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Singular Value Decomposition of Symmetric Complex Partitioned Matrices

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Abstract

We give an alternate proof to the singular value decomposition of symmetric complex partitioned matrices. Such decomposition can be obtained by finding a unitary and symmetric complex partitioned square root of a unitary and symmetric complex partitioned matrix. This square root can be found by looking at decompositions of normal complex partitioned matrices.

Keywords: singular value decomposition, complex partitioned matrix, quaternion matrix

1 Introduction

Let $\Phi = \begin{bmatrix} A & -\overline{B} \\ B & \overline{A} \end{bmatrix}$, where A,$B \in M_n(\mathbb{C})$, be the set of complex partitioned matrices. Complex partitioned matrices are used to study quaternion matrices. This is possible since the ring of quaternion matrices is isomorphic to the ring of complex partitioned matrices. Since complex partitioned matrices are complex matrices, they admit all decompositions of complex matrices. The question is: Which decompositions can be obtained using only complex partitioned matrices?

Over $M_n(\mathbb{C})$,a symmetric matrix A can be factored as $A = UDU^T$ for some unitary U and diagonal D with non-negative entries [Aut]. If A is complex partitioned, then it has such decomposition. Is it possible to have all factors in Φ? The answer is affirmative. In fact there are two ways to obtain the decomposition in Φ. One is to obtain orthonormal coneigenvectors of the matrix [JV2]. The other method is an adaptation of the method used by Autonne and this will be discussed in this paper. Autonne obtained the decomposition of A using its singular value decomposition(SVD). Let $A = UDV$ be its SVD, where $D = F \oplus 0$ with $F \in M_r(\mathbb{R})$. Then $A^T = V^T DU^T$ and $(\overline{V}U)D = D(\overline{V}U)^T$. By one of the properties of unitary matrices which will be stated as a lemma later on, we have $\overline{V}U = V_1 \oplus V_2$, where V_1 is unitary and symmetric which commutes with F. Then

$$A = V^T(\overline{V}U)DV = V^T(V_1 \oplus V_2)DV = V^T(V_1 \oplus I_{n-r})DV.$$

Over the complex matrices, $V_1 \oplus I_{n-r}$ has a unitary and symmetric square root, say S, that commutes with D. Hence

$$A = V^T S^2 DV = V^T SDSV = (SV)^T D(SV)$$

is the desired factorization. Thus, if such approach will be used to extend this decomposition to Φ, the question is: Is there a unitary and symmetric complex partitioned square root of a unitary and symmetric complex partitioned matrix? To answer this question we look at decompositions of normal matrices in Φ.

2 Normal and Symmetric Matrices

A matrix $A \in M_n(\mathbb{C})$ that is normal is diagonalizable via a unitary matrix. If in addition, A is symmetric, then A can be diagonalized using a real orthogonal matrix [H&J].

Let $A \in M_n(\mathbb{C})$ be unitary so that A is normal. Then there exists a unitary $Q \in M_n(\mathbb{C})$ such that $A = QDQ^*$, where $D = e^{i\theta_1} I_{n_1} \oplus e^{i\theta_2} I_{n_k} \oplus \cdots \oplus e^{i\theta_k} I_{n_k}$, with $n_1 + n_2 + \cdots + n_k = n$ and $\theta_i \in \mathbb{R}$ for each i. Let $D_1 = e^{i\theta_1/2} I_{n_1} \oplus e^{i\theta_2/2} I_{n_2} \oplus \cdots \oplus e^{i\theta_k/2} I_{n_k}$ and $A_1 = QD_1Q^*$. Then A_1 is a unitary square root of A. If in addition, A is symmetric, then Q can be chosen to be real orthogonal so that $A = QDQ^T$ and $A_1 = QD_1Q^T$. Hence, A_1 is symmetric. Then we have the following.

Theorem 2.1. *Let $A \in M_n(\mathbb{C})$ be unitary and symmetric. Then A has a unitary and symmetric square root.*

The following lemma (see [Aut]) characterizes matrices which commute with a diagonal matrix.

