MATIMYÁS MATEMATIKA
ISSN 0115-6926

Journal of the Mathematical Society of the Philippines
Vol. 31 Nos. 1-3 (2008) pp. 29-35

Some Convolution-Type Identities and the Combinatorics of A-Tableau

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Abstract

We derive some convolution-type identities for the (r, β)-Stirling numbers using the combinatorics of A-tableaux and, consequently, obtain interesting identities for some known Stirling-type numbers.

Keywords: (r, β)-Stirling numbers, A-tableau, convolution formula

1 Introduction

As defined in [8], an *A-tableau* is a list ϕ of column c of a Ferrer's diagram of a partition λ(by decreasing order of length) such that the lengths $|c|$ are part of the sequence $A = (a_i)_{i \geq 0}$, a strictly increasing sequence of nonnegative integers.

Note that an A-tableau can also be generated by fixing the number of columns whose lengths are in a sequence A. For example, the A-tableaux with exactly 3 columns whose lengths are part of $A = \{1, 2, 3, 4\}$ can be given in terms of multisets whose entries are column lengths (instead of columns) as follows

$$\begin{array}{ccccccc} \{4,4,4\} & \{4,4,3\} & \{4,4,2\} & \{4,4,1\} & \{4,3,3\} & \{4,3,2\} & \{4,3,1\} \\ \{4,2,2\} & \{4,2,1\} & \{4,1,1\} & \{3,3,3\} & \{3,3,2\} & \{3,3,1\} & \{3,2,2\} \\ \{3,2,1\} & \{3,1,1\} & \{2,2,2\} & \{2,2,1\} & \{2,1,1\} & \{1,1,1\} & \end{array}$$

Figure 2 below shows five of these tableaux corresponding to the multisets $\{4, 4, 4\}$, $\{4, 4, 3\}$, $\{3, 3, 3\}$, $\{3, 3, 2\}$, and $\{2, 2, 2\}$, respectively.

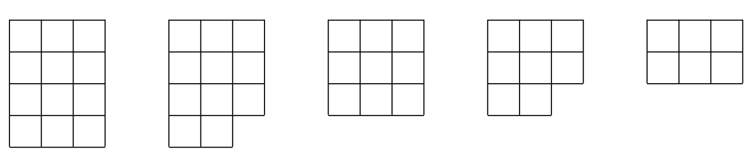

Figure 2. *Examples of A-tableaux with exactly 3 columns.*

This implies that the number of such A-tableaux is the same as the number of 3-element multisubsets of the multiset $\{\infty \cdot 1, \infty \cdot 2, \infty \cdot 3, \infty \cdot 4\}$ which is given by $H_3^4 = 20$ (see [4]).

In general, the number of r-element multisubsets of a multiset $M = \{\infty\cdot a_1, \infty\cdot a_2, \ldots, \infty\cdot a_n\}$ as given in [4] is

$$H_r^n = \binom{r+n-1}{r}.$$

Thus, if $T^A(k,r)$ denotes the set of A-tableaux with r columns whose lengths, not necessarily distinct, are in the set $\{0,1,2,\ldots,k\}$, then

$$|T^A(k,r)| = \binom{r+k}{r}. \tag{1}$$

In this paper, we express the (r,β)-Stirling numbers in terms of the weights of the columns of A-tableaux and derive some convolution-type identities using the combinatorics of A-tableaux.

2 (r,β)-Stirling numbers and Their Explicit Formula

The (r,β)**-Stirling numbers**, denoted by $\left\langle {n \atop k} \right\rangle_{\beta,r}$ were defined by means of the following linear transformation:

$$t^n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_{\beta,r} (t-r)_{\beta,k}$$

where

$$(t-r)_{\beta,k} = \prod_{i=0}^{k-1}(t-r-i\beta).$$

$(t)_{\beta,k}$ is called the **generalized factorial of t with increment** β, and as a convention $(t)_{\beta,k} = 0$ if $k \leq 0$. $\left\langle {n \atop k} \right\rangle_{\beta,r}$ are certain generalization of all second kind Stirling-type numbers. For instance, the classical Stirling numbers, the noncentral Stirling numbers, and the r-Stirling numbers of the second kind can be expressed in terms of $\left\langle {n \atop k} \right\rangle_{\beta,r}$ as follows:

$$\left\langle {n \atop k} \right\rangle_{1,0} = S(n,k), \text{ classical Stirling numbers of the second kind} \tag{2}$$

$$\left\langle {n \atop k} \right\rangle_{1,-a} = S_{(a)}(n,k), \text{ noncentral Stirling numbers of the second kind} \tag{3}$$

$$\left\langle {n \atop k} \right\rangle_{1,r} = \left\{ {n+r \atop k+r} \right\}_r, \text{ } r\text{-Stirling numbers of the second kind} \tag{4}$$

All other Stirling-type numbers of the second kind, like the weighted Stirling numbers and the degenerate Stirling numbers, can also be expressed in terms of $\left\langle {n \atop k} \right\rangle_{\beta,r}$.

