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Models which Realize Bimodal Cycles

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Abstract

Two families of some cubic maps which realize not only unimodal cycles but also bimodal cycles are studied. A class of bimodal cycles forced by (1342) is given.

Keywords: bimodal cycles, forcing relation on cycles.

1 Introduction

1.1 Sharkovsky's Theorem

In 1975 Li and Yorke published a famous paper "Period three implies chaos" [8]. Their main result given below states that the presence of a periodic point with least period 3 forces the presence of periodic points with all possible least periods.

Theorem 1.1 (Li-Yorke [8]) Let f be a continuous function from a closed interval I into itself. Assume there is a point $a \in I$ satisfying

$$f^3(a) \leq a < f(a) < f^2(a).$$

Then (i) for any $n \in \mathbb{R}$ there is a periodic point of least period n in I, and (ii) there exists an uncountable set of points without period, for which there is sensitive dependence on initial conditions. Nowadays we call statuses (i) and (ii) **chaotic in the sense of Li-Yorke**.

However we have to note that before 1975, Sharkovsky [13] already obtained the results which includes Theorem 1.1 (i) as a corollary. A brief history of the Sharkovsky's theorem is described in [4].

1.2 Extension of Sharkovsky's order

Unfortunately, the classification of periodic orbits by least period only is very coarse. If we look at the cyclic permutations (cycles) determined by periodic orbits, then the classification is very fine but the results are much weaker than for periods. First let us define cycles as follows.

Definition 1.1 (cycle) A cycle of length n (an n-cycle) is any bijection (cyclic permutation) $\theta : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that $1, \theta(1), \theta^2(1), \ldots, \theta^{n-1}(1)$ are all distinct. Any periodic orbit of f with least period n:

$$\mathcal{O} = \{x_1, x_2, \ldots, x_n\},$$

where $x_1 < x_2 < \cdots < x_n$ can be identified with a suitable cycle θ if

$$f^i(x_1) = x_{\theta^i(1)} \text{ for } i = 0, 1, \ldots, n-1.$$

Let us denote the set of all cycles of length n by C_n. By a cycle we mean any element of $C := \cup_{n \geq 1} C_n$.

Definition 1.2 (forcing relation on cycles, $\to$) Let $\theta, \eta \in C$ be cycles. Then define $\theta \to \eta$ (θ forces η) if and only if for every continuous function f from a closed interval I into itself, if f has an orbit of type θ then f also has an orbit of type η.

$$\begin{array}{ccccc}
 & & (1342) & & \\
 & \swarrow & & \searrow & \\
(1234) & & (1243) & & (1432) \\
\downarrow & \swarrow & & \searrow & \downarrow \\
(123) & & & & (132) \\
\downarrow & & & & \downarrow \\
(1324) & & & & (1423) \\
 & \searrow & & \swarrow & \\
 & & (12) & & \\
 & & \downarrow & & \\
 & & (1) & &
\end{array}$$

Forcing relation on 10 cycles of length 4 or less

Definition 1.3 (linear map) For a cycle $\theta \in C_n$, the θ-linear map $L_\theta : [1, n] \to [1, n]$ is defined by

$$\begin{array}{lll}
L_\theta(i) = \theta(i) & \text{for} & i \in \{1, 2, \ldots, n\}, \\
L_\theta \text{ is linear on } [j, j+1] & \text{for} & j \in \{1, 2, \ldots, n-1\}.
\end{array}$$

Here we note that the graph of L_θ consists of at most $n-1$ linear segments, each having a slope $m \in \mathbb{Z}$ satisfying $|m| \geq 1$.

It is well known that a cycle η is forced by θ if and only if L_θ has a periodic orbit of type η [1].

Definition 1.4 (modality) Given $\theta \in C_n$, $n \geq 3$ define $z(\theta)$ by:

$$z(\theta) := \begin{cases} \ \ |z| & \text{if } \theta(2) > \theta(1), \\ -|z| & \text{if } \theta(2) < \theta(1), \end{cases}$$

where $|z|$ is given by the number of i's such that

$$(\theta(i+1) - \theta(i))(\theta(i) - \theta(i-1)) < 0, \ 2 \leq i \leq n-1.$$

If $z(\theta) = k \in \mathbb{Z}$ then θ is called **k-modal cycle**. In particular if $z(\theta) = 1$ then θ is also called **unimodal cycle**, and if $z(\theta) = +2$ or $z(\theta) = -2$ then θ is also called **bimodal cycle**. Let us denote the set of all k-modal cycles of length n by $C_n(k)$. Let us also denote the set of all k-modal cycles by $C(k)$.

