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A Brief Expression of First Lyapunov Coefficient for Special Planar Systems

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Abstract

In this paper, we apply the Poincaré-Andronov-Hopf Bifurcation Theorem to two fundamental systems: generalized Liénard system and generalized Gause type predator-prey system, and derive a brief expression of the first Lyapunov coefficient κ (see below). As an example, we consider the system of the Chlorine Dioxide-Iodine-Malonic acid reaction in the end of the paper. We use the expression we suggest to show that both super- and sub-critical Hopf bifurcations may appear.

Keywords: Poincaré-Andronov-Hopf Bifurcation Theorem, generalized Liénard system, generalized Gause type, predator-prey system.

1 Introduction

Hopf bifurcation is a phenomenon which describes that a limit cycle will appear about any steady state which undergoes a transition from a stable to an unstable focus as some parameter varies [2]. There are two types of Hopf bifurcation. The one in which stable limit cycles occur about an unstable focus is called the supercritical Hopf bifurcation, while the other in which an unstable limit cycle occurs around a stable equilibrium is called the subcritical Hopf bifurcation. The subcritical case is always much more dramatic, and potentially dangerous in engineering applications. The way to distinguish super- and sub-critical Hopf bifurcation is the celebrated theorem named by Poincaré-Andronov-Hopf Bifurcation Theorem. Examples of such phenomenon can be found in the work of Poincaré [8]. The first specific study and formulation of a theorem was due to Andronov [1]. The work of Poincaré and Andronov was concerned with two-dimensional vector fields, while the theorem due to E. Hopf [3] is valid in n dimensions. For the sake of completeness, we state the theorem as follows. Consider the one parameter planar autonomous system:

$$\vec{x}\,' = \vec{G}(\vec{x}, \mu), \quad \vec{x} \in \mathbb{R}^2, \mu \in J \subseteq \mathbb{R}^1, \tag{1}$$

which satisfies the following assumptions:

A1. $\vec{G}(\vec{0}, \mu) = \vec{0}$ for all values μ near 0,

A2. $D\vec{G}(\vec{0}, \mu) = \begin{bmatrix} \alpha(\mu) & -\beta(\mu) \\ \beta(\mu) & \alpha(\mu) \end{bmatrix}$ with $\alpha(0) = 0 \neq \alpha'(0), \beta(0) \neq 0$.

Here, $0 \in J$ and $\vec{G} \in \mathcal{C}^3(\mathbb{R}^2\times\mathbb{R}, \mathbb{R}^2)$. Define $\vec{F}(\vec{x},\mu) = \vec{G}(\vec{x},\mu) - D\vec{G}(\vec{0},\mu)\vec{x} \equiv (f(\vec{x},\mu), g(\vec{x},\mu))^T$. Then the first Lyapunov coefficient of (1.1) is given by

$$\begin{aligned} 16\kappa \;&=\; (f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} + \beta^{-1}[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy})] \\ &\quad -\beta^{-1}[f_{xx}g_{xx} - f_{yy}g_{yy}]\ )|_{(\vec{0},0)}. \end{aligned} \tag{2}$$

Theorem 1.1 (Poincaré-Andronov-Hopf Bifurcation Theorem)**.**

1. *(Supercritical) If $\kappa < 0$, then there is an orbitally asymptotically stable limit cycle for $|\mu|$ sufficiently small.*
2. *(Subcritical) If $\kappa > 0$, then $\vec{0}$ is locally asymptotically stable and there is an orbitally unstable limit cycle for $|\mu|$ sufficiently small.*

In general, the computation of κ is not easy and it seems no way to simplify the calculation. But, for generalized Liénard system and generalized Gause type predator-prey system, it is possible to obtain a brief expression of κ. (See Theorems 2 and 4.) This is our goal in this paper.