Lemma 2.2. *Let $D = d_1 I_{n_1} \oplus \cdots \oplus d_k I_{n_k}$ with $n = n_1 + \cdots + n_k$ and $d_i \neq d_j$ for $i \neq j$. Then X commutes with D if and only if $X = X_1 \oplus \cdots \oplus X_k$, where $X_i \in M_{n_i}(\mathbb{C})$. In particular, if X is unitary, then each X_i is also unitary.*

By Lemma2.2, Theorem 2.1 can be restated as follows.

Theorem 2.3. *Let $A \in M_n(\mathbb{C})$ be unitary and symmetric. Then A has a unitary and symmetric square root which commutes with every matrix that commutes with A.*

Autonne's proof of the preceding theorem does not make use of the diagonalizability of a normal and symmetric matrix via a real orthogonal matrix. Instead he begins with the fact that a unitary matrix is diagonalizable via a unitary matrix and and the square root is constructed same as above. To determine if a normal and symmetric complex partitioned matrix is diagonalizable via a real and orthogonal complex partitioned matrix we need to look at decompositions of matrices of the form $\begin{bmatrix} A & \overline{B} \\ B & -\overline{A} \end{bmatrix}$, where $A,B \in M_n(\mathbb{C})$, using complex partitioned matrices. Let $\Psi = \left\{ \begin{bmatrix} A & -\overline{B} \\ B & \overline{A} \end{bmatrix}, \text{ where } A, B \in M_n(\mathbb{C}) \right\}$. Unlike Φ, the set Ψ is not a ring since it is not closed under matrix multiplication. In fact, given two matrices W_1 and W_2 in Ψ, we have $W_1 W_2 \in \Phi$. Notice also that

$$\begin{pmatrix} A & \overline{B} \\ B & -\overline{A} \end{pmatrix} = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$$

and that $W \in \Psi$ if and only if $iW \in \Phi$. Therefore, for any nonsingular $S \in \Phi$ and $W \in \Psi$, we have

$$S^{-1}(iW)S = i(S^{-1}WS)$$

which implies that $S^{-1}WS \in \Psi$. It is of interest to know the canonical forms of $W \in \Psi$ via elements of Φ.

Let $W \in \Psi$ be normal. Then $iW \in \Phi$ and hence, there exists a unitary $U \in \Phi$ such that $U^*(iW)U = D \oplus \overline{D}$, where D is diagonal whose entries have non-negative imaginary parts. This implies that

$$U^*WU = -iD \oplus -i\overline{D} = -iD \oplus -\left(\overline{-iD}\right)$$

Let $F = -iD$. Then the diagonal entries of F have non-negative real parts and $U^*(iW)U = F \oplus -\overline{F}$. This is summarized in the next theorem.

Theorem 2.4. *Let $W \in \Psi$ be normal. Then there exists a unitary $U \in \Phi$ such that $U^*WU = D \oplus -\overline{D}$, where D is diagonal whose entries have non-negative real parts.*

The preceding theorem implies that if $W \in \Psi$, then there are exactly $2n$ eigenvalues of W which are symmetrically located along the imaginary axis. Thus, we obtain the following decompositions of unitary, Hermitian, and real symmetric matrices in Ψ.

Corollary 2.5. *Let $W \in \Psi$ be unitary. Then there exists a unitary $U \in \Phi$ such that $U^*WU = D \oplus -\overline{D}$, where $D = e^{i\theta_1}I_{n_1} \oplus e^{i\theta_2}I_{n_2} \oplus \cdots \oplus e^{i\theta_k}I_{n_k}$ with $-\frac{\pi}{2} \leq \theta_j \leq \frac{\pi}{2}$ for $j = 1, 2, \cdots k$.*

If the matrix is Hermitian, we have the following.

Corollary 2.6. *Let $W \in \Psi$ be Hermitian. Then there exists a unitary $U \in \Phi$ such that $U^*WU = D \oplus -\overline{D}$, where D is real diagonal consisting only of non-negative entries.*

If $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ is an eigenvector of $W \in \Psi$ with corresponding eigenvalue λ, then $x^c = \begin{bmatrix} -\bar{x_2} & \bar{x_1} \end{bmatrix}^T$ is also an eigenvector of W corresponding to $-\bar{\lambda}$. If W is real symmetric, then x and x^c can be chosen so that both are real vectors.