Several properties of $\left\langle {n \atop k} \right\rangle_{\beta,r}$ like the generating functions, recurrence relations, and explicit formulas were already established by R. Corcino in [6]. One of these properties is the rational generating function given as follows

$$\sum_{n\geq 0} \left\langle {n \atop k} \right\rangle_{\beta,r} t^n = \frac{t^k}{\prod_{j=0}^{k}[1-(\beta j + r)t\,]}. \tag{5}$$

Note that (4) can be rewritten as

$$\sum_{n\geq k}\left\langle {n \atop k} \right\rangle_{\beta,r} t^{n-k} = \frac{1}{\prod_{j=0}^{k}\left[1-(\beta j+r)t\right]} = \prod_{j=0}^{k}\left(\frac{1}{1-(\beta j+r)t}\right).$$

Applying Newton's Binomial Theorem [4] we get

$$\sum_{n\geq k}\left\langle {n \atop k} \right\rangle_{\beta,r} t^{n-k} = \prod_{j=0}^{k}\left(\sum_{c_j\geq 0}(\beta j+r)^{c_j}t^{c_j}\right).$$

Rewriting the product of sums as sum of products (see [5] page 40), we obtain

$$\begin{aligned}\sum_{n\geq k}\left\langle {n \atop k} \right\rangle_{\beta,r} t^{n-k} &= \sum_{c_0,c_1,\ldots,c_k\geq 0}\left[\prod_{j=0}^{k}(\beta j+r)^{c_j}t^{c_j}\right]\\ &= \sum_{n\geq k}\left\{\sum_{c_0+c_1+\ldots+c_k=n-k}\left[\prod_{j=0}^{k}(\beta j+r)^{c_j}\right]\right\}t^{n-k}.\end{aligned}$$

Identifying the coefficients of t^{n-k}, we have the following explicit formula for $\left\langle {n \atop k} \right\rangle_{\beta,r}$

$$\left\langle {n \atop k} \right\rangle_{\beta,r} = \sum_{c_0+c_1+\ldots+c_k=n-k}\ \prod_{j=0}^{k}(\beta j+r)^{c_j},$$

which can be written further as stated in the following theorem.

Theorem 1. *The (r,β)-Stirling Numbers $\left\langle {n \atop k} \right\rangle_{\beta,r}$ equals*

$$\left\langle {n \atop k} \right\rangle_{\beta,r} = \sum_{0\leq j_1\leq j_2\leq\ldots\leq j_{n-k}\leq k}\left[\prod_{i=1}^{n-k}(j_i\beta+r)\right].$$

Theorem 1 is essential in relating the (r,β)-Stirling numbers to the concept of A-tableau. Let ω be a function from the set of nonnegative integers N^* to a ring K. Suppose Φ is an A-tableau with r columns of lengths $|c|\leq h$. Then, we set

$$\omega_A(\Phi) = \prod_{c\in\Phi}\omega(|c|).$$

Note that Φ might contain a finite number of columns whose lengths are zero since $0\in A = \{0,1,2,\ldots,k\}$ and if $\omega(0)\neq 0$.

From this point onward, whenever an A-tableau is mentioned, it is always associated with the sequence $A=\{0,1,2,\ldots,k\}$.

We are now ready to mention the following theorem.

Theorem 2. *Let $\omega : N^* \to K$ be the column weight according to length which is defined by $\omega(|c|) = |c|\beta + r$ where $|c|$ is the length of column c of an A-tableau in $T^A(k, n-k)$. Then*

$$\left\langle {n \atop k} \right\rangle_{\beta,r} = \sum_{\phi \in T^A(k,n-k)} \prod_{c \in \phi} \omega(|c|).$$