The rest of this paper is organized as follows. In Section 2, we will apply the Poincaré-Andronov-Hopf Bifurcation Theorem to the generalized Liénard and Gause type predator-prey systems and get our main results. Then we consider the system of the Chlorine Dioxide-Iodine-Malonic acid reaction in Section 3 to show that our finding is applicable.

2 Main Results

Let us consider the following one parameter generalized Liénard system:

$$\begin{cases} u'(t) = \Pi(v(t),\,\mu)\;-\;H(u(t),\,\mu) \\ v'(t) = -\Psi(u(t),\,\mu) \end{cases}, \tag{3}$$

where $0 \in J \subseteq \mathbb{R}$ and $H, \Psi, \Pi \in \mathcal{C}^3(\mathbb{R}\times J, \mathbb{R})$ satisfy the following assumptions:

L1. $H(0,\mu) = H'(0,0) = 0 \neq \frac{d}{d\mu}H'(0,0)$ for all $\mu \in J$.

L2. $\Psi(0,\mu) = 0 \neq \Psi'(0,\mu)$ for all $\mu \in J$.

L3. $\Pi(0,\mu) = 0 \neq \Pi'(0,\mu)$ for all $\mu \in J$.

L4. $\Psi'(0,0)\Pi'(0,0) > 0$.

Clearly, $\vec{0}$ is the only equilibrium of (3) and the Jacobian matrix of (3) at $\vec{0}$ is given by

$$A(\mu) = \begin{bmatrix} -H'(0,\mu) & \Pi'(0,\mu) \\ -\Psi'(0,\mu) & 0 \end{bmatrix}.$$

The characteristic polynomial and eigenvalues of $A(\mu)$ are given by $\lambda^2 + H'(0,\mu)\lambda + \Psi'(0,\mu)\Pi'(0,\mu)$ and $\left(-H'(0,\mu) \pm \sqrt{H'^2(0,\mu) - 4\Psi'(0,\mu)\Pi'(0,\mu)}\right)/2$, respectively. Hence $\vec{0}$ is stable if $H'(0,\mu) > 0$ and $\vec{0}$ is unstable if $H'(0,\mu) < 0$. From (L1)-(L4), we have $\Delta(\mu) \equiv 4\Psi'(0,\mu)\Pi'(0,\mu) -$

$H'^2(0,\mu) > 0$ if $|\mu| \ll 1$. This gives that $\vec{0}$ is a spiral focus (stable or unstable) for μ near 0. For convenience, we set

$$\alpha(\mu) = -\frac{1}{2}H'(0,\mu),\ \beta(\mu) = \frac{1}{2}\sqrt{\Delta(\mu)}$$

for $|\mu| \ll 1$. Clearly, $\alpha(0) = 0 \neq \alpha'(0)$ and $\beta(0) \neq 0$.

Let's now consider the following change of variables:

$$\begin{bmatrix} x \\ y \end{bmatrix} = P^{-1}(\mu)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ -\frac{\alpha(\mu)u+\Pi'(0,\mu)v}{\beta(\mu)} \end{bmatrix}$$

where

$$P(\mu) = \begin{bmatrix} 1 & 0 \\ \frac{-\alpha(\mu)}{\Pi'(0,0)} & \frac{-\beta(\mu)}{\Pi'(0,0)} \end{bmatrix} \text{ and } P^{-1}(\mu) = \begin{bmatrix} 1 & 0 \\ -\frac{\alpha(\mu)}{\beta(\mu)} & -\frac{\Pi'(0,\mu)}{\beta(\mu)} \end{bmatrix}.$$

Then one has

$$u = x, \quad \& \quad v = -\frac{\alpha(\mu)x + \beta(\mu)y}{\Pi'(0,\mu)}, \tag{4}$$

from which (3) can be reduced into

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} \alpha(\mu) & -\beta(\mu) \\ \beta(\mu) & \alpha(\mu) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f(x,y,\mu) \\ g(x,y,\mu) \end{bmatrix}, \tag{5}$$

where

$$\begin{bmatrix} f(x,y,\mu) \\ g(x,y,\mu) \end{bmatrix} = P^{-1}(\mu)\begin{bmatrix} \Pi(v,\mu) - H(u,\mu) + H'(0,\mu)u - \Pi'(0,\mu)v \\ \Psi'(0,\mu)u - \Psi(u,\mu) \end{bmatrix}. \tag{6}$$

Now, we are in a position to state and prove our first result.

