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On the Existence of Solutions to Quasilinear Elliptic Equations with Perturbed Coefficients

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Abstract

Let $Lu = -\sum_{i,j=1}^{N} a_{ij}(x,u)D_{ij}u + c(x,u)u$, $c(x,r) \geq \alpha > 0$. Consider the quasilinear elliptic equation $Lu = f(x,u,\nabla u)$ on a bounded smooth domain Ω in $\mathbb{R}^N$, where $|f(x,r,\xi)| \leq C_0 + h(|r|)|\xi|^\theta$, $0 < \theta < 2$ and h is a locally bounded function. It is shown that if the oscillations of $a_{ij}(x,r)$ with respect to r are sufficiently small on $[-C_0/\alpha, C_0/\alpha]$ uniformly for x in $\bar{\Omega}$, then there exists a solution u in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Keywords: quasilinear elliptic equation, $W^{2,p}$ estimate, $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ solution
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1 Introduction

Let Ω be a *bounded* $C^{1,1}$ domain in $\mathbb{R}^N$, $N \geq 3$, and let L_v, L, D_v, D be *elliptic operators* defined by

$$\begin{aligned} L_v u &= -\sum_{i,j=1}^{N} a_{ij}(x,v)D_{ij}u + c(x,v)u, \\ Lu &= L_u u, \\ D_v u &= -\sum_{i,j=1}^{N} D_i(a_{ij}(x,v)D_j u) + c(x,v)u, \text{ and} \\ Du &= D_u u, \end{aligned}$$

where $\sum_{i,j=1}^{N} a_{ij}(x,r)\xi_i\xi_j \geq \lambda|\xi|^2$ for some positive constant λ.

For a positive integer m and a real number p, $p \geq 1$, let

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \,\text{for}\, 0 < |\alpha| \leq m\},$$

$$\|u\|_{m,p} = (\int_\Omega \sum_{|\alpha| \leq m} |D^\alpha u|^p)^{\frac{1}{p}},$$

$W_0^{m,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in the space $W^{m,p}(\Omega)$ and

$$H_0^1(\Omega) = W_0^{1,2}(\Omega).$$

Consider the quasilinear elliptic equation

$$Lu = f(x, u, \nabla u). \tag{1}$$

Under the assumptions of a priori bounds and oscillations with respect to x of all possible solutions in $C^{2,\beta}(\bar{\Omega})$, O. A. Ladyzhenskaya and N. N. Ural'tseva proved the existence of solutions in $C^{2,\beta}(\bar{\Omega})$ for a general version of Equation (1) [7, p. 371]. If Ω is a bounded domain (without smoothness constraints on the boundary) and a_{ij}, c are L^∞ functions , L. Boccardo, F. Murat and J. P. Puel [1] set up a weak maximum principle for approximating solutions in $H_0^1(\Omega) \cap L^\infty(\Omega)$ to the elliptic equation in divergence form

$$Du = f(x, u, \nabla u), \tag{2}$$

where

$$|f(x, r, \xi)| \le C_0 + h(|r|)|\xi|^2 \tag{3}$$

with h being a locally bounded function defined on $\mathbb{R}^+$, and derived a solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$. Furthermore, all solutions in $H_0^1(\Omega) \cap L^\infty(\Omega)$ lie in $[-C_0/\alpha, C_0/\alpha]$.

It then arises readily to investigate $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ solutions to (1). Let a fixed point x in $\bar{\Omega}$ and an interval I in $\mathbb{R}$ be given. We denote as $\operatorname{osc} a_{ij}(x, r; I)$ *the oscillations of* $a_{ij}(x, r)$ *with respect to* r*, for* r *in* I, that is

$$\operatorname{osc} a_{ij}(x, r; I) = \sup\{|a_{ij}(x, r_1) - a_{ij}(x, r_2)| : r_1, r_2 \in I\},$$

and

$$\operatorname{osc} a(x, r; I) = \max_{1 \le i,j \le N} \operatorname{osc} a_{ij}(x, r; I).$$