Corollary 2.7. *Let $W \in \Psi$ be real symmetric. Then there exists a real orthogonal $U \in \Phi$ such that $U^TWU = D \oplus -D$, where D is real diagonal consisting only of non-negative entries.*

Let $W = \begin{bmatrix} A & -\overline{B} \\ B & \overline{A} \end{bmatrix}$ be normal and symmetric. Write $A = A_1 + iA_2$ and $B = B_1 + iB_2$, where $A_j, B_j \in M_n(\mathbb{R})$ for $j = 1, 2$. Then

$$W = \begin{bmatrix} A_1 + iA_2 & -\overline{(B_1 + iB_2)} \\ B_1 + iB_2 & \overline{A_1 + iA_2} \end{bmatrix} = \begin{bmatrix} A_1 & -B_1 \\ B_1 & A_1 \end{bmatrix} + i \begin{bmatrix} A_2 & B_2 \\ B_2 & -A_2 \end{bmatrix}.$$

Let $W_1 = \begin{bmatrix} A_1 & -B_1 \\ B_1 & A_1 \end{bmatrix}$ and $W_2 = \begin{bmatrix} A_2 & B_2 \\ B_2 & -A_2 \end{bmatrix}$. Then $W = W_1 + iW_2$. Since W is symmetric, W_1 and W_2 are real symmetric. By normality of W, then W_1 commutes with W_2. Since W_1 is real symmetric, there exists a real orthogonal $Q_1 \in \Phi$ such that $Q_1^TWQ_1 = D_1 \oplus D_1$,where $D_1 = a_1I_{n_1} \oplus a_2I_{n_2} \oplus \cdots \oplus a_kI_{n_k}$ and the a_i's are distinct real numbers [JV1]. Let $Q_1^TW_2Q_1 = \begin{bmatrix} S & T \\ T & -S \end{bmatrix}$, where $S, T \in M_n(R)$. Since W_1 commutes with W_2, then

$$(Q_1(D_1 \oplus D_1)Q_1^T)W_2 = W_2(Q_1(D_1 \oplus D_1)Q_1^T)$$

which implies that

$$(D_1 \oplus D_1)(Q_1^TW_2Q_1) = (Q_1^TW_2Q_1)(D_1 \oplus D_1)$$

that is

$$\begin{bmatrix} D_1 & 0 \\ 0 & D_1 \end{bmatrix} \begin{bmatrix} S & T \\ T & -S \end{bmatrix} = \begin{bmatrix} S & T \\ T & -S \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & D_1 \end{bmatrix}.$$

Hence, $D_1S = SD_1$ and $D_1T = TD_1$. By Lemma 2.2, S and T can be written as $S =$

$S_1 \oplus \cdots \oplus S_k$ and $T = T_1 \oplus \cdots \oplus T_k$, where $S_i, T_i \in M_{n_i}(R)$ for $i = 1, \ldots, k$. Therefore

$$Q_1^T W_2 Q_1 = \begin{bmatrix} S_1 & \cdots & 0 & T_1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & S_k & 0 & \cdots & T_k \\ T_1 & \cdots & 0 & -S_1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & T_k & 0 & \cdots & -S_k \end{bmatrix}.$$