Theorem 2.1. *If (L1)-(L3) hold then* $16\kappa = -\,\Psi'(0,0)\left(\frac{H''(u,\mu)}{\Psi'(u,\mu)}\right)'\Big|_{(0,0)}$.

Proof. From (4), (6) and chain rule, we have

$$f_x(x,y,\mu) = H'(0,\mu) - H'(u,\mu) + \alpha(\mu) - \alpha(\mu)\Pi'(v,\mu)/\Pi'(0,\mu)$$

and

$$f_y(x,y,\mu) = \beta(\mu)(\Pi'(0,\mu) - \Pi'(v,\mu))/\Pi'(0,\mu).$$

Then a straightforward computation yields

$$\begin{aligned}
f_{xx}(x,y,\mu) &= -H''(u,\mu) + \alpha^2(\mu)\Pi''(v,\mu)/\Pi'^2(0,\mu)\\
f_{xy}(x,y,\mu) &= \alpha(\mu)\beta(\mu)\Pi''(v,\mu)/\Pi'^2(0,\mu),\\
f_{yy}(x,y,\mu) &= \beta^2(\mu)\Pi''(v,\mu)/\Pi'^2(0,\mu),\\
f_{xxx}(x,y,\mu) &= -H'''(u,\mu) - \alpha^3(\mu)\Pi'''(v,\mu)/\Pi'^3(0,\mu),\\
f_{xxy}(x,y,\mu) &= -\alpha^2(\mu)\beta(\mu)\Pi'''(v,\mu)/\Pi'^3(0,\mu),\\
f_{xyy}(x,y,\mu) &= -\alpha(\mu)\beta^2(\mu)\Pi'''(v,\mu)/\Pi'^3(0,\mu),\\
f_{yyy}(x,y,\mu) &= -\beta^3(\mu)\Pi'''(v,\mu)/\Pi'^3(0,\mu).
\end{aligned}$$

On the other hand, since, by (6), $g(x,y,\mu) = -\frac{\alpha(\mu)}{\beta(\mu)}f(x,y,\mu) - \frac{\Pi'(0,\mu)}{\beta(\mu)}(\Psi'(0,\mu)u - \Psi(u,\mu))$, we have

$$
\begin{aligned}
&g_x(x,y,\mu) = -(\alpha(\mu)f_x + \Pi'(0,0)(\Psi'(0,\mu) - \Psi'(u,\mu)))/\beta(\mu),\\
&g_y(x,y,\mu) = -\alpha(\mu)f_y/\beta(\mu),\\
&g_{xx}(x,y,\mu) = -(\alpha(\mu)f_{xx} - \Pi'(0,0)\Psi''(u,\mu))/\beta(\mu),\\
&g_{xy}(x,y,\mu) = -\alpha(\mu)f_{xy}/\beta(\mu), \quad g_{yy}(x,y,\mu) = -\alpha(\mu)f_{yy}/\beta(\mu),\\
&g_{xxy}(x,y,\mu) = -\alpha(\mu)f_{xxy}/\beta(\mu), \quad g_{yyy}(x,y,\mu) = -\alpha(\mu)f_{yyy}/\beta(\mu).
\end{aligned}
$$

