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A Local Uncertainty Principle on Locally Compact Lie Groups

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Abstract

We will prove the following local uncertainty inequality on unimodular locally compact Lie groups G. Let $m : G \to \mathbf{C}$ be in $L^1(G) \cap L^2(G)$ such that $\int_{\widehat{G}} \|\widehat{m}(\lambda)\| < \infty$. Let Xm be also in $L^1(G) \cap L^2(G)$, where $X \in \mathfrak{g} = Lie(G)$ and such that $||X|| \geq 1$. For any compactly supported function m on G,

$$0 < C_1 \int_G |Xm(g)|dg + \frac{C_2}{|supp(m)|} \int_{supp(m)} |m(g)|dg,$$

where $C_1 = |supp(m)|^{-1}$, $C_2 = \|X\|$. This means that a function on a locally compact Lie group cannot be simultaneously smooth and has arbitrarily small average on sets on which it has sufficiently large derivatives.

Keywords: local uncertainty principle, locally compact, Lie group

1 Preliminaries

A **Lie group** is a group G with a compatible differentiable manifold structure. This means that the group operation $G \times G \to G, (x, y) \mapsto x \cdot y$ and group inversion $x \mapsto x^{-1}$ are differentiable mappings. Examples are the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$, the matrix groups $GL(n, \mathbb{C})$ and $SU(n)$, where the former is the group of n by n complex matrices with nonzero determinant and the latter is the group of n by n complex Hermitian matrices with determinant equal to 1.

For our purposes, we may take our Lie groups to be closed subgroups G of $GL(n, \mathbb{C})$. Locally ompact unimodular Lie groups, and more generally locally compact topological groups, have unique Haar measures. We may therefore define the linear spaces $L^1(G)$ and $L^2(G)$ of integrable and square integrable functions. $L^1(G)$ consists of the functions $f : G \to \mathbb{C}$ such that $\int_G |f(x)|dx < \infty$ while $L^2(G)$ consists of functions f for which $\int_G |f(x)|^2 dx < \infty$.

Let V be a complex vector space. A **representation** of the Lie group G in V is a strongly continuous homomorphism

$$\rho : G \longrightarrow GL(V)$$

of G to the group of invertible linear transformations of V. Thus, to each $g \in G$ there corresponds a linear mapping $\rho(g) : V \to V$ satisfying $\rho(e) = Id_V, \rho(gh) = \rho(g)\rho(h)$ and such that to each $\eta \in V$ the mapping $G \to V$ given by $g \mapsto \rho(g)\eta$ is continuous. We shall write (ρ, V) for a representation of G in V.

For closed subgroups $G \subset GL(n, \mathbb{C})$ the so-called standard representation is given by

$$\rho : G \longrightarrow GL(\mathbb{C}^n),$$

$$\rho(g)\eta = g \cdot \eta.$$

This representation is strongly continuous since the matrix entries of $\rho(g)$ depend polynomially on the matrix entries of g. Another example, which may be familiar from the theory of Fourier series is given as follows. Let $G = \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, V = L^2(V)$, and

$$\rho : G \longrightarrow GL(V),$$

$$(\rho(\theta)f)(t) = e^{i\theta} f(t - \theta).$$

As a final example, let G be the permutation group S_n on n letters and V is an n-dimensional vector space with basis $e_1, ..., e_n$. The we obtain a representation $T : G \to GL(V)$ given by

$$T(\sigma)e_i = e_{\sigma(i)}, \;\; i = 1, ..., n, \;\; \sigma \in S_n.$$

Given a representation (ρ, V) of G, if U is a subspace of V for which $\rho(g)u \in U, \;\; \forall g \in G$ and $\forall u \in U$, we call U an **invariant subspace** of V under ρ. The representation (ρ, V) of G is called **irreducible** if V has exactly two invariant subspaces, $\{0\}$ and V. One-dimensional representations are irreducible. The representation of the group S_n above is not irreducible because the one-dimensional subspace $\mathbb{C} \cdot (e_1 + ... + e_n)$is an invariant subspace.

If $V = \mathcal{H}$ is a Hilbert space, the representation $(\rho, V)$is said to be **unitary** if

$$(\rho(g)u, \rho(g)v) = (u, v), \;\; \forall g \in G, u, v \in \mathcal{H}.$$

Let (ρ_1, V_1) and (ρ_2, V_2)be two representations of G. We call them **equivalent** if there is a linear isomorphism $T : V_1 \to V_2$ satisfying the relation $T \circ \rho_1(g) = \rho_2(g) \circ T, \;\; \forall g \in G$. Write $\hat{G}$ for the set of equivalence classes of unitary irreducible representations of G. One of the main problems of the theory of representations is to determine $\hat{G}$.