In the light of the classical existence result of strong solutions in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ [3, p. 241] to the linear elliptic equation

$$L_0 u = -\sum_{i,j=1}^{N} a_{ij}(x) D_{ij} u + c(x) u = f(x), f \in L^p(\Omega), \tag{4}$$

[4] and [5] employed the perturbation method for $W^{2,p}$ estimate of Equation (4) together with the weak maximum principle of [1] when a_{ij}, c depend on both x and r, and derived the existence result in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ provided $\operatorname{osc} a(x, r; \mathbb{R})$ is sufficiently small uniformly for x in $\bar{\Omega}$ and

$$|f(x, r, \xi)| \le C_0 + h(|r|)|\xi|^\theta \text{ for } 0 < \theta < 2. \tag{5}$$

Recently, the result in [6] improved the existence theorem of solutions for

$$f(x, r, \xi) = o[|r| + h(|r|)|\xi|^2]. \tag{6}$$

This paper aims to extend the above results to the case that

$$|f(x, r, \xi)| \le C_0 + h(|r|)|\xi|^\theta, 0 < \theta < 2 \tag{7}$$

and

$$\operatorname{osc} a(x, r; [-C_0/\alpha, C_0/\alpha]) \le \frac{\lambda}{C_1}$$

for all x in $\bar{\Omega}$, where C_1 depends on N, p and the diffeomorphism (see Lemma 1). Our main result in Theorem 1 proves the existence of strong solutions to (1). We shall also remark in Corollary 2 that the existence result to (2) remains valid if a_{ij} are independent of r on $[-C_0/\alpha, C_0/\alpha]$. The main idea relies on the L^∞ estimate of solutions to (2) and (1). Proposition 1 then carries out a priori bounds for the solutions when $0 < \theta < 2$.

2 Preliminaries

Let a_{ij}, their derivatives $D_i a_{ij}$, $D_r a_{ij}$ and c are bounded Carathodory functions, and $c \geq \alpha$ for some positive number α. For simplicity, we denote

$$I_0 = [-C_0/\alpha, C_0/\alpha],$$

$$W(\Omega) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$

$$B_t = \{v \in W(\Omega) : \|v\|_{2,p} \leq t\}$$

and

$$g(v) = f(x, v, \nabla v)$$

and use C for a generic constant in this paper. Now assume that f is a Carathodory function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ satisfying (7) and, for $v \in W(\Omega)$, consider the equation

$$L_v u = g(v) \tag{8}$$

in Ω. If p is greater than N, then there exists a unique solution u in $W(\Omega)$. We start with a $W^{2,p}$ estimate for the solutions in $W(\Omega)$. It is well known [3] that if $u \in W^{2,p}(\Omega)$ and $1 < p < \infty$, one has

$$\|u\|_{2,p} \leq C(\|u\|_p + \|L_0 u\|_p).$$

For operators L_v, we quote the result from [4].

Lemma 1. *Let Ω be a bounded domain in $\mathbb{R}^N$ which is $C^{1,1}$ diffeomorphic to a ball in $\mathbb{R}^N$, and the coefficients $a_{ij} \in C^{0,1}(\bar{\Omega} \times \mathbb{R})$, $c \in L^\infty(\Omega \times \mathbb{R})$, $|a_{ij}|$, $|c| \leq \Lambda$ with Λ a positive constant, i, j= 1, $\ldots, N$. Then there exists a positive number C_1 (depending on N, p and the diffeomorphism) such that if*

$$\operatorname{osc} a(x, r; \mathbb{R}) \leq \frac{\lambda}{C_1}$$

for all x in $\bar{\Omega}$, one has the estimate

$$\|u\|_{2,p} \leq C(\|L_v u\|_p + \|u\|_p) \tag{9}$$

for every u in $W(\Omega)$ and $L_v u$ belonging to $L^p(\Omega)$, $1 < p < \infty$, where C is a constant (independent of v) depending on $N, p, \lambda, \Lambda, \partial\Omega, \Omega$, the diffeomorphism and the moduli of continuity of $a_{ij}(x, r)$ with respect to x in $\bar{\Omega}$.