Notice that $\alpha(0) = 0$ tells that $f_{xy} = f_{xyy} = f_{xxy} = g_y = g_{yy} = g_{yyy} = g_{xy} = g_{xxy} = 0$ if $\mu = 0$. As a consequence,

$$16\kappa = -H'''(0,0) + \frac{1}{\beta^2(0)}H''(0,0)\Pi'(0,0)\Psi''(0,0).$$

Since $2\alpha(0) = -H'(0,0) = 0$ and $4\beta^2(0) = \Delta(0) = 4\Psi'(0,0)\Pi'(0,0)$, we obtain the desired result. □

Observe that, from (L1) and (L2), there exists a $\delta > 0$ such that $H'(u,0) \neq 0$ and $\Psi(u,0) \neq 0$ for all $u \in (-\delta,\delta)$. Therefore, by Cauchy Mean Value Theorem, for any $u \in (-\delta,\delta) - \{0\}$, we have

$$\frac{H'(u,0)}{\Psi(u,0)} = \frac{H'(u,0) - H'(0,0)}{\Psi(u,0) - \Psi(0,0)} = \frac{H''(\xi,0)}{\Psi'(\xi,0)}, \tag{7}$$

for some ξ between 0 and u. On the other hand, one has by L'Hospital Rule

$$\lim_{u\to 0}\frac{H'(u,0)}{\Psi(u,0)} = \frac{H''(0,0)}{\Psi'(0,0)} \equiv A. \tag{8}$$

Note that

$$\frac{\frac{H'(u,0)}{\Psi(u,0)} - A}{u - 0} = \frac{\frac{H''(\xi,0)}{\Psi'(\xi,0)} - A}{\xi - 0}\cdot\frac{\xi}{u}. \tag{9}$$

The left hand side of (8) converges to $(\frac{H'(u,0)}{\Psi(u,0)})'|_{u=0}$ while the first part of the right hand side of (8) tends to $(\frac{H''(u,0)}{\Psi'(u,0)})'|_{u=0}$ if we let $u \to 0$. Since $\frac{\xi}{u}$ is always positive, $(\frac{H'(u,0)}{\Psi(u,0)})'|_{u=0}$ and $(\frac{H''(u,0)}{\Psi'(u,0)})'|_{u=0}$ have the same sign if $(\frac{H'(u,0)}{\Psi(u,0)})'|_{u=0} \neq 0$. Combining this result with Theorem 2.1, we have the following theorem.

Theorem 2.2. *If (L1)-(L3) hold and $(\frac{H'(u,0)}{\Psi(u,0)})'|_{u=0} \neq 0$, then*

$$\kappa \cdot \left(-\Psi'(0,0)\left(\frac{H'(u,\mu)}{\Psi(u,\mu)}\right)'\Bigg|_{(0,0)}\right) > 0.$$

Next, let us consider the following generalized Gause type predator-prey system:

$$\begin{cases} x'(t) = \varphi(x(t),\theta)(h(x(t),\theta) - \pi(y(t),\theta)),\\ y'(t) = \psi(x(t),\theta)\rho(y(t),\theta) \end{cases} \tag{10}$$

where φ, h, ψ, π, $\rho \in \mathcal{C}^3(\mathbb{R}_+ \times J, \mathbb{R})$, $\theta_* \in J \subseteq \mathbb{R}$ and for each $\theta \in J$ the following assumptions are satisfied.

G1. $\pi(0,\theta) = \rho(0,\theta) = 0 < \pi'(y,\theta)$ and $\rho'(y,\theta)\ \forall\, y \in \mathbb{R}_+$;

G2. $\varphi'(x,\theta) \geqslant 0 = \varphi(0,\theta)\ \forall\, x \in \mathbb{R}_+$;

G3. $\exists\, K_\theta > 0 \ni h(x,\theta)(x - K_\theta) < 0\ \forall\, x \in \mathbb{R}_+ - \{K_\theta\}$;

G4. $\exists\, \lambda = \lambda(\theta) \in (0, K_\theta) \ni \psi(x,\theta)(x-\lambda) > 0\ \forall\, 0 \leqslant x \neq \lambda$;

G5. $h((0,K_\theta],\theta) \subseteq \pi(\mathbb{R}_+,\theta)$ and $\delta = \delta(\theta) = \pi^{-1}(h(\lambda(\theta),\theta))$;

G6. $h'(\lambda(\theta_*),\theta_*) = 0 \neq \dfrac{d}{d\theta}(\varphi(\lambda(\theta),\theta)h'(\lambda(\theta),\theta))|_{\theta_*}$ and $\psi'(\lambda(\theta),\theta) > 0$.

