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On Some General Monge-Ampère Type Equations

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Abstract

We prove the existence and uniqueness of the classical solutions for some general Monge-Ampère type equations.

Keywords: comparison principle, Monge-Ampère type equation, cost function.

1 Introduction

Suppose that Ω is a domain in Euclidean n-space $\mathbb{R}^n$. Let Ω be uniformly convex with uniform radius $R > 0$ and $f = f(x)$ a positive function on Ω such that

$$f_0 := \inf_{\Omega} f > 0. \tag{1.1}$$

In this paper, we consider the Dirichlet problem for the Monge-Ampère type equation

$$\begin{cases} F[u] := \det(D^2u - A(Du)) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is smooth. The operator F in (1.2) is elliptic with respect to u whenever

$$D^2u - A(Du) > 0. \tag{1.3}$$

Let $\phi \in C^2(\overline{\Omega})$ be a uniformly convex function satisfying

$$\begin{cases} \det D^2\phi = 1, & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

The uniform convexity of ϕ ensures that there is a constant $\rho > 0$ such that

$$D^2\phi \geq \rho I, \text{ in } \Omega, \tag{1.5}$$

(see Gilbarg-Trudinger [2]), where I is the $n \times n$ identity matrix. Set

$$K_f := \frac{2}{\rho}\ \|f\|_\infty^{1/n} \tag{1.6}$$

and

$$\psi := K_f\ \phi. \tag{1.7}$$

Then we have

$$D^2\psi \geq 2\|f\|_\infty^{1/n} I. \tag{1.8}$$

This type of the equation $F[u] = f$ in (1.2) is connected to optimal transportation and studied by many authors. For the matrix A, we assume that

$$D^2\psi - A(D\psi) \geq s_0 I, \tag{1.9}$$

for some constant $s_0 > 0$ and

$$\left[\sum_{k=1}^n A_{ij,p_k}(p)\eta_k\right] \leq \varepsilon_0 |p|^\ell |\eta| I, \tag{1.10}$$

for all $p,\ \eta \in \mathbb{R}^n$, where $\ell \geq 1$ and ε_0 is a positive constant satifying

$$0 < \varepsilon_0 < 2^{-2(\ell+1)} \left(\frac{\rho}{R}\right) (K_f R)^{-\ell}, \tag{1.11}$$

as well as

$$\sum_{i,j,k,\ell=1}^n A_{ij,p_k p_\ell}(p)\eta_k\eta_\ell \geq 0, \tag{1.12}$$

for all $p, \eta \in \mathbb{R}^n$. It is equivalent to that the function $\sum\limits_{i,j=1}^n A_{ij}(p)$ is convex. Instead of the assumption (1.11), Liu-Trudinger-Wang [5] made more assumptions on the domain on Ω. Moreover, from (1.11), we have for $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n$,

$$\begin{aligned}\sum_{i,j} \Big(A_{ij}(2D\psi) - A_{ij}(D\psi)\Big)\xi_i\xi_j &\leq \xi^T \big(\varepsilon_0 |\sigma D\psi|^\ell |D\psi| I\big)\xi \\ &\leq \varepsilon_0\, 2^\ell\, |D\psi|^{\ell+1} |\,\xi|^2 \\ &\leq \|f\|_\infty^{1/n} I\end{aligned}$$

where $\sigma = \sigma(x) \in [0,1]$. That is

$$A(2D\psi) - A(D\psi) \leq \|f\|_\infty^{1/n} I. \tag{1.13}$$

Hence, by (1.9) and (1.13), we have

$$\begin{aligned}F[2\psi] &\geq \det\big(D^2\psi - A(D\psi)\big) + \det\Big(D^2\psi - \big(A(2D\psi) - A(D\psi)\big)\Big) \\ &\geq F[\psi] + \|f\|_\infty. \end{aligned} \tag{1.14}$$

