair of inverse functions, then

$$C_{t^n}(g(t))^k = \frac{k}{n} C_{t^n} \left(\frac{f(t)}{t}\right)^{-n},$$

and the following differentiation formula

$$\left[\frac{d^n}{dt^n}(\phi(t))^{-\alpha}\right]_{t=0} = \alpha \binom{n+\alpha}{n} \sum_{j=1}^{n} \frac{(-1)^j}{\alpha+j} \binom{n}{j} \left[\frac{d^n}{dt^n}(\phi(t))^j\right]_{t=0}$$

(see [6]). Throughout the paper, the parameters n and k are taken to be nonnegative integers.

In 1992, L.C. Hsu [8] constructed a general rule in deriving Schlömilch-type formula for the family of Stirling-type pair $\{A_1(n,k), A_2(n,k)\}$ satisfying the following generating functions

$$\frac{(f(t))^k}{k!} = \sum_{n=0}^{\infty} A_1(n,k) \frac{t^n}{n!} \tag{2}$$

$$\frac{(g(t))^k}{k!} = \sum_{n=0}^{\infty} A_2(n,k) \frac{t^n}{n!} \tag{3}$$

where $f(t)$ and $g(t)$ are inverse functions. The Schlomilch-type formula for $A_1(n,k)$ is given by

$$A_1(n,k) = \sum_{r=0}^{n-k} (-1)^r \binom{n-1+r}{n-k+r} \binom{2n-k}{n-k-r} A_2(n-k+r, r). \tag{4}$$

For instance, the Whitney numbers of the first and second kind satisfy the following relations

$$\frac{1}{k!}\left(\frac{\ln(mz+1)}{m}\right)^k = \sum_{n=0}^{\infty} w_m(n,k) \frac{z^n}{n!}$$

$$\frac{1}{k!}\left(\frac{e^{mz}-1}{m}\right)^k = \sum_{n=0}^{\infty} W_m(n,k) \frac{z^n}{n!}$$

where

$$W_m(n,k) = \frac{1}{m^k k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (m(k-j))^n. \tag{5}$$

It follows immediately from (4) and (5) that

$$w_m(n,k) = \sum_{h=0}^{n-k} \sum_{j=0}^{h} (-1)^{h+j} m^{n-k} \binom{h}{j} \binom{n+h-1}{n-k+h} \binom{2n-k}{n-k-h} \frac{(h-j)^{n-k+h}}{h!}. \tag{6}$$

However, there are known Stirling-type pairs that do not belong to the above family of Stirling-type pairs. The Schlomilch-type formula for these pairs of Stirling-type numbers cannot be derived using formula (4). In this paper, we use another approach to derive Schlomilch-type formulas for these type of Stirling numbers.

2 r-Stirling Numbers of the First Kind

In the 20th century, several mathematicians work on generalization and extension of the Stirling numbers which are related to combinatorial, probabilistic and statistical applications. Among them was A.Z. Broder [4] who defined a certain generalization of Stirling numbers by means of some combinatorial interpretations in terms of permutations and partitions which he called the *r-Stirling numbers*. Several properties have been established by Broder [4] for the r-Stirling numbers parallel to those of the classical Stirling numbers. These include recurrence relations, generating functions and convolution-type identities. However, there are some properties parallel to those of the classical Stirling numbers that have not been done yet. One of those is the Schlömilch-type formula for the r-Stirling numbers of the first kind. By looking at the structures of the generating functions of this pair of r-Stirling numbers, they do not satisfy (2) and (3). Hence, (4) cannot be used to derive Schlömilch-type formula for the r-Stirling numbers of the first kind.

