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McShane Integrability and Egoroff's Theorem

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Abstract

In this paper, we will give an alternative proof of Egoroff's Theorem without using concepts in measure theory.

Keywords: McShane integral, Egoroff's Theorem

1 Introduction

In classical measure theory, Egoroff's theorem establishes the implication of pointwise convergence of a sequence of measurable functions to uniform convergence. The usual proof of this theorem uses concepts in measure theory. In 1910, Carlo Severini gave the first proof. A year later, Dmitri Egorov independently published his results and the theorem became known under his name, after which many other versions and alternative proofs of the theorem arose (see [5, 6, 7]).

In the late 1960s, the McShane integral was introduced by E.J. McShane to overcome the difficulty of the Henstock integral as a nonabsolute integral. It is Riemann-type with the property: A function is McShane integrable if and only if its absolute value function is McShane integrable [8]. As a result of this property, the McShane integral is equivalent to the Lebesgue integral [1].

In this paper, we will give a version of Egoroff's Theorem using concepts in McShane integrability.

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2 Preliminaries

Let us revisit the following definitions.

Definition 2.1. A function $f : [a,b] \to \mathbb{R}$ is said to be *McShane integrable* to A on $[a,b]$ if for each $\epsilon > 0$, there exists $\delta(\xi) > 0$ on $[a,b]$ such that whenever $D = \{([u,v],\xi)\}$ is a McShane δ-fine division of $[a,b]$, we have

$$\left|(D)\sum f(\xi)(v-u) - A\right| < \epsilon.$$

Recall that by a *McShane δ-fine division* $D = \{([u,v];\xi)\}$ of $[a,b]$ we mean that $[u,v] \subseteq (\xi-\delta(\xi), \xi+\delta(\xi))$. If $f : [a,b] \to \mathbb{R}$ is McShane integrable to A, we write $A = (\mathcal{M})\int_a^b f(x)dx$, and call A as the *McShane integral* of f on $[a,b]$.

In what follows, unless otherwise stated, all integrals referred to are McShane integrals, and we write $\int_a^b f(x)dx = (\mathcal{M})\int_a^b f(x)dx$.

Definition 2.2. Let $E \subseteq \mathbb{R}$. The *characteristic function* $\mathbf{1}_E$ on E is defined by

$$\mathbf{1}_E = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

We now introduce the concept of an *integrable set* (see [4]).

Definition 2.3. Let $\mathcal{B}$ be the family of subsets $E \subseteq \mathbb{R}$ such that $\mathbf{1}_{E\cap[a,b]}$ is McShane integrable on $[a,b]$ for all closed intervals $[a,b]$ in $\mathbb{R}$. Let $E \in \mathcal{B}$. If $E \subseteq [a,b]$, then define

$$m(E) = \int_a^b \mathbf{1}_E(x)\, dx.$$

If $E \nsubseteq [a,b]$, define

$$m(E) = \lim_{n\to\infty} m(E \cap [-n,n]).$$

We refer to E as an *integrable set* and $\mathcal{B}$ the collection of all integrable sets in $\mathbb{R}$.

Yang in [4] showed that integrable sets are actually Lebesque measurable sets. In the same paper, Yang also showed that if $\{E_n : n \in \mathbb{N}\} \subseteq \mathcal{B}$ is a countable collection of pairwise disjoint integrable sets, then $\bigcup_{n=1}^{\infty} E_n$ is an integrable set and

$$m\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} m(E_n).$$

Hence, one can verify that if $\{E_n : n \in \mathbb{N}\} \subseteq \mathcal{B}$ is a countable collection of integrable sets, then $\bigcup_{n=1}^{\infty} E_n$ is an integrable set and

$$m\Big(\bigcup_{n=1}^{\infty} E_n\Big) \leq \sum_{n=1}^{\infty} m(E_n).$$

The following result and its proof can be found in [4].