Given $\alpha \in (0,1)$, let $\mathscr{M}_\alpha(\overline{\Omega})$ and X be defined by

$$\mathscr{M}_\alpha(\overline{\Omega}) := \Big\{ v \in C^{2,\alpha}(\overline{\Omega}) : D^2 v - A(Dv) > sI \text{ for some } s > 0 \Big\} \tag{1.15}$$

and

$$X := \Big\{ v \in \mathscr{M}_\alpha(\overline{\Omega}) : \mathfrak{F}[v] = \sigma \mathfrak{F}[\psi] \text{ for some constant } \sigma \in [0,1],\ v = 0 \text{ on } \partial\Omega \Big\} \tag{1.16}$$

where

$$\mathfrak{F}[u] := \frac{F[u]}{f(x)} - 1. \tag{1.17}$$

Then $X \neq \emptyset$ since $\psi \in X$. For each $v \in X$, from (1.14),

$$F[v] \leq f(x) + \sigma F[\psi] \leq F[2\psi]. \tag{1.18}$$

In conjunction with applications to optimal transportation, the equation (1.2) in general form is

$$F[u] := \det\big(D^2 u - A(x, Du)\big) = f(x, u, Du) \tag{1.19}$$

which comes from

$$\det DT_u = \psi(x, u, Du) > 0$$

where the mapping $T_u : \Omega \to \mathbb{R}^n$ is given by

$$T_u(\cdot) = Y(\cdot, Du(\cdot))$$

for some C^1 vector field $Y : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ with

$$|\det DT_u| = \psi(x, u, Du). \tag{1.20}$$

To write (1.20) in the form (1.19), we assume

$$\det Y_p \neq 0 \tag{1.21}$$

and obtain the elliptic solution of the equation (1.19) with

$$A(x, p) = -Y_p^{-1} Y_x. \tag{1.22}$$

When Y is generated by a cost function $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, satisfying

$$c_x\big(\cdot, Y(\cdot, p)\big) = p, \tag{1.23}$$

(1.23) gives the optimal transportation equation

$$F[u] = \det\Big(D^2 u - D_x^2 c\big(x, Y(x, Du)\big)\Big) = \frac{\psi(x, u, Du)}{|\det Y_p(x, Du)|}. \tag{1.24}$$

Here

$$c_{x,y}(x, Y(x, p)) = Y_p^{-1}(x, p)$$

and

$$c_{xx}(x, Y(x,p)) = -Y_p^{-1}(x,p)Y_x(x,p). \tag{1.25}$$

Ma-Trudinger-Wang [6] showed that the classical solution of (1.19) exits for the second boundary value problem. In this paper, we shall show that the solution of the Dirichlet problem (1.2) exists uniquely under the assumptions (1.9)-(1.12).

In section 2, we provide a comparison principle for the operator $F[u]$ in general form and establish a priori estimates of functions in X and their first and second derivatives in supremum norm. A priori estimates of $C^{2,\alpha}$ norms for functions in X are then obtained by Theorem 17.26 of Gilbarg-Trudinger [2]. Using the method of continuity, the Dirichlet problem (1.2) admits a solution u in $C^{2,\alpha}(\overline{\Omega})$.

2 Global a priori estimates for second derivatives

In this section, we establish the supremum of functions in X and their derivatives in $\overline{\Omega}$. Then by the method of continuity we are able to obtain the existence of solutions. To find those a priori estimates, the following comparison principle for the operator $F[u]$ plays a crucial role. The proof is suggested by Neil Trudinger.

Theorem 2.1. *Assume the matrix $A = A(x,z,p)$ satisfies*

$$\left[\frac{\partial A_{ij}(x,z,p)}{\partial z}\right] \geq 0$$

where $A = [A_{ij}]$. Let $F[u] = \det(D^2u - A(x,u,Du))$. Suppose the matrices $D^2u - A(x,u,Du)$ and $D^2v - A(x,v,Dv)$ are positive definite. If $F[u] \geq F[v]$ and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\overline{\Omega}$.