The r-Stirling numbers of the first kind, denoted by $\begin{bmatrix} n \\ m \end{bmatrix}_r$ are defined combinatorially as the number of permutations of the set $\{1, ..., n\}$ having m cycles such that the numbers $1, 2, 3, ..., r$ are in distinct cycles. The r-Stirling numbers of the first kind obey the triangular recurrence relation:

$$\begin{bmatrix} n \\ m \end{bmatrix}_r = (n-1)\begin{bmatrix} n-1 \\ m \end{bmatrix}_r + \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_r$$

where,

$$\begin{bmatrix} n \\ m \end{bmatrix}_r = \begin{cases} 0, & n < r \\ \delta_{m,r} & n = r \end{cases}$$

and

$$\delta_{m,r} = \begin{cases} 0, & r \neq m \\ 1, & r = m. \end{cases}$$

The following theorems in [4] are some of the properties of $\begin{bmatrix} n \\ m \end{bmatrix}_r$.

Theorem 1. *The r-Stirling numbers of the first kind satisfy*

$$\begin{bmatrix} n \\ m \end{bmatrix}_r = \frac{1}{r-1}\left(\begin{bmatrix} n \\ m-1 \end{bmatrix}_{r-1} - \begin{bmatrix} n \\ m-1 \end{bmatrix}_r\right), \qquad n \geq r > 1.$$

Theorem 2. *The r-Stirling numbers of the first kind satisfy*

$$\begin{bmatrix} n \\ n-m \end{bmatrix}_r = \sum_{r \leq i_1 < i_2 < ... < i_m < n} i_1 i_2 ... i_m, \qquad n, m \geq 0.$$

Theorem 3. *The r-Stirling numbers of the first kind have the horizontal generating function*

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix}_r z^k = \begin{cases} z^r(z+r)(z+r+1)...(z+n-1), & n \geq r \geq 0, \\ 0 & \textit{otherwise}. \end{cases}$$

Theorem 4. *The r-Stirling numbers of the first kind have the vertical exponential generating function*

$$\sum_n \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r \frac{z^n}{n!} = \begin{cases} \frac{1}{k!}\left(\frac{1}{1-z}\right)^r \left[\ln(\frac{1}{1-z})\right]^k, & k \geq 0, \\ 0 & \textit{otherwise}. \end{cases}$$

Note that when $r = 0$, the generating function in Theorem 4 reduces to

$$\sum_{n} |s(n,k)| \frac{z^n}{n!} = \frac{1}{k!} \left[\ln \left(\frac{1}{1-z} \right) \right]^k,$$

where $|s(n,k)|$ denotes the signless Stirling numbers of the first kind. This can further be written as

$$\sum_{n} (-1)^{n-k} s(n,k) \frac{z^n}{n!} = \frac{1}{k!} \left[\ln \left(\frac{1}{1-z} \right) \right]^k, \tag{7}$$

$$\sum_{n} s(n,k) \frac{(-z)^n}{n!} = \frac{1}{k!} \left[\ln(1-z) \right]^k. \tag{8}$$

To derive the Schlömilch-type formula for $\begin{bmatrix} n \\ k \end{bmatrix}_r$, we have to decompose first the generating function in Theorem 4 into product of two functions as follows

$$\sum_{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r \frac{z^n}{n!} = \left\{ \left(\frac{1}{1-z} \right)^r \right\} \left\{ \frac{1}{k!} \left[\ln \left(\frac{1}{1-z} \right) \right]^k \right\}.$$

The first function can be expressed as

$$\begin{aligned} \left(\frac{1}{1-z} \right)^r &= (1-z)^{-r} \\ &= \sum_{n \geq 0} \binom{-r}{n} z^n \\ &= \binom{-r}{0} (-z)^0 + \sum_{n>0} \binom{-r}{n} z^n. \end{aligned}$$