Remark 1. The magnitude of $\operatorname{osc} a(x, r; \mathbb{R})$ fulfilling the purpose of Lemma 1 can be found in [8, p. 23] if Ω is a ball in $\mathbb{R}^N$ and in [4, p. 191] if Ω is $C^{1,1}$ diffeomorphic to a ball in $\mathbb{R}^N$.

Remark 2. One can derive (9) using interior and exterior estimates similar to those in the proof of Theorem 9.11 and Theorem 9.13 [3] with $a_{ij}(x)$ replaced by $a_{ij}(x, v(x))$.

In view of Lemma 1, if u is a $W(\Omega)$ solution to (8), then

$$\|u\|_{2,p} \leq C(\|g(v)\|_p + \|u\|_p). \tag{10}$$

Furthermore, an application of the weak maximum principle of A. D. Aleksandrov [3, p. 220] implies that

$$\|u\|_\infty \leq C\|\frac{g(v)}{\mathcal{D}^*}\|_N,$$

where $\mathcal{D}^*$ is the geometric mean of the eigenvalues of the matrix $[a_{ij}]$, and C depends on N and the diameter of Ω. By ellipticity, $\mathcal{D}^* \geq \lambda > 0$, so

$$\|u\|_p \leq C\|g(v)\|_p. \tag{11}$$

Combining (10) and (11), one gets

$$\|u\|_{2,p} \leq C\|g(v)\|_p. \tag{12}$$

3 The Existence of Strong Solutions

In this section, we investigate the existence of strong solutions to (1) via the perturbation method, where f satisfies (7), $0 < \theta < 2$.

Lemma 2. *The map $\tilde{g}$ which assigns v in B_t to the $W(\Omega)$ solution u of (8) is continuous in $W^{1,p}(\Omega)$ if p is greater than N.*

Proof. See [4, p. 196]. □

Also, we quote the following result of Theorem 2.1 in [1, p. 28].

Lemma 3. *Assume that*

$$|f_1(x, r, \xi)| \leq C + k(|r|)|\xi|^2,$$

where k is an increasing function from $\mathbb{R}^+$ into $\mathbb{R}^+$. If the solution u to

$$Du = f_1(x, u, \nabla u)$$

is in $H_0^1(\Omega) \cap L^\infty(\Omega)$, then it varifies

$$-\frac{C_0}{\alpha} \leq u \leq \frac{C_0}{\alpha}$$

almost everywhere in Ω.

Remark 3. One can replace the increasing function k by a locally bounded function h in Lemma 3 by setting

$$k(|r|) = \sup\{h(|s|) : 0 \leq s \leq r\}.$$

Now, we reformulate (1),

$$Du = \tilde{f}(x, u, \nabla u),$$

where

$$\tilde{f}(x, u, \nabla u) = g(u) - [D_i a_{ij}(x, u) + D_r a_{ij}(x, u) D_i u] D_j u.$$

Proposition 1. *If $u \in W(\Omega)$ is a solution to (1) with the function f satisfying (7), then $u(x) \in I_0$ almost everywhere in Ω.*

Proof. Let ϵ be a sufficiently small positive number. By the inequality of arithmetic and gemetric means

$$\lambda_1 a_1 + \lambda_2 a_2 \geq a_1^{\lambda_1} a_2^{\lambda_2}$$

for positive numbers λ_1, λ_2 satifying

$$\lambda_1 + \lambda_2 = 1,$$

one gets

$$\lambda_1 \frac{\epsilon}{h(|r|)} + \lambda_2 [\frac{h(|r|)}{\epsilon}]^{\frac{\lambda_1}{\lambda_2}} |\xi|^2 \geq [\frac{\epsilon}{h(|r|)}]^{\lambda_1} [\frac{h(|r|)}{\epsilon}]^{\frac{\lambda_1}{\lambda_2} \cdot \lambda_2} |\xi|^{2\lambda_2}.$$