Proof. Let $G[u] := \ell\text{og}F[u] = \ell\text{og}\det(D^2u - A(x,u,Du))$, and

$$W = W(u) := D^2u - A(x,u,Du) > 0,$$

$$G^{ij} := \frac{\partial G}{\partial W_{ij}} \qquad \text{where} \qquad W = [W_{ij}].$$

Then $[G^{ij}] = W^{-1}$. By the fact that G is concave on the set of all positive matrices (see Gilbarg-Trudinger [2]), we have

$$\begin{aligned} 0 &\leq G[u] - G[v] \\ &\leq G^{ij}[u]\left(D_{ij}u - A_{ij}(x,u,Du) - D_{ij}v + A_{ij}(x,v,Dv)\right) \\ &= G^{ij}[u]\,D_{ij}(u-v) - G^{ij}[u]\left(A_{ij}(x,u,Du) - A_{ij}(x,v,Dv)\right). \end{aligned}$$

Since $[G^{ij}] = W^{-1} > 0$, by the weak maximum principle, we have

$$u - v \leq 0, \text{ in } \overline{\Omega}.$$

□

Now we are at the position of (1.18) that for all $v \in X$,

$$F[v] \le \|f\|_\infty + F[\psi], \text{ in } \Omega$$

and $v = 0 = \psi$ on $\partial\Omega$. By Theorem 2.1, the comparison principle, we have

$$2\psi \le v \le 0, \text{ in } \overline{\Omega}$$

which gives

$$\sup_{\overline{\Omega}} |v| \le 2K_f \|\phi\|_\infty, \qquad \text{for all } v \in X. \tag{2.1}$$

The convexity of both ψ and v implies

$$\sup_{\Omega} |Dv| \le 2K_f \sup_{\Omega} |D\phi| \le 2K_f R, \quad \text{for all } v \in X. \tag{2.2}$$

To establish the supremum bound of second order derivatives estimates, we need the following inequalities. From (1.10) and (1.11),

$$\begin{aligned} \left[\sum_{k=1}^{n} A_{ij,p_k}(Dv) D_k \psi\right] &\le \varepsilon_0 |Dv|^\ell |D\psi| I \\ &\le \varepsilon_0 (2K_f R)^\ell (K_f R) I \\ &\le \|f\|_\infty^{1/n} I \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} D^2\psi - [A_{ij,p_k}(Dv) D_k \psi] &\ge D^2\psi - \|f\|_\infty^{1/n} I \\ &\ge \|f\|_\infty^{1/n} I. \end{aligned} \tag{2.4}$$

Theorem 2.2. *For each $v \in X$, we have*

$$\sup_{\Omega} |D^2 v| \le C, \tag{2.5}$$

where C is a positive constant depending on $\sup_{\partial\Omega} |D^2 v|$, Ω, A, $\sup_{\Omega} |Dv|$.

Proof. Let $G(W) := \ell\text{og} \det W$, for symmetric positive definite matrix W. Then

$$G_{ij}(W) := \frac{\partial G(W)}{\partial W_{ij}} = W^{ij} \tag{2.6}$$

and

$$G_{ij,k\ell}(W) := \frac{\partial G(W)}{\partial W_{ij} \partial W_{k\ell}} = -W^{ik} W^{j\ell}, \tag{2.7}$$

where $[W^{ij}] = W^{-1}$. G is concave on the set $\{W \in \mathbb{S}^{n\times n} : W > 0\}$.