Theorem 2.4. [4] *If $A \subseteq [a,b]$ is an integrable set, then for each $\epsilon > 0$ there exists an open set $G \subseteq [a,b]$ such that $A \subseteq G$ and*

$$\int_a^b \mathbf{1}_{G\setminus A}(x)\, dx < \epsilon.$$

Definition 2.5. An integrable set $E \subseteq \mathbb{R}$ is said to have *variation zero* if

$$\int_a^b \mathbf{1}_E(x)\, dx = 0.$$

It is worth noting that a subset of a set of variation zero is again of variation zero.

Definition 2.6. A property is said to hold *almost everywhere* (abbreviated *a.e.*) on A if the set of points in A where it fails to hold is a set of variation zero.

Theorem 2.7. *Let $f : [a,b] \to \mathbb{R}$ be McShane integrable on $[a,b]$. If $g = f$ almost everywhere on $[a,b]$, then g is McShane integrable on $[a,b]$, and*

$$\int_a^b g(x)\, dx = \int_a^b f(x)\, dx.$$

Proof: Let $E = \{x \in [a,b] : f(x) \neq g(x)\}$, and let

$$A = \int_a^b f(x)\, dx.$$

For each $i \in \mathbb{N}$, let $E_i = \{x \in E : i-1 \leq |f(x) - g(x)| < i\}$. Then $E = \bigcup_{i=1}^{\infty} E_i$. Let $\epsilon > 0$. For each $i \in \mathbb{N}$, since E_i is of variation zero, there exists $\delta_i(\xi) > 0$ such that whenever $D_i = \{([u,v],\xi)\}$ is a McShane δ_i-fine division of $[a,b]$, we have

$$(D_i) \sum_{\xi \in E_i} (v-u) = \left|(D_i) \sum \mathbf{1}_{E_i}(\xi)(v-u)\right| < \frac{\epsilon}{i2^i}.$$

By the definition of A, there exists $\delta_f(\xi) > 0$ such that whenever $D = \{([u,v],\xi)\}$ is a McShane δ_f-fine division of $[a,b]$, we have

$$\left|(D) \sum f(\xi)(v-u) - A\right| < \epsilon.$$

Note that if $\xi \in E$, then there exists $i \in \mathbb{N}$ such that $\xi \in E_i$.

Define $\delta : [a,b] \to (0,+\infty)$ as the map $\delta(\xi) = \min\{\delta_f(\xi), \delta_i(\xi)\}$. Let $D = \{([u,v],\xi)\}$ be a McShane δ-fine division of $[a,b]$.

Then

$$\begin{aligned}
\Big|(D)\sum g(\xi)(v-u)-A\Big| &\leq (D)\sum \big|g(\xi)-f(\xi)\big|(v-u) \\
&\quad +\Big|(D)\sum f(\xi)(v-u)-A\Big| \\
&= (D)\sum_{\xi\in E} \big|g(\xi)-f(\xi)\big|(v-u) \\
&\quad +(D)\sum_{\xi\notin E} \big|g(\xi)-f(\xi)\big|(v-u) \\
&\quad +\Big|(D)\sum f(\xi)(v-u)-A\Big| \\
&= (D)\sum_{\xi\in E} \big|f(\xi)-g(\xi)\big|(v-u) \\
&\quad +\Big|(D)\sum f(\xi)(v-u)-A\Big| \\
&\leq (D)\sum_{\xi\in E} \Big((D_i)\sum \big|f(\xi)-g(\xi)\big|(v-u)\Big) \\
&\quad +\Big|(D)\sum f(\xi)(v-u)-A\Big| \\
&\leq (D)\sum_{\xi\in E} \Big(i\cdot\frac{\epsilon}{i2^i}\Big) + \Big|(D)\sum f(\xi)(v-u)-A\Big| \\
&< \epsilon+\epsilon \\
&= 2\epsilon.
\end{aligned}$$

□

In view of Theorem 2.7, the condition "$f_n(x) \longrightarrow f(x)$ on $[a,b]$" in all convergence theorems for the McShane integral (see [8]) can now be replaced by "$f_n(x) \longrightarrow f(x)$ *a.e.* on $[a,b]$".

The proof of the following result can be found in [1].

Theorem 2.8. [1] *If $f : [a,b] \to \mathbb{R}$ is McShane integrable on $[a,b]$, then there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of step functions such that $\varphi_n \to f$ almost everywhere on $[a,b]$ and*

$$\lim_{n\to\infty} \int_a^b |\varphi_n - f| = 0.$$

Next, we will show that the characteristic function on a set $X(f < c)$ is McShane integrable.