We will give here a couple of examples. For $G = \mathbb{R}$, $\hat{G} = \{\chi_\xi : \xi \in \mathbb{R}\}$, where $\chi_\xi : \mathbb{T} \to \mathbb{C}^\times = GL(1, \mathbb{C})$ is given by $\chi_\xi(x)Z = e^{i\xi x}z$. Restricted to the quotient group $\mathbb{T}, \chi_\xi : \mathbb{T} \to \mathbb{C}^\times = GL(1, \mathbb{C})$ is unitary if and only if $\xi \in \mathbb{Z}$. Thus $\hat{\mathbb{R}} = \mathbb{R}$ and $\hat{\mathbb{T}} = \mathbb{Z}$.

We now state the Peter-Weyl Theorem, which is the generalization to compact Lie groups of the Fourier series expansion of square integrable functions on the unit circle. This result provides a complete orthonormal basis of the space $L^2(G)$ of square integrable functions on the compact Lie group G in terms of the matrix elements of G. To state this result more precisely, let $\lambda \in \hat{G}$, and choose a representative $(\rho^\lambda, V_\lambda) \in \lambda$. Let $e_1, ..., e_{d(\lambda)}$ be a basis of V_λ. Then the entries of the matrix $(\rho^\lambda(g)_{ij})$ are functions $\rho^\lambda_{ij} : G \to \mathbb{C}, \;\; 1 \leq i, j \leq d(\lambda) = dim(V_\lambda)$. We call the ρ^λ_{ij} matrix elements. The conclusion of the Peter-Weyl Theorem is that the set

$$\{d(\lambda)^{1/2} \rho^\lambda_{ij} : \lambda \in \hat{G}, 1 \leq i, j \leq d(\lambda)\}$$

is a complete orthonormal basis of $L^2(G)$. There is still a much further genralization of the Fourier expansion to the case of locally compact unimodular Lie groups and will be given in the following section. It is the main tool in the proof of the main result of the paper.

2 Spectral Synthesis on Unimodular Groups

Let G be a unimodular Lie group and let $H = L^2(G,\mu)$. Let $g \mapsto T_g : H \longrightarrow H$ be the regular representation of $G \times G$ given by $T_{g_1,g_2}\varphi(x) = \varphi(g_1^{-1}xg_2)$. Then, there exists a direct integral decomposition

$$H \mapsto \hat{H} = \int_\Lambda \hat{H}(\lambda)d\rho(\lambda),\ T_g \mapsto \hat{T}_g = \int_\Lambda \hat{T}_g(\lambda)d\rho(\lambda).$$

There is a so-called Gelfand triplet $\Phi \subset H \subset \Phi'$ and a basis $e_{pq}(\lambda, g)$ in $H(\lambda)$ such that for $\varphi \in \Phi$,

$$\varphi(g) = \int_\Lambda d\rho(\lambda) \sum_{p,q=1}^{dim\hat{H}(\lambda)} \hat{\varphi}_{pq}(\lambda)e_{pq}(\lambda, g),\ \ \hat{\varphi}_{pq}(\lambda) = < \varphi, e_{pq}(\lambda) > .$$

For $\varphi, \psi \in \Phi$ the Plancherel equality has the form

$$\int_G \varphi(g)\hat{\psi}(g)d\mu(g) = \int_\Lambda d\rho(\lambda) \sum_{p,q=1}^{dim\hat{H}(\lambda)} \hat{\varphi}_{pq}(\lambda)\overline{\hat{\psi}_{pq}(\lambda)}$$

If $\varphi \in L^1(G)$, the operator $\pi : \mathcal{H}_\pi \longrightarrow \mathcal{H}_\pi$ given by $\pi(\varphi) = \int_G \varphi(x)\pi(x)dm(x)$, is a bounded linear operator.

3 Proof of Main Result

Theorem 1. *Let $m : G \to \mathbb{C}$ be in $L^1(G) \cap L^2(G)$ such that $\int_{\widehat{G}} \|\widehat{m}(\lambda)\| < \infty$. Let Xm be also in $L^1(G) \cap L^2(G)$, where $X \in \mathfrak{g} = Lie(G)$ and such that $||X|| \geq 1$. For any compactly supported function m on G,*

$$0 < C_1 \int_G |Xm(g)|dg + \frac{C_2}{|supp(m)|} \int_{supp(m)} |m(g)|dg,$$

where $C_1 = |supp(m)|^{-1}$, $C_2 = \|X\|$. This means that a function on a locally compact Lie group cannot be simultaneously smooth and has arbitrarily small average on sets on which it has sufficiently large derivatives.