For $v \in X$,

$$W(v) := D^2 v - A(Dv) > 0. \tag{2.8}$$

Then we have

$$G(W) = \overline{g^\sigma} \tag{2.9}$$

where

$$\overline{g^\sigma} := \left(1 - \sigma + \sigma \frac{F[\psi]}{f(x)}\right) f(x). \tag{2.10}$$

For each unit vector γ, we have

$$\overline{g^\sigma}_\gamma = \frac{\partial G(W)}{\partial \gamma} = W^{ij} D_\gamma(W_{ij}) \tag{2.11}$$

and

$$\begin{aligned} \overline{g^\sigma}_{\gamma\gamma} = \frac{\partial^2 G(W)}{\partial \gamma^2} &= W^{ij,k\ell} D_\gamma(W_{ij}) D_\gamma(W_{k\ell}) + W^{ij} D_{\gamma\gamma}(W_{ij}) \\ &\le W^{ij} D_{\gamma\gamma}(W_{ij}), \end{aligned} \tag{2.12}$$

by (2.7). Moreover,

$$\begin{aligned} D_{\gamma\gamma} W_{ij} &= D_{\gamma\gamma}\big(D_{ij} v - A_{ij}(Dv)\big) \\ &= D_{ij}(D_{\gamma\gamma} v) - A_{ij,p_k}(Dv) D_k(D_{\gamma\gamma} v) - A_{ij,p_k p_\ell} D_{k\gamma} v D_{\ell\gamma} v. \end{aligned}$$

Hence, we have by (2.7)

$$D_{\gamma\gamma} W_{ij} \le D_{ij}(D_{\gamma\gamma} v) - A_{ij,p_k}(Dv) D_k(D_{\gamma\gamma} v). \tag{2.13}$$

(2.12) and (2.13) ensure that

$$\overline{g^\sigma}_{\gamma\gamma} \le W^{ij} D_{ij}(D_{\gamma\gamma} v) - W^{ij} A_{ij,p_k} D_k(D_{\gamma\gamma} v). \tag{2.14}$$

Define a linear elliptic operator by

$$L(\cdot) := W^{ij} D_{ij}(\cdot) - W^{ij} A_{ij,p_k}(Dv) D_k(\cdot).$$

Then

$$\begin{aligned} L(D_{\gamma\gamma} v) &\ge \overline{g^\sigma}_{\gamma\gamma} \\ &\ge -C_1 \end{aligned} \tag{2.15}$$

where C_1 is a positive number depending on f. Moreover,

$$\begin{aligned} L(v) &= W^{ij}\big(W_{ij} + A_{ij}(Dv)\big) - W^{ij} A_{ij,p_k}(Dv) D_k v \\ &= n + W^{ij} A_{ij}(Dv) - A_{ij,p_k}(Dv) D_k v \\ &\ge n - C_2 \mathscr{T} \end{aligned} \tag{2.16}$$

where $\mathscr{T}$ is the trace of W^{-1} and C_2 is a positive number depending on A, A_{p_k}, and $\sup_\Omega |Dv|$.

For the function ψ defined on (1.7),

$$\begin{aligned} L(\psi) &= W^{ij} D_{ij}\psi - W_{ij} A_{ij,p_k}(Dv) D_k\psi \\ &= \operatorname{tr}\Big(W^{-1}\Big[D_{ij}\psi - A_{ij,p_k}(Dv)D_k\psi\Big]\Big) \\ &\geq \|f\|_\infty^{1/n} \mathscr{T}. \end{aligned} \tag{2.17}$$

Choose $k_1 = C_1/n$ and $k_2 = C_1 C_2 \|f\|_\infty^{-1/n}$ to obtain

$$L(D_{\gamma\gamma}v + k_1 v + k_2\psi) \geq 0. \tag{2.18}$$

By the weak maximum principle, we have

$$\sup_\Omega \big(D_{\gamma\gamma}v + k_1 v + k_2\psi\big) \leq \sup_{\partial\Omega} \big(D_{\gamma\gamma}v + k_1 v + k_2\psi\big) \tag{2.19}$$

which implies

$$\sup_\Omega D_{\gamma\gamma} v \leq C \tag{2.20}$$

where $C > 0$ depending on $\sup_{\partial\Omega} |D^2 v|$, A, f and Ω. On the other hand, for each unit vector γ,

$$\begin{aligned} D_{\gamma\gamma} v &= (W_{ij} + A_{ij}(Dv))\gamma_i\gamma_j \\ &\geq A_{ij}(Dv)\gamma_i\gamma_j \\ &\geq -C \end{aligned}$$

where $C > 0$ depends on A, f and Ω. Then there holds

$$\sup_\Omega |D^2 v| \leq C$$

where $C > 0$ depends on $\sup_{\partial\Omega} |D_{\gamma\gamma} v|$, A, f and Ω. □

Next, we want to estimate $\sup_{\partial\Omega} |D^2 u|$. The main idea of the proof follows from Trudinger [7].

