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Two Edge-Disjoint Heterochromatic Spanning Trees in Colored Complete Graphs

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Abstract

In 1996, Brualdi and Hollingsworth proved the existence of two edge-disjoint heterochromatic spanning trees in a complete graph whose edges are colored in such a way that each color induces a perfect matching. They proved the theorem by using Rado's theorem in Matroid Theory. In this paper, we prove that every properly edge-colored complete graph, whose order might be odd, has two edge-disjoint heterochromatic spanning trees by a graph theoretical method.

Keywords: heterochromatic spanning tree, multicolored spanning tree

A subgraph of an edge-colored graph is said to be *heterochromatic* (or *multicolored*) if all its edges have distinct colors. An edge-coloring of a graph is said to be *proper* if all the edges incident with the same vertex are colored with distinct colors. In 1996, Brualdi and Hollingsworth [1] proved the existence of two edge-disjoint heterochromatic spanning trees in every edge-colored complete graph K_{2n} $(n \geq 3)$ where each color induces a perfect matching. They proved this theorem by using Rado's theorem in Matroid Theory. In this paper, we generalize this theorem as the following Theorem 1 and give an elementary graph-theoretical proof to it. Some results related to our theorem can be found in [2] and [3], where the result of [3] does not imply our theorem because of plural spanning trees.

Theorem 1. *Every properly edge-colored complete graph K_n $(n \geq 5)$ has two edge-disjoint heterochromatic spanning trees.*

Proof. Let $c : E(K_n) \to \{1(\text{red}), 2(\text{blue}), 3, 4, \ldots, r\}$ be a proper edge-coloring of K_n. In this proof, "disjoint" means "edge-disjoint"

We first prove that the theorem is true for K_5 by induction on r, which is the number of colors. Let $V(K_5) = \{x, y, z, a, b\}$. Since the coloring is proper, $r \geq 5$. If $r = 5$ then the coloring is unique, and thus without loss of generality, we may assume $c(xy) = c(ab) = 1$, $c(xz) = c(ya) = 2$, $c(zb) = c(xa) = 3$, $c(yz) = c(xb) = 4$, $c(yb) = c(za) = 5$ (See Figure 1). Then we can find two disjoint heterochromatic spanning trees $T_1 = xy + ya + az + zb$ and $T_2 = xz + zy + yb + ba$. Hence we may assume $r \geq 6$.

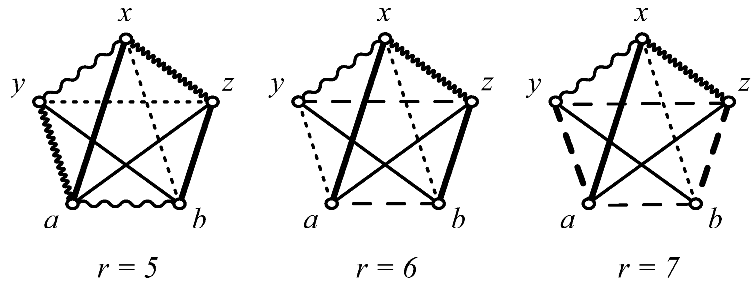

Figure 1: A complete graph K_5 colored with r colors.

If an edge e of K_5 is colored with color C_1 and no other edge of K_5 is colored with C_1, then we call such an edge e a *single colored edge*. Since $r \geq 6$, there exist at least two single colored edges, and so without loss of generality, we may assume that there exist exactly one red edge and one blue edge in K_5. If these red and blue edges are non-adjacent (i.e., independent), then by regarding a blue edge as a red edge, we can apply the inductive hypothesis to the resulting edge-colored complete graph, and hence we can obtain two disjoint heterochromatic spanning trees. Therefore we may assume that all the single colored edges are adjacent, and so there are at most four single colored edges. Moreover we may assume that $6 \leq r \leq 7$ since otherwise there are at least six single colored edges. Let xy and xz be single colored edges. We may assume that $c(xy) = 1$(red) and $c(xz) = 2$(blue). If $r = 6$ then there are exactly two single colored edges, and so without loss of generality, we may assume $c(xy) = 1$, $c(xz) = 2$, $c(ab) = c(yz) = 3$, $c(xb) = c(ya) = 4$, $c(za) = c(yb) = 5$. $c(xa) = c(zb) = 6$. If $r = 7$ then there are exactly four single colored edges, and without loss of generality, we may assume $c(xy) = 1$, $c(xz) = 2$, $c(ab) = c(yz) = 3$, $c(xb) = 4$, $c(za) = c(yb) = 5$, $c(xa) = 6$, $c(ya) = c(zb) = 7$. In each case, we can easily find the desired two disjoint heterochromatic spanning trees (see Figure 1).