The following facts are well known: $< C, \rightarrow>$ is not a linear ordering; $< C(1), \rightarrow>$ and $< C(-1), \rightarrow>$ are linear orderings; $\theta = (123 \cdots n)$ is the largest element of $C_n(1)$; and $\theta^* = (1n \cdots 32)$ is the largest element of $C_n(-1)$ (See [2]).

The family of some quadratic maps like the logistic maps [9],[12], or the tent maps or the truncated tent maps [1],[4] realize cycles of length n for any $n \in \mathbb{N}$. But those realized cycles are all unimodal cycles.

In the next section, we study two different models which realize bimodal cycles $\theta_1 = (1243)$ and $\theta_2 = (1342)$ respectively. In section 3, we study the essential difference of these two bimodal cycles.

2 Models which Realize Bimodal Cycles

2.1 A model for Batesian mimicry

Let us consider the family of the maps $G_a : [-1, 1] \to [-1, 1]$ defined by

$$G_a(x) := ax^3 + (1 - a)x, \tag{1}$$

where $a \in [0, 4]$. This map was presented as a simple mathematical model for Batesian mimicry [3].

2.1.1 3-Cycles

To find the value of a at which a 3-cycle created in a tangent bifurcation, we have to solve the following system of equations:

$$\begin{cases} aX^3 + (1-a)X & = & Y, \\ aY^3 + (1-a)Y & = & Z, \\ aZ^3 + (1-a)Z & = & X, \\ (3aX^2 + 1 - a)(3aY^2 + 1 - a)(3aZ^2 + 1 - a) & = & 1, \end{cases}$$

where $-1 \le X, Y, Z \le 1$, and $0 < a < 4$.

Alternatively, considering the discriminant of $G_a^3(x) - x = 0$, we get the value of a as positive roots of the polynomial:

$$4a^8 - 48a^7 + 189a^6 - 144a^5 - 828a^4 + 2052a^3 - 972a^2 - 2700a + 4644 = 0$$

which are approximately $a = 3.69964$ and $a = 3.92487$ [3].

2.1.2 $\theta_1 = (1243)$

From the symmetry of both the graph of $y = G_a(x)$ and $y = L_{\theta_1}(x)$, $\theta_1 := (1243)$, we can easily find the value of a at which a 4-cycle created in a tangent bifurcation and whose orbit $\mathcal{O}$ realizes $\theta_1 = (1243)$. Indeed at the value $a = 1 + 2\sqrt{2} \approx 3.828427$, the orbit of 4-cycle

$$\begin{aligned} & \left\{ -\sqrt{\tfrac{1}{14}\left(12 - 3\sqrt{2} + \sqrt{3(18 - 8\sqrt{2}}\right)},\ -\sqrt{\tfrac{1}{14}\left(12 - 3\sqrt{2} - \sqrt{3(18 - 8\sqrt{2}}\right)}, \right. \\ & \qquad \left. \sqrt{\tfrac{1}{14}\left(12 - 3\sqrt{2} - \sqrt{3(18 - 8\sqrt{2}}\right)},\ \sqrt{\tfrac{1}{14}\left(12 - 3\sqrt{2} + \sqrt{3(18 - 8\sqrt{2}}\right)} \right\} \\ \approx \ & \{-0.934882, -0.483931, 0.483931, 0.934882\} \end{aligned}$$

realizes $\theta_1 = (1243)$.

2.2 A model for business cycles

Next let us consider the family of the maps $H_a : [-1,1] \to [-1,1]$ defined by

$$H_a(x) := ax - (a+1)x^3, \tag{2}$$

where $a \in [0,3]$. This map was presented as a mathematical model for business cycles [6],[7].

2.2.1 3-cycles

To find the value of a at which a 3-cycle created in a tangent bifurcation, we have to solve the following system of equations:

$$\begin{cases} aX - (a+1)X^3 & = & Y, \\ aY - (a+1)Y^3 & = & Z, \\ aZ - (a+1)Z^3 & = & X, \\ (a-2(a+1)X^2)(a-2(a+1)Y^2)(a-2(a+1)Z^2) & = & 1, \end{cases}$$

where $-1 \le X, Y, Z \le 1$, and $0 < a < 3$.