Theorem 2.3. *For each $v \in X$,*

$$\sup_{\partial\Omega} |D^2| v \leq C \tag{2.21}$$

where $C > 0$ depends on $\partial\Omega$ and f and A.

Proof. Take any $x_0 \in \partial\Omega$. Without loss of generality, we may suppose that x_0 is the origin and the x_n axis points the inward normal direction. Let ω be a function defined in a neighborhood of the origin in $\mathbb{R}^{n-1}$ such that if $x' = (x_1, x_2, \cdots, x_{n-1})$ then $(x', \omega(x')) \in \partial\Omega$. That is, if $x_n = \omega(x')$, then $(x', x_n) \in \partial\Omega$. Then it is clear that

$$\omega(0) = 0 \text{ and } D\omega(0) = 0. \tag{2.22}$$

Since $\partial\Omega \in C^3$, we may assume $\omega \in C^3$ and

$$\omega(x') = \Omega_{\alpha\beta} x_\alpha x_\beta + O(|x'|^3) \text{ in a neighborhood of } 0, \tag{2.23}$$

where $1 \le \alpha, \beta \le n-1$, $\Omega_{\alpha\beta} := \frac{1}{2} D_{\alpha\beta}\omega(0)$. Following the proof in Trudinger[7], p.38, there holds

$$|D_{\alpha\beta} v(0)| \le C, \tag{2.24}$$

for all α, β, $1 \le \alpha, \beta \le n-1$, where $C > 0$ depending on $\partial\Omega$, f and the matrix A.

Next, for $\beta = 1, 2, 3, \cdots, n-1$, from the definition of $G(v)$, we differentiate $G(v)$ with respect to x_k to obtain

$$G^{ij}(v) D_{ij}(D_k v) = \overline{g^\sigma}_k + G^{ij}(v) A_{ij,p_\ell} D_\ell (D_k v) \tag{2.25}$$

where $[G^{ij}] := W^{-1}$, and $A_{ij,p_\ell} := \dfrac{\partial A_{ij}(Dv)}{\partial p_\ell}$. Define a new function by

$$v^\beta := D_\beta v + \omega_\beta D_n v, \;\; \beta = 1, 2, \cdots, n-1 \tag{2.26}$$

and a linear operator

$$L(\cdot) := G^{ij}(v) D_{ij}(\cdot) - G^{ij}(v) A_{ij,p_\ell}(Dv) D_\ell(\cdot). \tag{2.27}$$

Hence for $\beta = 1, 2, \cdots, n-1$, we have

$$Lv^\beta = G^{ij}(v) D_{ij} v^\beta - G^{ij}(v) A_{ij,p_\ell}(Dv) D_\ell v^\beta. \tag{2.28}$$

The operator L is uniformly elliptic since the matrix W^{-1} is uniformly positive. The first term of the right hand side in (2.28) is

$$\begin{aligned} G^{ij} D_{ij} v^\beta = \overline{g^\sigma}_\beta &+ G^{ij} A_{ij,p_\ell} D_\ell (D_\beta v) \\ &+ \omega_\beta \Big(\overline{g^\sigma}_n + G^{ij} A_{ij,p_\ell} D_\ell (D_n v) \Big) \\ &+ \Big(2G^{ij} \omega_{\beta i} D_{jn} v \Big) + \Big(\omega_{\beta ij} D_n v \Big). \end{aligned} \tag{2.29}$$