By Newton's Binomial Theorem, we get

$$\begin{aligned} \left(\frac{1}{1-z} \right)^r &= 1 + \sum_{n>0} \frac{(-r)(-r-1)...(-r-n+1)}{n!} (-z)^n \\ &= 1 + \sum_{n>0} (-1)^n \frac{(r)(r+1)...(r+n-1)}{n!} (-1)^n z^n \\ &= 1 + \sum_{n>0} (r)(r+1)...(r+n-1) \frac{z^n}{n!} \end{aligned}$$

and by the definition of a rising factorial,

$$\begin{aligned} \left(\frac{1}{1-z} \right)^r &= 1 + \sum_{n>0} r^{\overline{n}} \frac{z^n}{n!} \\ &= \sum_{n \geq 0} r^{\overline{n}} \frac{z^n}{n!} \end{aligned}$$

where $r^{\overline{n}} = r(r+1)...(r+n-1)$. The second function can be written as

$$\begin{aligned}\frac{1}{k!}\left[\ln\left(\frac{1}{1-z}\right)\right]^k &= \frac{1}{k!}\left[\ln(1-z)^{-1}\right]^k \\ &= \frac{1}{k!}\left[(-1)\ln(1-z)\right]^k \\ &= \frac{1}{k!}(-1)^k\left[\ln^k(1-z)\right] \\ &= (-1)^k\left[\frac{\ln^k(1-z)}{k!}\right].\end{aligned}$$

By (8), we have

$$\begin{aligned}\frac{1}{k!}\left[\ln\left(\frac{1}{1-z}\right)\right]^k &= (-1)^k\sum_{n\geq k} s(n,k)\frac{(-z)^n}{n!} \\ &= (-1)^k\sum_{n\geq k} s(n,k)\frac{(-1)^n z^n}{n!} \\ &= \sum_{n\geq k}(-1)^{n+k}s(n,k)\frac{z^n}{n!}.\end{aligned}$$

Hence, using Cauchy's Rule for the product of two power series, we have

$$\begin{aligned}\sum_n \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r \frac{z^n}{n!} &= \left\{\left(\frac{1}{1-z}\right)^r\right\}\left\{\frac{1}{k!}\left[\ln\left(\frac{1}{1-z}\right)\right]^k\right\} \\ &= \sum_{n\geq 0}\left\{\sum_{m=k}^{n}\frac{r^{\overline{n-m}}z^{n-m}}{(n-m)!}(-1)^{m+k}\frac{s(m,k)z^m}{m!}\right\}\end{aligned}$$

$$\begin{aligned}&= \sum_{n\geq 0}\left\{\sum_{m=k}^{n}\frac{z^{n-m+m}}{(n-m)!m!}(-1)^{m+k}s(m,k)r^{\overline{n-m}}\right\} \\ &= \sum_{n\geq 0}\left\{\sum_{m=k}^{n}\frac{(-1)^{m+k}}{(n-m)!m!}s(m,k)r^{\overline{n-m}}\right\}z^n.\end{aligned}$$

Comparing the coefficients of z^n, we get

$$\frac{\begin{bmatrix} n+r \\ k+r \end{bmatrix}_r}{n!} = \sum_{m=k}^{n}\frac{(-1)^{m+k}}{(n-m)!m!}s(m,k)r^{\overline{n-m}}.$$

Equivalently,

$$\begin{aligned}\begin{bmatrix} n+r \\ k+r \end{bmatrix}_r &= n!\sum_{m=k}^{n}\frac{(-1)^{m+k}}{(n-m)!m!}s(m,k)r^{\overline{n-m}} \\ &= \sum_{m=k}^{n}\frac{n!(-1)^{m+k}}{(n-m)!m!}s(m,k)r^{\overline{n-m}} \\ &= \sum_{m=k}^{n}(-1)^{m+k}\binom{n}{m}s(m,k)r^{\overline{n-m}}.\end{aligned}$$

Using the Schlömilch formula in (1), we have the following theorem.