Alternatively, considering the discriminant of $H_a^3(x) - x = 0$, we get the value of a as positive roots of the polynomial:

$$4a^8 + 16a^7 - 35a^6 - 206a^5 - 113a^4 + 376a^3 + 715a^2 + 1690a + 2197 = 0$$

which are approximately $a = 2.45044$ and $a = 2.98177$.

2.2.2 $\theta_2 = (1342)$

From the symmetry of both the graph of $y = H_a(x)$ and $y = L_{\theta_2}(x)$, $\theta_2 := (1342)$, we can easily find the value of a at which a 4-cycle created in a tangent bifurcation and whose orbit $\mathcal{O}$ realizes $\theta_2 = (1342)$. Indeed at the value $a = 2\sqrt{2} \approx 2.828427$, the orbit of 4-cycle

$$\begin{aligned} &\left\{-\sqrt{\tfrac{1}{14}\left(12 - 3\sqrt{2} + \sqrt{3(18 - 8\sqrt{2})}\right)},\ -\sqrt{\tfrac{1}{14}\left(12 - 3\sqrt{2} - \sqrt{3(18 - 8\sqrt{2})}\right)},\right. \\ &\qquad \left.\sqrt{\tfrac{1}{14}\left(12 - 3\sqrt{2} - \sqrt{3(18 - 8\sqrt{2})}\right)},\ \sqrt{\tfrac{1}{14}\left(12 - 3\sqrt{2} + \sqrt{3(18 - 8\sqrt{2})}\right)}\right\} \\ \approx\ &\{-0.934882, -0.483931, 0.483931, 0.934882\} \end{aligned}$$

realizes $\theta_2 = (1342)$.

3 Difference Between the Two Bimodal Cycles

These cycles $\theta_1 = (1243)$ and $\theta_2 = (1342)$ are both bimodal and self-conjugate:

Definition 3.1 (conjugate of cycle) For a cycle $\theta \in C_n$, the conjugate (inverse pattern) of θ, which is denoted by θ^*, is defined by

$$\theta^*(i) := n + 1 - \theta(n+1-i) \text{ for } i = 1, 2, \ldots, n.$$

In particular, if $\theta^*(i) = \theta(i)$ for any $i = 1, 2, \ldots, n$, then we call it self-conjugate. Obviously, $\theta^{**} = \theta$ for any cycle $\theta \in C_n$.

Proposition 3.1 *For any cycle $\theta \in C_n$, the graph of $y = L_\theta(x)$ is point symmetric to that of $y = L_{\theta^*}(x)$ with respect to the point $(x, y) = (\frac{n+1}{2}, \frac{n+1}{2})$.*

Proof.
From the definition of θ^*, it follows that

$$\frac{(i, \theta^*(i)) + (n+1-i, \theta(n+1-i))}{2} = \frac{(n+1, n+1)}{2}$$

for any $i = 1, 2, \ldots, n$. □

3.1 Cycles with linear map having exactly one fixed point

To show the difference between $\theta_1 = (1243)$ and $\theta_2 = (1342)$, first we present the result which is given by I. Mulvey in [10].

Definition 3.2 (RL-pattern) Let $\eta \in C_n$ be any cycle of length n. The RL-pattern for η is the sequence

$$G(\eta) = G_1 G_2 \cdots G_n \in \{R, L\}^n$$

defined by

$$G_i := \begin{cases} R & \text{if} \quad \eta^i(1) > \eta^{i-1}(1), \\ L & \text{if} \quad \eta^i(1) < \eta^{i-1}(1). \end{cases}$$

Let us denote the length of the longest string of consecutive R's in the RL-pattern for η by $R(\eta)$, and L's by $L(\eta)$ respectively. Obviously, every RL-pattern begins with an R and ends with an L.