The third term in (2.29) is

$$\begin{aligned} 2G^{ij} \omega_{\beta i} D_{jn} v &= 2\omega_{\beta i} G^{ij} W_{jn} + 2G^{ij} \omega_{\beta i} A_{jn} \\ &= 2\omega_{\beta n} + 2G^{ij} \omega_{\beta i} A_{jn}. \end{aligned} \tag{2.30}$$

The second term of the right hand side in (2.29) becomes

$$G^{ij} A_{ij,p_\ell} D_\ell v^\beta = G^{ij} A_{ij,p_\ell} (D_{\ell\beta} v + \omega_{\alpha\beta} D_n v + \omega_\beta D_\ell (D_n v)). \tag{2.31}$$

To summarize, we have derived

$$Lv^\beta = \overline{g^\sigma}_\beta + \omega_\beta \overline{g^\sigma}_n + 2\omega_{\beta n} + 2G^{ij} \omega_{\beta i} A_{jn} + \omega_{\beta ij} D_n v. \tag{2.32}$$

Since $\mathscr{T} := \mathrm{tr}[G^{ij}] \ge n(\det W^{-1})^{1/n} = n/(\overline{g^\sigma})^{1/n}$, we finally obtain

$$|Lv^\beta| \le C\mathscr{T}, \tag{2.33}$$

where $C = C(\partial\Omega, f, \sup_\Omega |Dv|)$. Near the origin, we set

$$\xi(x) := \frac{1}{4}\left(\Omega_{\alpha m} - \frac{1}{2R}\delta_{\alpha m}\right) x_\alpha x_m - x_n + \frac{1}{8R}x_n^2 \tag{2.34}$$

where the sum is taken over $1 \le \alpha, m \le n-1$. Then

$$D_{ij}\xi = \frac{1}{2}\left(\Omega_{\alpha m} - \frac{1}{2R}\delta_{\alpha m}\right)(\delta_{i\alpha}\delta_{jm}) + \frac{1}{4R}\delta_{in}\delta_{jn}.$$

$$D_\ell \xi = \frac{1}{4}\left(\Omega_{\alpha m} - \frac{1}{2R}\delta_{\alpha m}\right)(\delta_{\ell\alpha}x_m + \delta_{\ell m}x_\alpha) - \delta_{\ell n} + \frac{1}{4R}\delta_{\ell n}x_n.$$

By calculation,

$$\begin{aligned} L\xi &= G^{ij}D_{ij}\xi - G^{ij}A_{ij,p_\ell}D_\ell\xi \\ &= \frac{1}{2}G^{\alpha m}\left(\Omega_{\alpha m} - \frac{1}{2R}\delta_{\alpha m}\right) + 2\left(\frac{1}{8R} + k_1\right)G^{nn} - G^{ij}A_{ij,p_\ell}D_\ell\xi \end{aligned}$$

where the sum is taken over $1 \le \alpha$, $m \le n-1$, and $1 \le i, j, \ell \le n$. Hence,

$$\begin{aligned} L\xi &= G^{ij}D_{ij}\xi - G^{ij}A_{ij,p_\ell}D_\ell\xi \\ &= \frac{1}{2}G^{\alpha m}\left(\Omega_{\alpha m} - \frac{1}{2R}\delta_{\alpha m}\right) + 2\left(\frac{1}{8R} + k_1\right)G^{nn} \\ &\quad + \frac{1}{2}G^{ij}A_{ij,p_\alpha}\left(\Omega_{\alpha m} - \frac{1}{2R}\delta_{\alpha m}\right)x_m \\ &\quad - G^{ij}A_{ij,p_n} + 2G^{ij}A_{ij,p_n}\left(\frac{1}{8R} + k_1\right)x_n \\ &\ge \frac{1}{4R}(\mathscr{T} - G^{nn}) - \frac{1}{8R}G^{nn} + G^{ij}A_{ij,p_n} - C'\varepsilon \end{aligned} \tag{2.35}$$