Theorem 5. *The Schlömilch-type formula for r-Stirling numbers of the first kind is given by*

$$\begin{bmatrix} n+r \\ k+r \end{bmatrix}_r = \sum_{m=k}^{n} \sum_{h=0}^{m-k} \sum_{j=0}^{h} (-1)^{m+k+h+j} \binom{n}{m} \binom{h}{j} \binom{m-1+h}{m-k+h} \binom{2m-k}{m-k-h} \frac{(h-j)^{m-k+h}}{h!} r^{\overline{n-m}}.$$

Taking $r=0$, $r^{\overline{n-m}} = \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases}$. Hence, Theorem 5 reduces to

$$(-1)^{n+k} \begin{bmatrix} n \\ k \end{bmatrix}_0 = \sum_{h=0}^{n-k} \sum_{j=0}^{h} (-1)^{h+j} \binom{h}{j} \binom{n-1+h}{n-k+h} \binom{2n-k}{n-k-h} \frac{(h-j)^{n-k+h}}{h!}$$

$$s(n,k) = \sum_{h=0}^{n-k} \sum_{j=0}^{h} (-1)^{h+j} \binom{h}{j} \binom{n-1+h}{n-k+h} \binom{2n-k}{n-k-h} \frac{(h-j)^{n-k+h}}{h!}$$

which is exactly the Schlömilch formula in (1).

3 r-Whitney Numbers of the First Kind

Recently, the paper [11] by I. Mezo introduced further generalization of r-Stirling numbers and Whitney numbers [1, 2, 3] which is called r-*Whitney numbers*. The main objective of the paper [11] is to obtain a new formula for Bernoulli polynomials which is expressed in terms of r-Whitney numbers. Several subsequent studies on r-Whitney numbers came out including the work of Cheon and Jung [5] and Merca [10]. However, we observe that Schlömilch-type formula for r-Whitney numbers of the first kind has not been considered yet. In this paper, we derive this formula using the same method in deriving the Schlömilch-type formula for r-Stirling numbers of the first kind.

The r-Whitney numbers of the first kind, denoted by $w_{m,r}(n,k)$, are defined as the coefficients of the expansion of the following relation:

$$(x+r|m)_n = \sum (-r)^{n+k} w_{m,r}(n,k) x^k,$$

where $(x+r|m)_n$ is the generalized factorial of x with increment m which is given by

$$(x+r|m)_n = \prod_{i=0}^{n-1} (x+r-im).$$

One of the properties established in [11] is the following vertical generating function

$$\sum_{n=k}^{\infty} w_{m,r}(n,k) \frac{x^n}{n!} = (1+mx)^{\frac{-r}{m}} \frac{ln^k(1+mx)}{m^k k!}. \tag{9}$$

Now, note that this exponential generating function is expressed as product of two functions.

The first function can be expressed as

$$\begin{aligned}(1+mx)^{\frac{-r}{m}} &= \sum_{n\geq 0}\binom{\frac{-r}{m}}{n}m^n x^n\\ &= \sum_{n\geq 0}\frac{(\frac{-r}{m})_n}{n!}m^n x^n\\ &= \sum_{n\geq 0}(-r|m)_n\frac{x^n}{n!}\end{aligned}$$

and the second function can be written as

$$\begin{aligned}\frac{\ln^k(1+mx)}{m^k k!} &= \frac{1}{m^k}\sum_{n\geq k}s(n,k)\frac{(mx)^n}{n!}\\ &= \frac{1}{m^k}\sum_{n\geq k}m^n s(n,k)\frac{x^n}{n!}.\end{aligned}$$

Hence, using Cauchy's Rule for the product of two power series, we have

$$\begin{aligned}\sum_{n=k}^{\infty}w_{m,r}(n,k)\frac{x^n}{n!} &= (1+mx)^{\frac{-r}{m}}\frac{\ln^k(1+mx)}{m^k k!}\\ &= \left(\sum_{n\geq 0}(-r|m)_n\frac{x^n}{n!}\right)\left(\frac{1}{m^k}\sum_{n\geq k}m^n s(n,k)\frac{x^n}{n!}\right)\\ &= \sum_{n\geq 0}\left(\sum_{i=0}(-r|m)_{n-i}\binom{n}{i}m^{i-k}s(i,k)\right)\frac{x^n}{n!}.\end{aligned}$$

Thus, comparing the coefficients of $\frac{x^n}{n!}$, we have

$$w_{m,r}(n,k) = \sum_{i=k}^{n}(-r|m)_{n-i}\binom{n}{i}m^{i-k}s(i,k).$$

Using equation (1), we obtain the following theorem.