When λ_2 equals to $\theta/2$,

$$h(|r|)|\xi|^\theta \leq \epsilon + h_1(|r|)|\xi|^2,$$

where h_1 is a locally bounded function, which in turn implies that

$$|\tilde{f}| \leq C_0 + \epsilon + h_2(|r|)|\xi|^2$$

for some locally bounded function h_2. Since ϵ is arbitrary, applying Remark 3 and Lemma 3, one concludes that $u(x)$ lies in I_0 almost everywhere in Ω for every solution $u \in W(\Omega)$ to Equation (1) with the function f satisfying (7). □

It then suffices to examine the existence of strong solutions to (1) with $a_{ij}(x,r)$ replaced by

$$b_{ij}(x,r) = \begin{cases} a_{ij}(x, \frac{-C_0}{\alpha-\alpha_0}), & \text{if } r < \frac{-C_0}{\alpha} \\ a_{ij}(x,r), & \text{if } \frac{-C_0}{\alpha} \leq r \leq \frac{C_0}{\alpha} \\ a_{ij}(x, \frac{C_0}{\alpha-\alpha_0}), & \text{if } r > \frac{C_0}{\alpha} \end{cases}. \tag{13}$$

Denote $\tilde{L}_v$ and $\tilde{L}$ the elliptic operators defined by

$$\tilde{L}_v u = -\sum_{i,j=1}^{N} b_{ij}(x,v) D_{ij} u + c(x,v)u$$

and

$$\tilde{L}u = \tilde{L}_u u.$$

Consider now the equation

$$\tilde{L}u = g(u) \tag{14}$$

in Ω. Let g_n be the truncation of g by $\pm n$. For v in $W^{1,p}(\Omega)$, the Dirichlet problem

$$\tilde{L}_v u = g_n(v) \tag{15}$$

has a unique solution $u_{n,v}$ in $W(\Omega)$. We note here that

$$\operatorname{osc} b_{ij}(x,r;\mathbb{R}) = \operatorname{osc} a_{ij}(x,r;I_0).$$

So if

$$\operatorname{osc} a(x,r;I_0) \leq \frac{\lambda}{C_1}$$

for x in $\bar{\Omega}$, the estimate (9) holds. Lemma 2 and the Schauder fixed point theorem imply that there exists a $W(\Omega)$ solution u_n to the truncated equation

$$\tilde{L}u = g_n(u). \tag{16}$$

We proceed to the $W^{2,p}$ estimate of (u_n).

Lemma 4. *If*

$$\operatorname{osc} a(x, r; I_0) \leq \frac{\lambda}{C_1}$$

for all x in $\bar{\Omega}$, then the approximating solutions (u_n) to (14) are $W^{2,p}$ bounded.

Proof. Since f satisfies (7),

$$|g_n(u_n)| \leq C_0 + \alpha_0 |u_n| + h(|u_n|)(C_\epsilon + \epsilon |\nabla u_n|^2).$$

Thus, by Proposition 1,

$$|g_n(u_n)| \leq C + \epsilon |\nabla u_n|^2.$$

Also, because each u_n belongs to $L^\infty(\Omega) \cap W^{2,p}(\Omega)$, from the Gagliardo-Nirenberg interpolation theorem [2, p. 194], we obtain

$$\|\nabla u_n\|_{2p}^2 \leq C \|u_n\|_\infty \|u_n\|_{2,p}.$$

So

$$\|g_n(u_n)\|_p \leq C + \epsilon \|u_n\|_{2,p}. \tag{17}$$

Combining (9) and (17), one deduces that

$$\begin{aligned} \|u_n\|_{2,p} &\leq C(\|u_n\|_p + \|g_n(u_n)\|_p) \\ &\leq C + \epsilon \|u_n\|_{2,p}. \end{aligned}$$

We get a $W^{2,p}$ bounded sequence (u_n) if ϵ is sufficiently small. □

Once L^∞ and $W^{2,p}$ bounds are established, the existence of solutions in $W(\Omega)$ can be deduced as in the proof of Theorem 3.1 [4, p. 201]. For the sake of completion, we quote the proof in Theorem 1.