Theorem 2.9. *Let c be any real number. If $f : [a,b] \to \mathbb{R}$ is McShane integrable on $[a,b]$ and*

$$X = X(f < c) = \{x \in [a,b] : f(x) < c\},$$

then the characteristic function $\mathbf{1}_X$ on X is McShane integrable.

Proof: By Theorem 2.8, there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of step functions such that $\varphi_n \to f$ on $[a,b]$ except on a set S of variation zero. Let

$$X(\varphi_i < c - \tfrac{1}{k}) = \{x \in [a,b] : \varphi_i(x) < c - \tfrac{1}{k}\}$$

where $c \in \mathbb{R}$. Then for each i and k, the function $\mathbf{1}_{X(\varphi_i<c-\frac{1}{k})}$ is McShane integrable on $[a,b]$, being a step function. Thus, $\mathbf{1}_{\bigcap_{i=n}^{m} X(\varphi_i<c-\frac{1}{k})}$ is also McShane integrable on $[a,b]$, for any m. Let

$$Y = \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty} X(\varphi_i < c - \tfrac{1}{k}).$$

Note that if $A \subseteq B$, then $\mathbf{1}_A \leq \mathbf{1}_B$. Hence, for any fixed n

$$\mathbf{1}_{\bigcap_{i=n}^{m+1} X(\varphi_i<c-\frac{1}{k})} \leq \mathbf{1}_{\bigcap_{i=n}^{m} X(\varphi_i<c-\frac{1}{k})}, \quad \text{for each } m \geq n;$$

since $\bigcap_{i=n}^{m+1} X(\varphi_i < c - \frac{1}{k}) \subseteq \bigcap_{i=n}^{m} X(\varphi_i < c - \frac{1}{k})$. It can be seen that for each $x \in [a,b]$

$$\lim_{m\to\infty} \mathbf{1}_{\bigcap_{i=n}^{m} X(\varphi_i<c-\frac{1}{k})}(x) = \mathbf{1}_{\bigcap_{i=n}^{\infty} X(\varphi_i<c-\frac{1}{k})}(x).$$

Hence, $\{\mathbf{1}_{\bigcap_{i=n}^{m} X(\varphi_i<c-\frac{1}{k})}\}_m$ is a decreasing sequence of McShane integrable functions on $[a,b]$. Thus, by the Monotone Convergence Theorem, $\mathbf{1}_{\bigcap_{i=n}^{\infty} X(\varphi_i<c-\frac{1}{k})}$ is McShane integrable on $[a,b]$, for each n.

On the other hand, $\{\mathbf{1}_{\bigcup_{n=1}^{m}\bigcap_{i=n}^{\infty} X(\varphi_i<c-\frac{1}{k})}\}_m$ is an increasing sequence of McShane integrable functions on $[a,b]$ and for each $x \in [a,b]$

$$\lim_{m\to\infty} \mathbf{1}_{\bigcup_{n=1}^{m}\bigcap_{i=n}^{\infty} X(\varphi_i<c-\frac{1}{k})}(x) = \mathbf{1}_{\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty} X(\varphi_i<c-\frac{1}{k})}(x).$$

Again, by the Monotone Convergence Theorem, $\mathbf{1}_{\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty} X(\varphi_i<c-\frac{1}{k})}$ is McShane integrable on $[a,b]$.

Similarly, $\{\mathbf{1}_{\bigcup_{k=1}^{m}\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty} X(\varphi_i<c-\frac{1}{k})}\}_m$ is an increasing sequence of McShane integrable functions on $[a,b]$ and for each $x \in [a,b]$

$$\lim_{m\to\infty} \mathbf{1}_{\bigcup_{k=1}^{m}\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty} X(\varphi_i<c-\frac{1}{k})}(x) = \mathbf{1}_{\bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty} X(\varphi_i<c-\frac{1}{k})}(x) = \mathbf{1}_Y;$$

and hence, by the Monotone Convergence Theorem, $\mathbf{1}_Y$ is McShane integrable on $[a,b]$.