Theorem 6. *The Schlömilch-type formula for r-Whitney numbers of the first kind is given by*

$$\begin{aligned}w_{m,r}(n,k) = \sum_{i=k}^{n}\sum_{h=0}^{i-k}\sum_{j=0}^{h}&(-1)^{h+j}(-r|m)_{n-i}m^{i-k}\binom{n}{i}\binom{h}{j}\binom{i+h-1}{i-k+h}\binom{2i-k}{i-k-h}\times\\ &\times\frac{(h-j)^{i-k+h}}{h!}.\end{aligned}$$

Note that when $r=0$,

$$(-r|m)_{n-i} = \begin{cases}0, & n\neq i\\ 1, & n=i.\end{cases}$$

Hence, we obtain

$$w_m(n,k) = w_{m,0}(n,k) = \sum_{h=0}^{n-k}\sum_{j=0}^{h}(-1)^{h+j}m^{n-k}\binom{h}{j}\binom{n+h-1}{n-k+h}\binom{2n-k}{n-k-h}\frac{(h-j)^{n-k+h}}{h!}$$

the Schlömilch-type formula for Whitney numbers of the first kind.

4 Unified Generalization

The unified generalization of Stirling numbers, denoted by $S(n,k;\alpha,\beta,\gamma)$, was defined by L.C. Hsu and P. J-S. Shuie [9] by means of the following relation

$$(t|\alpha)_n = \sum_{k=0}^{n} S(n,k;\alpha,\beta,\gamma)(t-\gamma|\beta)_k, \tag{10}$$

where α,β,γ may be real or complex numbers. The unified generalization satisfies the following explicit formula

$$S(n,k;\alpha,\beta,\gamma) = \frac{1}{\beta^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\beta j+\gamma|\alpha)_n$$

and exponential generating function

$$\frac{(1+\alpha z)^{\gamma/\alpha}}{k!}\left(\frac{(1+\alpha z)^{\beta/\alpha}-1}{\beta}\right)^k = \sum_{n=0}^{\infty} S(n,k;\alpha,\beta,\gamma)\frac{z^n}{n!} \tag{11}$$

(see [7, 9]). Clearly, the method established in (2), (3) and (4) to derive Schlömilch-type formulas cannot be applied directly to obtain such formula for $S(n,k;\alpha,\beta,\gamma)$. However, one can easily do it by applying the approach used in Sections 2 and 3. That is, one has to define first a kind of generalization of Whitney numbers by means of the following generating functions

$$\frac{1}{k!}\left(\frac{(1+\alpha z)^{\beta/\alpha}-1}{\beta}\right)^k = \sum_{n=0}^{\infty} w(n,k;\alpha,\beta)\frac{z^n}{n!} \tag{12}$$

$$\frac{1}{k!}\left(\frac{(1+\beta z)^{\alpha/\beta}-1}{\alpha}\right)^k = \sum_{n=0}^{\infty} W(n,k;\alpha,\beta)\frac{z^n}{n!} \tag{13}$$

and establish some necessary properties, particularly, the Schlömilch-type formula for $w(n,k;\alpha,\beta)$. It can easily be verified that when $\beta \to 0$,

$$\frac{(1+\alpha z)^{\beta/\alpha}-1}{\beta} \to \frac{\ln(1+\alpha z)}{\alpha} \quad \text{and} \quad \frac{(1+\beta z)^{\alpha/\beta}-1}{\alpha} \to \frac{e^{\alpha z}-1}{\alpha}.$$

These facts imply that the generating functions for Whitney numbers of the first and second kind can be deduced, respectively, from the generating functions in (12) and (13) by letting $\beta \to 0$. This confirms that the pair $\{w_m(n,k;\alpha,\beta), W_m(n,k;\alpha,\beta)\}$ is a kind of generalization of the pair of Whitney numbers.