Let $\theta \in C_n$ be a cycle of length n such that L_θ has exactly one fixed point. Let us denote this unique fixed point by $p_1 \in (1, n)$, and let

$$E_1 := \{x < p_1 | L_\theta(x) = p_1\}.$$

If $E_1 \neq \phi$, let us define p_2 by $p_2 := \max\{E_1\}$. Inductively, for $i > 1$ define E_i as follows:

$$E_i := \{x < p_i | L_\theta(x) = p_i\}.$$

If $E_i \neq \phi$, let us also define p_{i+1} by $p_{i+1} := \max\{E_i\}$. We see that for some $i \geq 1, E_i = \phi$ since L_θ has exactly one fixed point.

Similarly let

$$F_1 := \{x > p_1 | L_\theta(x) = p_1\}.$$

If $F_1 \neq \phi$, let us define q_2 by $q_2 := \min\{F_1\}$. Inductively, for $j > 1$ define F_j as follows:

$$F_j := \{x > q_j | L_\theta(x) = q_j\}.$$

If $F_j \neq \phi$, let us also define q_{j+1} by $q_{j+1} := \min\{F_j\}$. We see that for some $j \geq 1, F_j = \phi$ since L_θ has exactly one fixed point.

We are now ready to have the following definitions:

Definition 3.3 (step number and backward step number) Let $\theta \in C_n$ be a cycle of length n such that L_θ has exactly one fixed point. The step number of θ, denoted by $S(\theta)$, is the smallest value of i for which $E_i = \phi$. The backward step number of θ, denoted by $S^*(\theta)$, is the smallest value of j for which $F_j = \phi$.

Lemma 3.1 *Let $\theta \in C_n$ be a cycle of length n such that L_θ has exactly one fixed point. Then it follows that*

$$S(\theta) = S^*(\theta^*).$$

Putting θ^* for θ, we also have $S(\theta^*) = S^*(\theta^{**}) = S^*(\theta)$.

Proof.

Let us denote the unique fixed point of L_θ by $p_1 \in (1, n)$. The graph of $y = L_\theta(x)$ is point symmetric to that of $y = L_{\theta^*}(x)$ with respect to the point $(x, y) = (\frac{n+1}{2}, \frac{n+1}{2})$. Thus L_{θ^*} also has exactly one fixed point $q_1 \in (1, n)$ which is point symmetric to p_1 with respect to the point $(x, y) = (\frac{n+1}{2}, \frac{n+1}{2})$. Inductively p_i $(i \geq 2)$, the maximum of $E_{i-1} := \{x < p_{i-1} | L_\theta(x) = p_{i-1}\}$, is point symmetric to q_i $(i \geq 2)$, the minimum of $F_{i-1} := \{x > q_{i-1} | L_{\theta^*}(x) = q_{i-1}\}$, with respect to the point $(x, y) = (\frac{n+1}{2}, \frac{n+1}{2})$. Therefore, the step number S of θ is same as the back step number S^* of θ^*. □

Proposition 3.2 Let $\theta \in C_n$, $n \geq 2$ be a cycle such that L_θ has exactly one fixed point. Let $\eta \in C_m$, $m \in \mathbb{N}$ be any cycle. Then the following statements hold:

(a) If $R(\eta) > S(\theta)$ then θ does not force η.
(b) If $R(\eta) > S^*(\theta^*)$ then θ^* does not force η.
(c) If $L(\eta) > S^*(\theta)$ then θ does not force η.
(d) If $L(\eta) > S^*(\theta^*)$ then θ^* does not force η.

Proof.

The first statement (a) was proven by I. Mulvey in [10]. Using Lemma 3.1, we can prove the remaining statements (b)-(d) in similar fashion. □

Remark 3.1 *For $\theta_1 = (1243)$, we have $S(\theta_1) = 2$ and $S^*(\theta_1) = 2$. Therefore θ_1 does not force any cycle η with $R(\eta) > 2$ nor $L(\eta) > 2$. In particular, θ_1 does not force $\eta = (123 \cdots n)$, the largest element of $C_n(1)$, nor $\eta^* = (1n \cdots 32)$, the largest element of $C_n(-1)$, for any $n \geq 4$. On the other hand, the orbit $\mathcal{O}_\eta = \{15/13, 30/13, 40/13\}$ realizes $\eta = (123)$, and the orbit $\mathcal{O}_{\eta^*} = \{25/13, 35/13, 50/13\}$ realizes $\eta^* = (132)$.*

Therefore at the value $a = 1 + 2\sqrt{2} \approx 3.828427$, though the system (1) is chaotic in the sense of Li-Yorke, it seems that there still exists a large class of ± 1-modal cycles which are not realized.