Let $W_{l_i} = \begin{bmatrix} S_i & T_i \\ T_i & -S_i \end{bmatrix}$ for $i = 1, \ldots, k$. Since W_2 is symmetric, W_{l_i} is real and symmetric for each i. By Corrolary 2.7, we can find a real orthogonal complex partitioned matrix Q_{l_i} such that $Q_{l_i}^T W_{l_i} Q_{l_i} = F_i \oplus -F_i$, where F_i is diagonal and has non-negative entries for $i = 1, \ldots, k$. Let $Q_{l_i} = \begin{bmatrix} C_i & -E_i \\ E_i & C_i \end{bmatrix}$, where $C_i, E_i \in M_{n_i}(R)$ for $i = 1, \ldots, k$. Let

$$Q_2 = \begin{bmatrix} C_1 & \cdots & 0 & -E_1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & C_k & 0 & \cdots & -E_k \\ E_1 & \cdots & 0 & C_1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & E_k & 0 & \cdots & C_k \end{bmatrix},$$

then $Q_2 \in \Phi$ is real orthogonal and $Q_2^T(Q_1^T W_2 Q_1)Q_2 = F \oplus -F$, where $F = F_1 \oplus \cdots \oplus F_k$. Since $E_1 \oplus \cdots \oplus E_k$ and $C_1 \oplus \cdots \oplus C_k$ commute with D_1, then Q_2 commutes with $D_1 \oplus D_1$. Hence

$$\begin{aligned} Q_2^T(Q_1^T W Q_1)Q_2 &= Q_2^T(D_1 \oplus D_1)Q_2 + iQ_2^T(Q_1^T W_2 Q_1)Q_2 \\ &= (D_1 \oplus D_1) + i(F \oplus -F) \\ &= (D_1 + iF) \oplus (D_1 + \overline{iF}) \\ &= (D_1 + iF) \oplus (\overline{D_1 + iF}) \end{aligned}$$

We summarize this in the following.

Theorem 2.8. *Let $W \in \Phi$ be normal and symmetric. Then there exists a real orthogonal matrix $Q \in \Phi$ such that $Q^T W Q = D \oplus \overline{D}$, where D is diagonal whose entries have non-negative imaginary parts.*

Let $W \in \Phi$ be unitary. Then there exists a unitary $Q \in \Phi$ such that $Q^* W Q = D \oplus \overline{D}$, where $D = e^{i\theta_1} I_{n_1} \oplus e^{i\theta_2} I_{n_2} \oplus \cdots \oplus e^{i\theta_k} I_{n_k}$ with $n_1 + n_2 + \cdots + n_k = n$. Let $D_1 = e^{i\theta_1/2} I_{n_1} \oplus e^{i\theta_2/2} I_{n_2} \oplus \cdots \oplus e^{i\theta_k/2} I_{n_k}$ and $W_1 = Q(D_1 \oplus \overline{D_1})Q^*$. Then W_1 is a unitary square root of W. Moreover, by the construction of D and D_1 and Lemma 2.2, every matrix which commutes with W commutes with W_1. If, in addition, W is symmetric, then Q can be chosen so that it is real orthogonal and this implies that W and W_1, can be written as $W = Q(D \oplus \overline{D})Q^T$ and $W_1 = Q(D_1 \oplus \overline{D_1})Q^T$. Hence, W_1 is symmetric.

Theorem 2.9. *Let $W \in \Phi$ be unitary and symmetric. Then W has a unitary and symmetric square root which commutes with every matrix that commutes with W.*

It was shown in [Aut] that every unitary matrix $U \in M_n(\mathbb{C})$ can be written as $U = Q_1 D Q_2$, where Q_1 and Q_2 are both real orthogonal and D is diagonal. The following shows that same factorization holds true in Φ.

Theorem 2.10. *Let $U \in \Phi$ be unitary. Then U can be written as $U = Q_1(D \oplus \overline{D})Q_2$, where Q_1 and Q_2 are both real orthogonal in Φ and $D = diag(e^{i\theta_1}, \ldots, e^{i\theta_n})$.*

Proof. Suppose $U \in \Phi$ is unitary. Then $UU^T \in \Phi$ and symmetric. By Theorem 2.8, we can find a real orthogonal matrix $Q \in \Phi$ such that $Q^T UU^T Q = D \oplus \overline{D}$, where $D = diag(e^{i\theta_1}, \ldots, e^{i\theta_n})$. Let $F = D_1 \oplus \overline{D_1}$, where $D_1 = diag(e^{i\theta_1/2}, \ldots, e^{i\theta_n/2})$. Then

$$(F^{-1}Q^T U)(F^{-1}Q^T U)^T = F^{-1}(Q^T UU^T Q)F^{-1} = F^{-1}(D \oplus \overline{D})F^{-1} = I_{2n}.$$