Theorem 1. *Let Ω be $C^{1,1}$ smooth in $\mathbb{R}^N$, $N \geq 3$, $a_{ij} \in C^{0,1}(\bar{\Omega} \times \mathbb{R})$, a_{ij}, $D_i a_{ij}$, $D_r a_{ij}$, $c \in L^\infty(\Omega \times \mathbb{R})$. Then there exists a solution u in $W(\Omega)$ to Equation (1) provided*

$$\operatorname{osc} a(x, r; I_0) \leq \frac{\lambda}{C_1}$$

for all x in $\bar{\Omega}$.

Proof. By Lemma 4, we get $W^{2,p}$ bounded approximating solutions to (14). It then follows from the compact imbedding $W^{2,p}(\Omega) \to W^{1,p}(\Omega)$ that there exists a subsequence, still denoted by (u_n), such that $u_n \to u$, $\nabla u_n \to \nabla u$ almost everywhere and $u_n \to u$ in $W_0^{1,p}(\Omega)$. Now, since $\|u_n\|_{2,p}$ are bounded by some positive quantity t and the set B_t is closed in $W^{1,p}(\Omega)$, the limit u of (u_n) belongs to $W^{2,p}(\Omega)$. By passing to the limit and using the Vitali Convergence Theorem, one deduces that $\tilde{L} u_n \to \tilde{L} u$ in $\mathfrak{D}'(\Omega)$ and $g_n(u_n) \to g(u)$ in $L^1(\Omega)$ [1] which proves that u is a $W(\Omega)$ solution to (14).

Finally, since solutions to (1) lie in I_0 almost everywhere in Ω, $a_{ij}(x, u(x))$ are equal to $b_{ij}(x, u(x))$. One concludes that u is in fact a solution to (1). □

Corollary 1. *There exists a solution in $W(\Omega)$ to the elliptic equation (2) if a_{ij} are independent of r for $|r| \leq C_0/\alpha$.*

Proof. In view of Proposition 1, consider Equation (2) with a_{ij} truncated by b_{ij} in (13), that is

$$-\sum_{i,j=1}^{N} D_i(b_{ij}(x,u)D_ju) + c(x,u)u = g(u). \tag{18}$$

Since a_{ij} are independent of r and $b_{ij}(x,r)$ equal to $a_{ij}(x,r)$ for r in I_0, $b_{ij}(x,r)$ can be written by $b_{ij}(x)$ on I_0. Hence, (18) can be reformulated as

$$-\sum_{i,j=1}^{N} b_{ij}(x)D_{ij}u + c(x,u)u = \hat{f}(x,u,\nabla u), \tag{19}$$

where

$$\hat{f}(x,u,\nabla u) = g(u) + \sum_{i,j=1}^{N} D_ib_{ij}(x)D_ju.$$

One can then get a $W^{2,p}$ bounded approximating solution sequence (u_n) to

$$-\sum_{i,j=1}^{N} b_{ij}(x)D_{ij}u + c(x,u)u = \hat{f}_n(x,u,\nabla u)$$

without assuming small oscillations of $b_{ij}(x,r)$ to r. Applying the proof in Theorem 1, there exists a $W(\Omega)$ solution u to (19). Finally, since solutions to (2) lie in I_0 almost everywhere in Ω and $a_{ij}(x,u(x))$ equal to $b_{ij}(x)$ for $|u(x)| \leq C_0/\alpha$, the solution $u(x)$ to (19) in fact is a solution to (2). □

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