Remark 3.1 leads us to the following generalization:

Theorem 3.1 *Any cycle $\theta \in C_n$, $n \geq 2$ with linear map L_θ has exactly one fixed point can not be an upper bound for $C(+1)$ nor $C(-1)$.*

Proof.

If L_θ has exactly one fixed point then $S(\theta)$ and $S^*(\theta)$ can be defined. Let $S(\theta) = k$ and $S^*(\theta) = l$ for some $k, l \in \mathbb{N}$. Then from Proposition 3.2, θ does not force any cycle η with $R(\eta) > k$ nor $L(\eta) > l$. In particular, θ does not force $\eta = (123 \cdots k+2)$, the largest element of $C_{k+2}(1)$, nor $\eta'^* = (1l+2 \cdots 32)$, the largest element of $C_{l+2}(-1)$. □

Example 3.1 *Indeed there are some cycles which can not be upper bound for $C(+1)$ nor $C(-1)$, though their modalities are big enough. For example, the linear map L_θ for -8-modal cycle $\theta = (1, 8, 2, 7, 5, 10, 3, 9, 4, 6) \in C_{10}(-8)$ has exactly one fixed point and $S(\theta) = S^*(\theta) = 1$. Thus it can not be an upper bound for $C(+1)$ nor $C(-1)$ at all.*

With Lemma 3.2 below, we have the following statement as a corollary of Theorem 3.1:

Corollary 3.1 *Any* $+2$*-modal cycle* $\theta \in C(+2)$ *can not be an upper bound for* $C(+1)$ *nor* $C(-1)$.

Lemma 3.2 *If* $\theta \in C_n(+2)$*, then* L_θ *has exactly one fixed point.*

Proof. Let us define the upper triangular space T_+ and the lower triangular space T_- as follows:

$$\begin{array}{rcl} T_+ & := & \{(x,y) \in \mathbb{N}\times\mathbb{N} \mid 1 \le x, y \le n, x < y\}, \\ T_- & := & \{(x,y) \in \mathbb{N}\times\mathbb{N} \mid 1 \le x, y \le n, x > y\}. \end{array}$$

Then for any $\theta \in C_n$, the graph of $y = L_\theta(x)$ begins with the point $(x,y) = (1,\theta(1)) \in T_+$ and ends with the point $(x,y) = (n,\theta(n)) \in T_-$. Thus from the continuity of the graph, there always exists at least one intersection point of $y = L_\theta(x)$ and $y = x$, which is corresponding to the fixed point of $L_\theta(x)$. Moreover for any $\theta \in C_n$ there always exist an odd number of intersection points of $y = L_\theta(x)$ and $y = x$. Suppose there exist more than three distinct intersection points of $y = L_\theta(x)$ and $y = x$ for $+2$-modal cycle $\theta \in C_n$. Then for some h, i, j, where $1 \le h < h+1 \le i < i+1 \le j < j+1 \le n$ it follows that

$$\begin{array}{lll} (h,\theta(h)) \in T_+, & (h+1,\theta(h+1)) \in T_-, & \theta(h+1)-\theta(h) \le -1, \\ (i,\theta(i)) \in T_-, & (i+1,\theta(i+1)) \in T_+, & \theta(i+1)-\theta(i) > 1, \\ (j,\theta(j)) \in T_+, & (j+1,\theta(j+1)) \in T_-, & \theta(j+1)-\theta(j) \le -1. \end{array}$$

This contradict "θ is $+2$-modal cycle". □

3.2 Class of bimodal cycles forced by $\theta_1 = (1243)$

In this subsection let us show that θ_1 forces a class of $+2$-modal cycles whose linear maps have exactly one fixed point.

The following is a key lemma to show the presence of periodic orbits [5].

Lemma 3.3 Let f be a continuous function from a closed interval I into itself. Let J_i, $0 \le i \le n-1$, be closed subintervals of I. If $f(J_i) \supset J_{i+1}$ for all $0 \le i \le n-2$ and $f(J_{n-1}) \supset J_0$, then there exists a periodic point $y \in J_0$ of f such that $f^i(y) \in J_i$ for all $1 \le i \le n-1$ and $f^n(y) = y$.