Obviously, $e_\star = (\lambda,\delta)$ is the only positive equilibrium of (10) and the Jacobian matrix of the system (10) at $e_\star$ is

$$J = \begin{bmatrix} \varphi(\lambda,\theta)h'(\lambda,\theta) & -\varphi(\lambda,\theta)\pi'(\delta,\theta) \\ \rho(\delta,\theta)\psi'(\lambda,\theta) & 0 \end{bmatrix}.$$

The characteristic equation and eigenvalues are given by

$$s^2 - \varphi(\lambda,\theta)h'(\lambda,\theta)s + \varphi(\lambda,\theta)\pi'(\delta,\theta)\rho(\delta,\theta)\psi'(\lambda,\theta) = 0$$

and

$$\left(\varphi(\lambda,\theta)h'(\lambda,\theta) \pm \sqrt{[\varphi(\lambda,\theta)h'(\lambda,\theta)]^2 - 4\varphi(\lambda,\theta)\pi'(\delta,\theta)\rho(\delta,\theta)\psi'(\lambda,\theta)}\right)/2,$$

respectively. Hence $e_\star$ is stable if $h'(\lambda,\theta) < 0$ and $e_\star$ is unstable if $h'(\lambda,\theta) > 0$. Let $\Phi(x) \equiv \int_\lambda^x (\varphi(\xi))^{-1} d\xi$ and $Q(y) \equiv \int_\delta^y (\rho(\eta))^{-1} d\eta$. Then, by (G1) and (G2), $\Phi'(x) > 0$ and $Q'(y) > 0$ for all $x,\, y \in \mathbb{R}_+$. Hence, Φ^{-1} and Q^{-1} exist. Now, consider the change of variable $(x,y) \longrightarrow (u,v)$ where

$$u = \Phi(x), \quad v = Q(y).$$

Then, $(u,v) = \vec{0}$ if and only if $(x,y) = e_\star$, and we have

$$u'(t) = \Pi(v(t),\mu) - H(u(t),\mu), \quad v'(t) = -\Psi(u(t),\mu)$$

where

$$\Pi(v,\mu) = \pi(\delta,\mu+\theta_*) - \pi(y,\mu+\theta_*),$$

$$H(u,\mu) = h(\lambda,\mu+\theta_*) - h(x,\mu+\theta_*),$$

and

$$\Psi(u,\mu) = -\psi(x,\mu+\theta_*).$$

Since it is easy to see that $\Psi'(u,\mu) = -\psi'(x,\theta)\varphi(x,\theta)$, $H'(u,\mu) = -h'(x,\theta)\varphi(x,\theta)$ and $\Pi'(v,\mu) = -\pi'(y,\theta)\rho(y,\theta)$, we have $\Psi'(0,0)\Pi'(0,0) > 0$ and $H(0,\mu) = H'(0,0) = 0 \neq \frac{d}{d\mu}H'(0,0)$. Therefore, the result

$$16\kappa = \varphi^2(\lambda(\theta_*),\theta_*)\psi'(\lambda(\theta_*),\theta_*)\left(\frac{(\varphi(x,\theta)h'(x,\theta))'}{\psi'(x,\theta)}\right)'|_{(\lambda(\theta_*),\theta_*)}.$$

follows from the fact $H''(u,\mu) = -(\varphi(x,\theta)h'(x,\theta))'\varphi(x,\theta)$, and Theorem 2.1. This proves the following theorem.