Proof: Let ϕ be an A-tableau in $T^A(k, n-k)$. Then ϕ has exactly $n-k$ columns, say $c_1, c_2, \ldots, c_{n-k}$ whose lengths are $j_1, j_2, \ldots, j_{n-k}$, respectively. Now for each column $c_i \in \phi$, $i = 1, 2, \ldots, n-k$ we have $|c_i| = j_i$ and $\omega(|c_i|) = j_i\beta + r$. Then

$$\prod_{c \in \phi} \omega(|c|) = \prod_{i=1}^{n-k} \omega(|c_i|) = \prod_{i=1}^{n-k} (j_i\beta + r),$$

and hence

$$\sum_{\phi \in T^A(k,n-k)} \prod_{c \in \phi} \omega(|c|) = \sum_{0 \le j_1 \le j_2 \le \ldots \le j_{n-k} \le k} \prod_{i=1}^{n-k} (j_i\beta + r).$$

Using Theorem 1, we obtain the desired result. $\square$

3 Convolution-Type Identities for $\left\langle {n \atop k} \right\rangle_{\beta,r}$

Suppose

ϕ_1 is a tableau with $k-m$ columns whose lengths are in the set $\{0, 1, \ldots, m\}$, and

ϕ_2 be a tableau with $n-k-j$ columns whose lengths are in the set $\{m+1, m+2, \ldots, m+j+1\}$

Then

$$\phi_1 \in T^{A_1}(m, k-m) \text{ and } \phi_2 \in T^{A_2}(j, n-k-j)$$

where $A_1 = \{0, 1, \ldots, m\}$ and $A_2 = \{m+1, m+2, \ldots, m+j+1\}$. Notice that by joining the columns of ϕ_1 and ϕ_2, we obtain an A-tableau ϕ with $n-m-j$ columns whose lengths are in the set $A = A_1 \cup A_2 = \{0, 1, \ldots, m+j+1\}$. That is, $\phi \in T^A(m+j+1, n-m-j)$. Then,

$$\sum_{\phi \in T^A(m+j+1,n-m-j)} \omega_A(\phi)$$

$$= \sum_{k=m}^{n-j} \left\{ \sum_{\phi_1 \in T^{A_1}(m,\, k-m)} \omega_{A_1}(\phi_1) \right\} \left\{ \sum_{\phi_2 \in T^{A_2}(j,\, n-k-j)} \omega_{A_2}(\phi_2) \right\}.$$

Note that

$$\begin{aligned}\sum_{\phi_2\in T^{A_2}(j,\,n-k-j)}\omega_{A_2}(\phi_2) &= \sum_{\phi_2\in T^{A_2}(j,\,n-k-j)}\prod_{c\in\phi_2}(|c|\beta+r)\\ &= \sum_{m+1\le g_1\le\ldots\le g_{\,n-k-j}\le\, m+j+1}\ \prod_{i=1}^{n-k-j}(g_i\beta+r)\\ &= \sum_{0\le g_1\le\ldots\le g_{\,n-k-j}\le j}\ \prod_{i=1}^{n-k-j}(g_i\beta+(m+1)\beta+r).\end{aligned}$$

Thus,

$$\sum_{0\le g_1\le\ldots\le g_{n-m-j}\le m+j+1}\ \prod_{i=1}^{n-m-j}(g_i\beta+r)$$

$$=\sum_{k=m}^{n-j}\left\{\sum_{0\le g_1\le\ldots\le g_{k-m}\le m}\ \prod_{i=1}^{k-m}(g_i\beta+r)\right\}\left\{\sum_{0\le g_1\le\ldots\le g_{n-k-j}\le j}\ \prod_{i=1}^{n-k-j}(g_i\beta+(m+1)\beta+r)\right\}.$$

By Theorem 1, we obtain the following identity.

Theorem 3. *The (r,β)-Stirling numbers satisfy the following second form of convolution formula*

$$\left\langle {n+1 \atop m+j+1} \right\rangle_{\beta,r} = \sum_{k=0}^{n}\left\langle {k \atop m} \right\rangle_{\beta,r}\left\langle {n-k \atop j} \right\rangle_{\beta,(m+1)\beta+r}.$$

Note that when $\beta=1$, Theorem 4 gives

$$\left\langle {n+1 \atop m+j+1} \right\rangle_{1,r} = \sum_{k=0}^{n}\left\langle {k \atop m} \right\rangle_{1,r}\left\langle {n-k \atop j} \right\rangle_{1,m+r+1}$$

and by equation (3), we obtain the following corollary.