We are now ready to show the following claim:

Theorem 3.2 $\theta_1 = (1243)$ *forces a class of* $+2$*-modal cycles whose linear maps have exactly one fixed point.*

Proof.

For $\theta_1 = (1243)$, let us consider the following labeled digraph $G_{\theta_1} :=< G, \longrightarrow, sgn >$ [2],[11]:

(a) $G := \{I_1, J_{21}, I_{22}, I_3\}$, where $I_1 := [1,2], J_{21} := [2,2.5], J_{22} := [2.5,3], I_3 := [3,4]$;

(b) $I_1 \longrightarrow J_{21}, I_1 \longrightarrow J_{22}, I_1 \longrightarrow I_3, J_{21} \longrightarrow J_{22}, J_{21} \longrightarrow I_3, J_{22} \longrightarrow I_1, J_{22} \longrightarrow J_{21}$, $I_3 \longrightarrow I_1, I_3 \longrightarrow J_{22}$;

(c) $sgn(I_1) = +1, sgn(J_{21}) = -1, sgn(J_{22}) = -1, sgn(I_3) = +1$.

Next, for any $m, n \in \mathbf{N}$, let us consider a closed walk of length $m+n+2$:

$$\bar{a}_{m,n} = (I_1, \overbrace{\ldots, J_{21}}^{m}, I_3, \overbrace{\ldots, J_{22}}^{n}) \in G^{m+n+2},$$

where $\overbrace{(\ldots, J_{21})}^{m}$ is given by $(\widehat{J_{21}, J_{22}}, \ldots, \widehat{J_{21}, J_{22}}, J_{21})$ or $(\widehat{J_{22}, J_{21}}, \ldots, \widehat{J_{22}, J_{21}})$, and $\overbrace{(\ldots, J_{22})}^{n}$ is given by $(\widehat{J_{22}, J_{21}}, \ldots, \widehat{J_{22}, J_{21}}, J_{22})$ or $(\widehat{J_{21}, J_{22}}, \ldots, \widehat{J_{21}, J_{22}})$.

Then, from Lemma 3.3, there exists a periodic point $y \in I_1$ of L_{θ_1} such that $L^i_{\theta_1}(y) \in J_{21} \cup J_{22}$ for all $1 \le i \le m$, $L^{m+1}_{\theta_1}(y) \in I_3$, $L^i_{\theta_1}(y) \in J_{21} \cup J_{22}$ for all $m+2 \le i \le m+n+1$, and $L^{m+n+2}_{\theta_1}(y) = y$. Thus $\mathcal{O}_{m,n} = \{y, L_{\theta_1}(y), \ldots, L^{m+n+1}_{\theta_1}(y)\}$ is a periodic orbit with least period $m+n+2$ since $\bar{a}_{m,n}$ is nonrepetitive.

Next let us consider the cycle $\eta_{m,n} \in C_{m+n+2}$ which is realized by this periodic orbit $\mathcal{O}_{m,n}$. Let us rewrite $\mathcal{O}_{m,n}$ as follows:

$$\mathcal{O}_{m,n} = \{x_1, x_2, x_3, \ldots, x_l, x_{l+1}, \ldots, x_{m+n}, x_{m+n+1}, x_{m+n+2}\}$$

where $x_1 < x_2 < \cdots < x_l < x_{l+1} < \cdots < x_{m+n+1} < x_{m+n+2}$. Then $x_1 = y \in I_1$, $x_2 = L^m_{\theta_1}(y), x_3, \ldots, x_l \in J_{21}$, $x_{l+1}, \ldots, x_{m+n}, x_{m+n+1} = L^{m+n+1}_{\theta_1}(y) \in J_{22}$, and $x_{m+n+2} = L^{m+1}_{\theta_1}(y) \in I_3$ for some l. Then it follows that

$$\begin{array}{lclcl} L_{\theta_1}(x_2) & = & L^{m+1}_{\theta_1}(y) & = & x_{m+n+2}, \\ L_{\theta_1}(x_{m+n+1}) & = & L^{m+n+2}_{\theta_1}(y) = y & = & x_1. \end{array}$$