It can be seen that $X \setminus S = Y \setminus S$, implying that $X \setminus Y \subseteq S$. Hence, $X \setminus Y$ is also a set of variation zero. Therefore, the characteristic function $\mathbf{1}_X$ on X is McShane integrable. □

Since $\{x \in [a,b] : f(x) \geq c\} = [a,b] \setminus \{x \in [a,b] : f(x) < c\}$, we remark from Theorem 2.9 that the characteristic function $\mathbf{1}_{X(f\geq c)}$ on $\{x \in [a,b] : f(x) \geq c\}$ is also McShane integrable. Similarly, $\mathbf{1}_{X(f>c)}$ and $\mathbf{1}_{X(f\leq c)}$ are McShane integrable. Furthermore, if X is a countable union of sets of the forms:

$$\{x \in [a,b] : f(x) < c\}, \; \{x \in [a,b] : f(x) \geq c\},$$

$$\{x \in [a,b] : f(x) > c\}, \;\text{ or }\; \{x \in [a,b] : f(x) \leq c\},$$

then $\mathbf{1}_X$ is McShane integrable.

3 Main Results

Here, we will state and prove Egoroff's Theorem without using concepts in measure theory.

Theorem 3.1. *If $\{f_n\}$ is a sequence of McShane integrable functions on $[a,b]$ and $f_n(x) \to f(x)$ as $n \to \infty$ a.e. on $[a,b]$, then given $\epsilon > 0$ and $\delta > 0$ there exists an integrable set $A \subseteq [a,b]$ with $m(A) < \delta$ and a positive integer N such that for each $x \in [a,b]\setminus A$ and each $n \geq N$, we have*

$$|f_n(x) - f(x)| < \epsilon.$$

Proof: We may assume that $\lim_{n\to\infty} f_n(x) = f(x)$ on $[a,b]$. Let $\epsilon > 0$ and

$$G_i \quad = \quad \{x \in [a,b] : |f_i(x) - f(x)| \geq \epsilon\}.$$

For each $n \in \mathbb{N}$, let

$$E_n = \bigcup_{i=n}^{\infty} G_i = \{x \in [a,b] : |f_i(x) - f(x)| \geq \epsilon \text{ for some } i \geq n\}.$$

Since $E_{n+1} \subseteq E_n$, Theorem 2.9 implies that $\{\mathbf{1}_{E_n}\}_n$ is a monotone decreasing sequence of McShane integrable functions.

Let $x \in [a,b]$. By the convergence of $\{f_n(x)\}$ to $f(x)$, there exists a positive integer N_x such that $x \notin E_n$, and consequently $\mathbf{1}_{E_n}(x) = 0$, for all $n \geq N_x$. Thus,

$$\lim_{n\to\infty} \mathbf{1}_{E_n}(x) = 0.$$

By the Monotone Convergence Theorem,

$$\lim_{n\to\infty} m(E_n) = \lim_{n\to\infty} (\mathcal{M}) \int_a^b \mathbf{1}_{E_n} = 0.$$

Thus, given $\delta > 0$ there exists a positive integer N such that $m(E_N) < \delta$. Take

$$A = E_N = \{x \in [a,b] : |f_i(x) - f(x)| \geq \epsilon \text{ for some } i \geq N\}.$$

Note that

$$[a,b]\setminus A = [a,b]\setminus E_N = \{x \in [a,b] : |f_i(x) - f(x)| < \epsilon \text{ for every } i \geq N\}.$$

Hence, for each $x \in [a,b]\setminus A$ and $n \geq N$, we have

$$|f_n(x) - f(x)| < \epsilon.$$

□

The following is Egoroff's Theorem.