Corollary 1. *The r-Stirling numbers of the second kind satisfy the following convolution-type identity:*

$$\left\{ {n+r+1 \atop m+j+r+1} \right\}_{r} = \sum_{k=m}^{n-j}\left\{ {k+r \atop m+r} \right\}_{r}\left\{ {n-k+m+r+1 \atop j+m+r+1} \right\}_{m+r+1}.$$

Moreover, if $\beta=1$ and $r=0$, then Theorem 4 becomes

$$\left\langle {n+1 \atop m+j+1} \right\rangle_{1,0} = \sum_{k=0}^{n}\left\langle {k \atop m} \right\rangle_{1,0}\left\langle {n-k \atop j} \right\rangle_{1,m+1}$$

and by relations (1) & (3), we have the following corollary.

Corollary 2. *The Stirling numbers of the second kind satisfy the following identity:*

$$S(n+1, m+j+1) = \sum_{k=m}^{n-j} S(k,m) \left\{ \begin{matrix} n-k+m+1 \\ j+m+1 \end{matrix} \right\}_{m+1}.$$

Also, with $\beta = 1$ and $r = -a$ we have

$$\left\langle \begin{matrix} n+1 \\ m+j+1 \end{matrix} \right\rangle_{1,-a} = \sum_{k=0}^{n} \left\langle \begin{matrix} k \\ m \end{matrix} \right\rangle_{1,-a} \left\langle \begin{matrix} n-k \\ j \end{matrix} \right\rangle_{1,m-a+1}.$$

By (2) & (3) we obtain the succeeding corollary.

Corollary 3. *The non-central Stirling numbers of the second kind satisfy the following identity:*

$$S_{(a)}(n+1, m+j+1) = \sum_{k=m}^{n-j} S_{(a)}(k,m) \left\{ \begin{matrix} n-k+m-a+1 \\ j+m-a+1 \end{matrix} \right\}_{m-a+1}.$$

If $\beta = 0$ and $r = 1$, Theorem 4 together with equation (3) of [6] gives

$$\binom{n+1}{m+j+1} = \sum_{k=m}^{n-j} \binom{k}{m} \binom{n-k}{j}.$$

The next theorem provides another form of convolution-type identity.

Theorem 4. *The (r, β)-Stirling numbers satisfy the following convolution-type identity:*

$$\left\langle \begin{matrix} m+j \\ n \end{matrix} \right\rangle_{\beta,r} = \sum_{k=m}^{n-j} \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle_{\beta,r} \left\langle \begin{matrix} j \\ n-k \end{matrix} \right\rangle_{\beta,k\beta+r}.$$

Proof: Let

ϕ_1 be a tableau with $m-k$ columns whose lengths are in $A_1 = \{0, 1, \ldots, k\}$, and

ϕ_2 be a tableau with $j-n+k$ columns whose lengths are in $A_2 = \{k, k+1, \ldots, n\}$.

Then $\phi_1 \in T^{A_1}(k, m-k)$ and $\phi_2 \in T^{A_2}(n-k, j-n+k)$. Using the same argument above, we can easily obtain the convolution formula. □

Setting $\beta = 1$, Theorem 5 yields

$$\left\langle \begin{matrix} m+j \\ n \end{matrix} \right\rangle_{1,r} = \sum_{k=m}^{n-j} \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle_{1,r} \left\langle \begin{matrix} j \\ n-k \end{matrix} \right\rangle_{1,k+r},$$

which consequently gives the following corollary.

Corollary 4. *The r-Stirling numbers of the second kind satisfy the following convolution-type identity*

$$\left\{ \begin{matrix} m+j+r \\ n+r \end{matrix} \right\}_r = \sum_{k=m}^{n-j} \left\{ \begin{matrix} m+r \\ k+r \end{matrix} \right\}_r \left\{ \begin{matrix} j+k+r \\ n+r \end{matrix} \right\}_{k+r}.$$

Now, when $r = 0$, this gives the following corollary.

Corollary 5. *The classical Stirling numbers of the second kind satisfy the following identity*

$$S(m+j,n) = \sum_{k=m}^{n-j} S(m,k) \left\{ \begin{matrix} j+k \\ n \end{matrix} \right\}_k.$$

Moreover, when $\beta = 1$ and $r = -a$ we have

Corollary 6. *The non-central Stirling numbers of the second kind satisfy the following identity*

$$S_{(a)}(m+j,n) = \sum_{k=m}^{n-j} S_{(a)}(m,k) \left\{ \begin{matrix} j+k-a \\ n-a \end{matrix} \right\}_{k-a}.$$

It is worth mentioning that the identities in Corollaries 6-10 are not known in the literature of the classical Stirling numbers, r-Stirling numbers, and non-central Stirling numbers of the second kind.

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