Namely the cycle $\eta_{m,n}$ has the form of

$$\eta_{m,n} = \begin{pmatrix} 1 & 2 & 3 & \cdots & m+n & m+n+1 & m+n+2 \\ \cdots & m+n+2 & \cdots & \cdots & \cdots & 1 & \cdots \end{pmatrix}.$$

Moreover from $x_2 < \cdots < x_l < x_{l+1} < \cdots < x_{m+n+1}$ and $sgn(J_{21}) = sgn(J_{22}) = -1$, it follows that $L_{\theta_1}(x_2) > \cdots > L_{\theta_1}(x_l) > L_{\theta_1}(x_{l+1}) > \cdots > L_{\theta_1}(x_{m+n+1})$. Namely the cycle $\eta_{m,n}(i)$ is monotone decreasing for $2 \le i \le m+n+1$. Therefore, for any $m, n \in \mathbf{N}$, the cycle $\eta_{m,n}$ is a $+2$-modal cycle and from Lemma 3.2 $L_{\eta_{m,n}}$ has exactly one fixed point. □

3.3 Cycles with linear map having plural fixed points

Opposite to $\theta_1 = (1243)$, $\theta_2 = (1342)$ forces both $\eta = (123\cdots n)$ and $\eta^* = (1n\cdots 32)$, for any $n \in \mathbb{N}$. Namely $\theta_2 = (1342)$ is an upper bound for the set of all ± 1-modal cycles $C(\pm 1)$.

More generally, it was proven in [5] that $\theta_2 = (1342)$ forces a class of -2-modal cycles whose linear maps have three distinct fixed points. Indeed

$$\begin{array}{rcl} \varphi_{m,n} & = & (1\ m+1\ m+2 \cdots\ m+n\ m\ m-1\ m-2 \cdots 2) \\ & = & \begin{pmatrix} 1 & 2 & 3 & \cdots & m & m+1 & m+2 & \cdots & m+n-1 & m+n \\ m+1 & 1 & 2 & \cdots & m-1 & m+2 & m+3 & \cdots & m+n & m \end{pmatrix} \end{array}$$

and $\varphi^*_{m,n} = \varphi_{n,m}$ where $m, n \ge 2$, which are forced by $\theta_2 = (1342)$, force both $\eta = (123\cdots n)$, and $\eta^* = (1n\cdots 32)$, for any $n \in \mathbb{N}$.

Remark 3.2 *$\theta_2 = (1342)$ forces both $\eta = (123\cdots n)$ and $\eta^* = (1n\cdots 32)$, for any $n \in \mathbb{N}$.*

Therefore at the value $a = 2\sqrt{2} \approx 2.828427$ the system (2) is chaotic in the sense of Li-Yorke, and all ± 1-modal cycles are already realized.

4 Final Remark

Remark 4.1 *$X_1 = -1, X_2 = 0$, and $X_3 = 1$ are fixed points of (1) for all $a \in [0,4]$. $X_1 = -1$ and $X_3 = 1$ are on the both edges of the interval $[-1,1]$ and unstable for all $a \in [0,4]$. Any periodic orbit does not bifurcate from these two fixed points. The unique fixed point of L_θ where θ is any cycle realized by (1) with a suitable value $a \in [0,4]$ is corresponding to $X_2 = 0$. (1) realizes some $+2$-modal cycles, because (1) is a cubic polynomial of x and the coefficient of x^3 is positive.*

On the other hand, $Y_2 = 0$ is a fixed point of (2) for all $a \in [0,3]$. It is stable for $0 \le a < 1$ and unstable for $1 < a < 3$. $Y_1 = -\sqrt{(a-1)/(a+1)}$ and $Y_3 = \sqrt{(a-1)/(a+1)}$ are fixed points of (2) bifurcated from $Y_2 = 0$ at the value $a = 1$. Three distinct fixed points of L_θ where θ is a cycle realized by (2) with a suitable value $a \in (1,3]$ is corresponding to Y_1, Y_2, and Y_3. (2) realizes some -2-modal cycles, because (2) is a cubic polynomial of x and the coefficient of x^3 is negative.

So the modality of the cycle and the number of fixed points of the linear map of the cycle are determined by the original system which realizes the cycle.

Conversely from the observation in Sections 2 and 3, we can conclude that not only modality of the cycle but also the number of fixed points of the linear map of the cycle affect how much chaotic the system is.

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