The following proof cannot remove the condition given by the last part of the last sentence, given by $||X|| \geq 1$. Let G be a locally compact unimodular Lie group with Haar measure m, let $\hat{G}$ be its unitary dual, μ the Plancherel measure on $\hat{G}$. For $\pi \in \hat{G}$, let $\mathcal{H}_\pi$ be the corresponding representation Hilbert space. The Plancherel formula is

$$\int_{\hat{G}} Tr[\pi(f)^*\pi(f)]d\mu(\pi) = \int_G |f(x)|^2 dm(x),$$

for $f \in L^1(G) \cap L^2(G)$. Write $h(t, x) = m(g \cdot exp\, tX)f(x)$, where X is an element of the Lie algebra $\mathfrak{g}$ of the Lie group G.

Then, we have the following.

$$\int_G |m(g \cdot exp\, tX)f(x)|^2 dm(x) = \int_{\hat{G}} Tr[\pi(h)^* \pi(h)] d\mu(h).$$

The derivative of the left hand side is $\int_G \frac{\partial}{\partial t}|h(t,x)|^2 dx$ and is equal to

$$2\int_G Xm(x)f(x)dm(x) = 2\int_{\hat{G}} Tr\pi(Xm \cdot f)^* \pi(Xm \cdot f) d\mu(\pi).$$

This fact will be used below.

Now the Fourier series expansion of f is given by

$$f(g) = \int_\Lambda d\rho(\lambda) \sum_{p,q=1}^{dim \hat{\mathcal{H}}(\lambda)} \hat{f}_{pq}(\lambda) e_{pq}(\lambda, g),$$

where $\hat{f}_{pq}(\lambda) = \int_G f(g)\overline{e_{pq}(\lambda, g)} d\mu(g)$ is the Fourier transform of f_{pq}, as above. Write $m(g \cdot exp\, tX)f(x)$ in its Fourier series

$$\int_\Lambda d\mu(\pi) \sum_{p,q=1}^{dim \hat{\mathcal{H}}(\pi)} \hat{m}_{pq}(\pi) e^\pi_{pq}(g \cdot exp\, tX) f(x).$$

Taking derivatives

$$\frac{d}{dt} m(g \cdot tX)f(x) = \int_\Lambda d\mu(\pi) \sum_{p,q=1}^{dim \hat{\mathcal{H}}(\pi)} \hat{m}_{pq}(\pi) e^\pi_{pq}(g \cdot exp\, tX) d\, e^\pi_{pq}(X) f(x)$$

so that

$$Xm(x)f(x) = \int_\Lambda d\mu(\pi) \sum_{p,q=1}^{dim \hat{\mathcal{H}}(\pi)} \hat{m}_{pq}(\pi) e^\pi_{pq}(x) X(f \circ e^\pi_{pq})(x)$$

when evaluated at $t = 0$. We break the sum into two,

$$Xm(x)f(x) = \int_\Lambda \hat{m}_{p_0,q_0}(\pi) e^\pi_{P_0,q_0}(x) d\mu(\pi) +$$
$$\int_\Lambda d\mu(\pi) \sum_{p,q \neq p_0,q_0}^{dim \hat{\mathcal{H}}(\pi)} \hat{m}_{pq}(\pi) e^\pi_{pq}(x) X(f \circ e^\pi_{pq})(x).$$

Moving terms and applying the triangle inequality, we obtain

$$\int_G \int_\Lambda \Big| \hat{m}_{p_0,q_0}(\pi) e^\pi_{p_0,q_0}(x) \Big| d\mu(\pi) \leq \int_G \Big| Xm(x)f(x) \Big| +$$
$$\int_G \int_\Lambda d\mu(\pi) \sum_{p,q=p_0,q_0}^{dim \hat{\mathcal{H}}(\pi)} \Big| \hat{m}_{pq}(\pi) e^\pi_{pq}(x) X(f \circ e^\pi_{pq})(x) \Big|.$$

Then

$$\sup_{p_0,q_0,\pi} \int_G \int_\Lambda |\hat{m}_{p_0,q_0}(\pi)| \leq \int_G \Big| Xm(x) f(x) \Big| + \int_{\text{supp}(m)} |m(x)| ||X|| |f(e^{\pi}_{pq}(x))| dx,$$

where we used the fact that the operator norm of X is $||X|| \geq 1$.
Take

$$f(x) = \begin{cases} |\text{supp}(m)|^{-1} & \text{if } x \in \text{supp}(m) \\ 0 & \text{if otherwise.} \end{cases}$$

Therefore,

$$0 < |supp(m)|^{-1} \int_G |Xm(x)| dx + \frac{\|X\|}{|supp(m)|} \int_{supp(m)} |m(x)| dx. \quad \square$$

References

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