Theorem 2.3. *If (G1)-(G6) hold for system (10), then*

$$16\kappa = \varphi^2(\lambda(\theta_*),\theta_*)\psi'(\lambda(\theta_*),\theta_*)\left(\frac{(\varphi(x,\theta)h'(x,\theta))'}{\psi'(x,\theta)}\right)'|_{(\lambda(\theta_*),\theta_*)}.$$

Similarly, by using of Cauchy Mean Value Theorem and L'Hospital Rule as in Theorem 3, we have the following result.

Theorem 2.4. *If (G1)-(G6) hold and* $\left(\frac{\varphi(x,\theta)h'(x,\theta)}{\psi(x,\theta)}\right)'|_{(\lambda(\theta_*),\theta_*)} \neq 0$*, then*

$$\kappa\cdot\varphi^2(\lambda(\theta_*),\,\theta_*)\psi'(\lambda(\theta_*),\,\theta_*)\left(\frac{\varphi(x,\theta)h'(x,\theta)}{\psi(x,\theta)}\right)'\Bigg|_{(\lambda(\theta_*),\,\theta_*)} > 0.$$

3 Example

In this section, we will determine the criticality of the Hopf bifurcation of the following example through the method introduced in Section 2.

Example Lengyel et al. ([5], [6]) proposed a model of oscillation reaction, the chlorine dioxide-iodine-malonic acid (ClO_2-I_2-MA) reaction.

$$\begin{array}{l} \mathrm{MA} + \mathrm{I_2} \to \mathrm{IMA} + \mathrm{I^-} + \mathrm{H^+} \\ \mathrm{ClO_2} + \mathrm{I^-} \to \mathrm{ClO_2^-} + \frac{1}{2}\mathrm{I_2} \\ \mathrm{ClO_2} + 4\mathrm{I^-} + 4\mathrm{H^+} \to \mathrm{Cl^-} + 2\mathrm{I_2} + 2\mathrm{H_2O} \end{array}$$

Through their simulations, they reduced the system to a two-variable model. After appropriate nondimensionalization, the model becomes

$$\begin{cases} x' = a - x - \dfrac{4xy}{1+x^2} = f(x,y) \\ y' = bx\left(1 - \dfrac{bxy}{1+x^2}\right) = g(x,y) \end{cases}, \tag{11}$$

where x and y denote the dimensionless concentrations of I^- and ClO_2^- and $a, b > 0$ are parameters.

Theorem 3.1. *The Hopf bifurcation of system* (11) *is supercritical if* $0 < a < \frac{675+25\sqrt{769}}{4}$ *while it is subcritical if* $a > \frac{675+25\sqrt{769}}{4}$.

Proof. To apply our results, we make the change of variables $v = x - \frac{4y}{b}$ and $dt = \varphi(x)d\tau$ so that system (3.1) is converted into the Liénard system

$$\frac{dx}{d\tau} = v - H(x), \quad \frac{dv}{d\tau} = -\Psi(x)$$

where

$$\varphi(x) = \frac{bx}{1+x^2},\ H(x) = x - \frac{1}{(1+x^2)(1-\frac{a}{x})},\ \Psi(x) = -\frac{1}{b}(1+x^2)(5-\frac{a}{x}).$$

Note that the bifurcation occurs at the points (x_*, y_*) such that $f = g = 0,\ f_x + g_y = 0$, and $f_x g_y - f_y g_x > 0$, from which follows

$$x_* = \frac{a}{5},\ y_* = 1 + \frac{a^2}{25},\ b = \frac{3a}{5} - \frac{25}{a},\ a > \frac{5\sqrt{15}}{3}.$$

According to Theorem 2, a direct computation gives

$$16\kappa = -\Psi(x_*) \left(\frac{H''(x)}{\Psi'(x)} \right)' |_{x=x_*} = \frac{(2a^4 - 675a^2 - 3125)}{125(a^2+25)}.$$

Thus, we are done. □

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