Corollary 3.2. (Egoroff's Theorem) *If $\{f_n\}$ is a sequence of McShane integrable functions on $[a,b]$ and $f_n(x) \to f(x)$ as $n \to \infty$ a.e. on $[a,b]$, then given $\delta > 0$ there exists a subset $A \subseteq [a,b]$ with $m(A) < \delta$ such that $\{f_n\}$ converges to f uniformly on $[a,b]\setminus A$.*

Proof: Let $\delta > 0$. By Theorem 3.1, for each n there exists an integrable set $A_n \subseteq [a,b]$ with $m(A_n) < \frac{\delta}{2^n}$ and a positive integer K_n such that for each $x \in [a,b]\setminus A_n$ and each $k \geq K_n$, we have

$$|f_k(x) - f(x)| < \tfrac{1}{n}.$$

We may assume that $\{K_n : n \in \mathbb{N}\}$ is increasing. Let $A = \bigcup_{n=1}^{\infty} A_n$. Then $A \subseteq [a,b]$ is an integrable set and

$$m(A) = m\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \sum_{n=1}^{\infty} m(A_n) < \sum_{n=1}^{\infty} \frac{\delta}{2^n} = \delta.$$

We will now show that $f_n \to f$ uniformly on $[a,b]\setminus A$. Let $\epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \epsilon$. Corresponding to n_0, there exists $K_{n_0} \in \mathbb{N}$ such that for all $k \geq K_{n_0}$ and $x \in [a,b]\setminus A_{n_0}$, we have

$$|f_k(x) - f(x)| < \tfrac{1}{n_0} < \epsilon.$$

Note that $[a,b]\setminus A \subseteq [a,b]\setminus A_{n_0}$. Hence, the result follows. □

The following result is a special version of Egoroff's Theorem.

Corollary 3.3. *If $\{f_n\}$ is a sequence of McShane integrable functions on $[a,b]$ and $f_n(x) \to f(x)$ as $n \to \infty$ a.e. on $[a,b]$, then given $\delta > 0$ there exists an open set $G \subseteq [a,b]$ with $m(G) < \delta$ such that $\{f_n\}$ converges to f uniformly on $[a,b]\setminus G$.*

Proof: Let $\delta > 0$. By Corollary 3.2, there exists a subset $A \subseteq [a,b]$ with $m(A) < \frac{\delta}{2}$ such that $\{f_n\}$ converges to f uniformly on $[a,b]\setminus A$. By Theorem 2.4, there exists an open set $G \subseteq [a,b]$ such that $A \subseteq G$ and

$$(\mathcal{M})\int_a^b \mathbf{1}_{G\setminus A}(x)\, dx < \frac{\delta}{2},$$

that is, $m(G\setminus A) < \frac{\delta}{2}$. Hence, we have

$$\begin{aligned} m(G) &= (\mathcal{M})\int_a^b \mathbf{1}_G(x)\, dx \\ &= (\mathcal{M})\int_a^b \mathbf{1}_A(x)\, dx + (\mathcal{M})\int_a^b \mathbf{1}_{G\setminus A}(x)\, dx \\ &= m(A) + m(G\setminus A) \\ &< \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta. \end{aligned}$$

Moreover, $[a,b]\setminus G \subseteq [a,b]\setminus A$, thus $\{f_n\}$ also converges to f uniformly on $[a,b]\setminus G$. □

References

[1] T. S. Chew, *The Riemann-type Integral that Includes Lebesgue-Stieltjes and Stochastic Integrals*, Lecture, Chulalongkorn University, 2004.

[2] P. Y. Lee, *Lanzhou Lectures on Henstock Integration* Vol. **2**, World Scientific Publishing Co., 1989.

[3] H. L. Royden, *Real Analysis 3ed*, Macmillan Publishing Company, 1989.

[4] C. H. Yang, *Measure Theory and the Henstock integral*, Acad. Exer., NUS, Singapore, 1997.

[5] J. H. Ma, S. Y. Wen and Z. Y. Wen, Egoroff's theorem and maximal run length, *Monatshefte fur MathematiK* **151** (4) (2007) 287-292.

[6] J. Kuwabe, The Egorff property and Egoroff theorem in Riesz space-valued nonadditive measure theory, *Fuzzy Sets and systems* **158** (1) (2007) 50-57.

[7] M. Yasuda, On Egoroff's theorems on finite monotone nonadditive measure space, *Fuzzy Sets and Systems* **153** (1) (2005) 71-78.

[8] R. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, American Mathematical Society, 1994.