where the sum is taken over $1 \le \alpha, m \le n-1$, and $1 \le i, j \le n$. Hence for small $\varepsilon > 0$,

$$\begin{aligned} L\xi &\ge \mathscr{T}(1/8R - \varepsilon_0 2^\ell K_f^\ell R^\ell) - C'\varepsilon \\ &\ge \lambda_0\mathscr{T} - C'\varepsilon \\ &\ge \frac{1}{2}\lambda_0\mathscr{T} \end{aligned} \tag{2.36}$$

for some constant $\lambda_0 > 0$. On $\partial\Omega \cap B_\varepsilon(0)$,

$$x_n = \Omega_{\alpha m}x_\alpha x_m + O(|x'|^3),$$

we have

$$\begin{aligned} L\xi &= \frac{-3}{4}\Omega_{\alpha m}x_\alpha x_m - \frac{1}{8R}|x'|^2 + O(|x'|^3) \le \frac{-7}{8R}|x'|^2 + O(|x'|^3) \\ &\le \frac{-1}{2R}|x'|^2, \end{aligned} \tag{2.37}$$

for all small $\varepsilon > 0$. If $x \in \partial B_\varepsilon(0) \cap \Omega$, then

$$\varepsilon^2 = |x'|^2 + x_n^2, \text{ and } x_n \geq \omega(|x'|).$$

If $\varepsilon > 0$ is small then

$$\begin{aligned}\xi(x) &= \frac{-3}{4R}|x'| - \frac{1}{8R}|x'|^2 + \frac{1}{16R}x_n^2 + O(|x'|^3) \\ &\leq \frac{-1}{2R}|x'|^2, \end{aligned} \tag{2.38}$$

for $2|x'|^2 \geq x_n^2$. On the other case, $2|x'|^2 < x_n^2$, we have $-x_n \leq -\sqrt{\frac{2}{3}}\varepsilon$. Thus for small $\varepsilon > 0$,

$$\begin{aligned}\xi(x) &\leq \frac{1}{8R}\varepsilon^2 - \sqrt{\frac{2}{3}}\varepsilon \\ &\leq -\lambda_1\varepsilon, \end{aligned} \tag{2.39}$$

for some $\lambda_1 > 0$. Taking into accounts (2.37)$-$(2.39) and the maximum principle, $\xi(x) \leq v(x)$ in $B_\varepsilon(0) \cap \Omega$ for some small $\varepsilon > 0$. Since $\xi(0) = v(0)$, we obtain a bound for $D_{\gamma\gamma}v(0)$ depending $\partial\Omega$ and f. This proves Theorem 2.3. □

We are now able to apply Theorem 17.8, 17.16, 17.26 of Gilbarg-Trudinger [2] and obtain the following theorem.

Theorem 2.4. *Let $\partial\Omega \in C^4$. Then the Dirichlet problem* (1.2) *admit a classical solution $u \in C^{2,\alpha}(\overline{\Omega})$.*

Remark 2.5.

(a) If $c(x,y) = -\frac{1}{h}\sqrt{1 + h^2|x-y|^2}$, then

$$A(x,p) = -h\sqrt{1-|p|^2}(I - p \otimes p).$$

A satisfies (1.12) *whenever $p \perp \eta$ and it does satisfy (A3) condition of [6, 10], where*

$$(A3) \qquad \sum_{i,j,k,\ell=1}^{n} A_{ij,p_k p_\ell}\xi_i\xi_j\eta_i\eta_j \geq c_0|\xi|^2|\eta|^2,$$

for some positive constant c_0. The above condition is called (A3w) if $c_0 = 0$.

(b) Theorem 2.2 and Theorem 2.3 could hold if the assumption (1.12) *is replaced by (A3w) condition.*

(c) Liu-Trudinger [5] also proved the interior second derivative estimates of Pogorelov type for general Monge-Ampère type equations when the matrix A only satisfies the weaker condition (A3w).

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