We prove the theorem by induction on n. Suppose that $n \geq 6$ and the theorem holds for K_m with $5 \leq m < n$. For an edge-colored graph H, we define $c(H) = \{c(e) \mid e \in E(H)\}$. Let x be a vertex of $G = K_n$. Since $G - x$ is a properly edge-colored complete graph K_{n-1}, $G - x$ has two disjoint heterochromatic spanning trees T_1 and T_2 by the inductive hypothesis. Since $|E(T_1)| = n - 2$ and $\deg_G(x) = n - 1$, G has an edge xy such that $c(xy) \notin c(T_1)$. Thus $T_1 + xy$ is a heterochromatic spanning tree of G (see Figure 2).

If $\{c(xv) \mid v \in V(G) - \{x, y\}\} \neq c(T_2)$ then G has an edge xz $(z \neq y)$ such that $c(xz) \notin c(T_2)$. Then we obtain two disjoint heterochromatic spanning trees $T_1 + xy$ and $T_2 + xz$. Hence we may assume that $\{c(xv) \mid v \in V(G) - \{x, y\}\} = c(T_2)$. Then $T_2 + xy$ is a heterochromatic spanning tree of G. If $\{c(xv) \mid v \in V(G) - \{x, y\}\} \neq c(T_1)$ then G has an edge xz $(z \neq y)$ with $c(xz) \notin c(T_1)$. Thus we can obtain two disjoint heterochromatic spanning trees $T_2 + xy$ and $T_1 + xz$. Therefore we may assume that

$$\{c(xv) \mid v \in V(G) - \{x, y\}\} = c(T_1) = c(T_2).$$

Let u be a vertex in $V(G) - \{x, y\}$. Since $|E(T_2)| = n - 2$ and $\deg_G(u) = n - 1$, we can find an edge uw with $c(uw) \notin c(T_2)$. Since $c(T_1) = c(T_2)$, we have $uw \notin T_1$. Moreover we have $w \neq x$ as $c(ux) \in c(T_2)$. Thus $T_2 + uw$ has a cycle C containing uw. Let e be an edge of $C - uw$. Since $\{c(xv) \mid v \in V(G) - \{x, y\}\} = c(T_2)$ and $e \in T_2$, there exists an edge xv

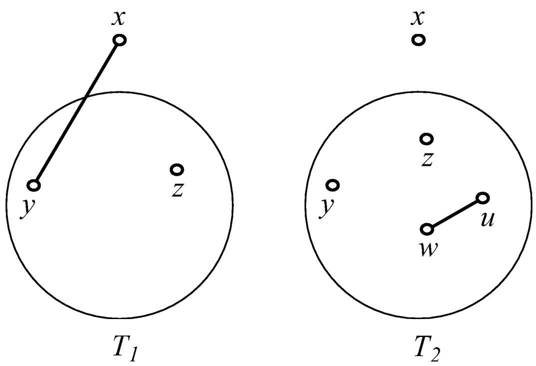

Figure 2: $T_1 + xy$ is a heterochromatic spanning tree of G.

$(\neq xy)$ with $c(xv) = c(e)$. Therefore we can obtain two disjoint heterochromatic spanning trees $T_1 + xy$ and $T_2 + uw - e + xv$. □

Brualdi and Hollingsworth [1] conjectured the existence of n edge-disjoint heterochromatic spanning trees in any edge-colored complete graph K_{2n} $(n \geq 3)$ where each color induces a perfect matching. This conjecture implies that we can partition the edges of K_{2n} into n edge-disjoint heterochromatic spanning trees. We conclude this paper by giving a more general conjecture related to the above conjecture by considering proper edge-coloring.

Conjecture 2. *Every properly edge-colored complete graph K_n $(n \geq 5)$ has $\lfloor n/2 \rfloor$ edge-disjoint heterochromatic spanning trees.*

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