Thus, $F^{-1}Q^T U$ is orthogonal and unitary, hence it must be real. If we let $Q_1 = Q$ and $Q_2 = F^{-1}Q^T U$, then $Q_1 F Q_2 = U$ □

3 The Symmetric Matrix

This section presents Autonne's decomposition holds true in Φ by adapting his approach. But first we need to characterize $n \times n$ unitary matrices U and V such that $F = UFV$, where $F = D \oplus 0$ with $D = d_1 I_{n_1} \oplus \cdots \oplus d_k I_{n_k} \in M_r(\mathbb{R})$ and $d_1 > \cdots > d_k > 0$ (see [Aut]).

Lemma 3.1. *The complex unitary matrices U and V that satisfy $F = UFV$ commute with F and have the forms $U = U_1 \oplus U_2$ and $V = V_1 \oplus V_2$.Consequently, U_1 commutes with D and $U_1 V_1 = I_r$. Moreover, U_2 and V_2 are arbitrary unitary matrices.*

The next lemma can be easily shown using the trace of a matrix.

Lemma 3.2. *Let A and B be complex matrices such that $AA^* + BB^* = 0$. Then $A = B = 0$.*

Let $W \in \Phi$ and $W = U(D \oplus D)V$ be its SVD, where $D_1 = a_1 I_{n_1} \oplus a_2 I_{n_2} \oplus \cdots \oplus a_k I_{n_k} \in M_r(R)$ and the a_i's are positive. Then

$$U(D \oplus D)V = W = W^T = V^T(D \oplus D)U^T$$

which implies that

$$\overline{V}U(D \oplus D)V\overline{U} = D \oplus D.$$

Let $Z = \overline{V}U$. Then $Z \in \Phi$ is unitary and $Z(D \oplus D)\overline{Z} = D \oplus D$. Write

$$Z = \begin{bmatrix} Z_{11} & Z_{12} & -\overline{Z_{31}} & -\overline{Z_{32}} \\ Z_{21} & Z_{22} & -\overline{Z_{41}} & -\overline{Z_{42}} \\ Z_{31} & Z_{32} & \overline{Z_{11}} & \overline{Z_{12}} \\ Z_{41} & Z_{42} & \overline{Z_{21}} & \overline{Z_{22}} \end{bmatrix}$$

conformal to

$$D \oplus D = \begin{bmatrix} D_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $Z(D \oplus D) = (D \oplus D)Z^T$, then $Z_{21}D_1 = 0$ and $Z_{41}D_1 = 0$. Since D_1 is nonsingular, $Z_{21} = 0$ and $Z_{41} = 0$. Therefore, we can write Z as

$$Z = \begin{bmatrix} Z_{11} & Z_{12} & -\overline{Z_{31}} & -\overline{Z_{32}} \\ 0 & Z_{22} & 0 & -\overline{Z_{42}} \\ Z_{31} & Z_{32} & \overline{Z_{11}} & \overline{Z_{12}} \\ 0 & Z_{42} & 0 & \overline{Z_{22}} \end{bmatrix}$$

Since $ZZ^* = Z^*Z$, then

$$Z_{11}Z_{11}^* + Z_{12}Z_{12}^* + \overline{Z_{31}}Z_{31}^T + \overline{Z_{32}}Z_{32}^T = Z_{11}^*Z_{11} + Z_{31}^*Z_{31}.$$

Since $tr(A) = tr(A^T)$ and $tr(AA^*) = tr(A^*A)$ for any complex matrix, $tr(Z_{12}Z_{12}^* + Z_{32}Z_{32}^*) = 0$. By Lemma 3.2, $Z_{12} = Z_{32} = 0$. Therefore, Z can be rewritten as

$$Z = \begin{bmatrix} Z_{11} & 0 & -\overline{Z_{31}} & 0 \\ 0 & Z_{22} & 0 & -\overline{Z_{42}} \\ Z_{31} & 0 & \overline{Z_{11}} & 0 \\ 0 & Z_{42} & 0 & \overline{Z_{22}} \end{bmatrix}.$$