We observe that the generating functions in (12) and (13) belong to the family of generating functions defined in (2) and (3). Hence, by making use of relation (4), we obtain

$$w(n,k;\alpha,\beta)=\sum_{r=0}^{n-k}(-1)^r\binom{n-1+r}{n-k+r}\binom{2n-k}{n-k-r}W(n-k+r,r;\alpha,\beta)$$

where

$$W(n-k+r,r;\alpha,\beta)=S(n-k+r,r;\beta,\alpha,0)=\frac{1}{\alpha^r r!}\sum_{j=0}^{r}(-1)^j\binom{r}{j}(\alpha(r-j)|\beta)_{n-k+r}.$$

It follows that, for $\alpha\neq 0$,

$$w(n,k;\alpha,\beta)=\sum_{r=0}^{n-k}\sum_{j=0}^{r}(-1)^{r+j}\binom{r}{j}\binom{n-1+r}{n-k+r}\binom{2n-k}{n-k-r}\frac{(\alpha(r-j)|\beta)_{n-k+r}}{\alpha^r r!}. \tag{14}$$

Furthermore, using the method in Section 2 and 3 with

$$(1+\alpha z)^{\gamma/\alpha}=\sum_{n=0}^{\infty}(\gamma|\alpha)_n\frac{z^n}{n!}$$

$$\frac{1}{k!}\left(\frac{(1+\alpha z)^{\beta/\alpha}-1}{\beta}\right)^k=\sum_{n=0}^{\infty}w(n,k;\alpha,\beta)\frac{z^n}{n!},$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty}S(n,k;\alpha,\beta,\gamma)\frac{z^n}{n!}&=\left(\sum_{n=0}^{\infty}(\gamma|\alpha)_n\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}w(n,k;\alpha,\beta)\frac{z^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\sum_{i=0}^{n}(\gamma|\alpha)_{n-i}\frac{z^{n-i}}{(n-i)!}w(i,k;\alpha,\beta)\frac{z^i}{i!}\\&=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{h=0}^{i-k}\sum_{j=0}^{h}(-1)^{h+j}\binom{n}{i}\binom{h}{j}\binom{i-1+h}{i-k+h}\binom{2i-k}{i-k-h}\times\\&\quad\times\frac{(\gamma|\alpha)_{n-i}(\alpha(h-j)|\beta)_{i-k+h}}{\alpha^h h!}\frac{z^n}{n!}.\end{aligned}$$

Comparing coefficients completes the proof of the following theorem.

Theorem 7. *For $\alpha\neq 0$, the Schlömilch-type formula for the unified generalization of Stirling numbers is given by*

$$\begin{aligned}S(n,k;\alpha,\beta,\gamma)=&\sum_{i=0}^{n}\sum_{h=0}^{i-k}\sum_{j=0}^{h}(-1)^{h+j}\binom{n}{i}\binom{h}{j}\binom{i-1+h}{i-k+h}\binom{2i-k}{i-k-h}\times\\&\times\frac{(\gamma|\alpha)_{n-i}(\alpha(h-j)|\beta)_{i-k+h}}{\alpha^h h!}.\end{aligned}$$

Note that, when $\alpha=m$, $\beta=0$ and $\gamma=-r$, Theorem 7 yields

$$\begin{aligned}S(n,k;m,0,-r)=&\sum_{i=k}^{n}\sum_{h=0}^{i-k}\sum_{j=0}^{h}(-1)^{h+j}\binom{n}{i}\binom{h}{j}\binom{i+h-1}{i+h-k}\binom{2i-k}{i-h-k}\times\\&\times\frac{(-r|m)_{n-i}(m(h-j))^{i-k+h}}{m^h h!},\end{aligned}$$

which is exactly the Schlömilch-type formula for r-Whitney numbers of the first kind in Theorem 6.

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