Let $Z_1 = \begin{bmatrix} Z_{11} & -\overline{Z_{31}} \\ Z_{31} & \overline{Z_{11}} \end{bmatrix}$. Then $Z(D\oplus D)\overline{Z} = D\oplus D$ implies that $Z_1(D_1\oplus D_1)\overline{Z_1} = D_1\oplus D_1$. By Lemma 3.1, Z_1 commutes with $D_1 \oplus D_1$ and $Z_1\overline{Z_1} = I$. Since Z is unitary, Z_1 must be unitary. Hence, $Z_1^* = Z_1^{-1} = \overline{Z_1}$, which implies $Z_1^T = Z_1$.Therefore, $Z_1 \in \Phi$ is unitary and symmetric. Since Z_1 commutes with $D_1 \oplus D_1$, Z_{11} and Z_{31} commute with D_1. Therefore, we can write $Z_{11} = S_1 \oplus \cdots \oplus S_k$ and $Z_{31} = T_1 \oplus \cdots \oplus T_k$, where $S_i, T_i \in M_{n_i}(\mathbb{C})$ for $i = 1, \ldots, k$. This implies that

$$Z_1 = \begin{bmatrix} S_1 & \cdots & 0 & -\overline{T_1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & S_k & 0 & \cdots & -\overline{T_k} \\ T_1 & \cdots & 0 & \overline{S_1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & T_k & 0 & \cdots & \overline{S_k} \end{bmatrix}.$$

Let $A_i = \begin{bmatrix} S_i & -\overline{T_i} \\ T_i & \overline{S_i} \end{bmatrix}$ for $i = 1, \ldots, k$. Then each A_i is unitary and symmetric. By Theorem 2.9, A_i has a unitary and symmetric complex partitioned square root, say B_i. Let $B_i = \begin{bmatrix} X_i & -\overline{Y_i} \\ Y_i & \overline{X_i} \end{bmatrix}$ and

$$X = \begin{bmatrix} X_1 & \cdots & 0 & -\overline{Y_1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & X_k & 0 & \cdots & -\overline{Y_k} \\ Y_1 & \cdots & 0 & \overline{X_1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & Y_k & 0 & \cdots & \overline{X_k} \end{bmatrix}.$$

Then X is unitary and symmetric which commutes with $D_1 \oplus D_1$. Let

$$S = \begin{bmatrix} X_1 & \cdots & 0 & 0 & -\overline{Y_1} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & X_k & 0 & 0 & \cdots & -\overline{Y_k} & 0 \\ 0 & \cdots & 0 & I_{n-r} & 0 & \cdots & 0 & 0 \\ Y_1 & \cdots & 0 & 0 & \overline{X_1} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & Y_k & 0 & 0 & \cdots & \overline{X_k} & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & I_{n-r} \end{bmatrix}$$

By direct calculation, we get $Z(D \oplus D) = S^2(D \oplus D)$. Since X is symmetric and commutes with $D_1 \oplus D_1$, then S is symmetric and commutes with $D \oplus D$ and this implies that

$$Z(D \oplus D) = S(D \oplus D)S = S^T(D \oplus D)S.$$

Since $V^T\overline{V} = I_{2n}$, we have

$$W = U(D \oplus D)V = (V^T\overline{V})[U(D \oplus D)V].$$

Since $Z = \overline{V}U$, then

$$W = V^T Z(D \oplus D)V = V^T[S^T(D \oplus D)S]V = (SV)^T(D \oplus D)(SV).$$

If we let $Q = (SV)^T$, then $Q \in \Phi$ is unitary and $W = Q(D \oplus D)Q^T$. This is summarized in the following.

Theorem 3.3. *Let $W \in \Phi$ be symmetric. Then there exists a unitary $Q \in \Phi$ such that $W = Q(D \oplus D)Q^T$, where D is diagonal and consists only of non-negative entries.*

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