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On Cycle Derivatives of Complete Graphs and Complete Bipartite Graphs

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Abstract

The concept of cycle derivative of a graph was introduced by the first two authors in [2] and [3]. The *(first) cycle derivative* of a graph G, denoted by G', is obtained by treating the induced cycles (originally called *prime cycles*) of the graph G as vertices of G' and where two vertices are adjacent if and only if they are induced cycles with a common edge. Here, we consider the cycle derivatives of the complete graph K_n and the complete bipartite graph $K_{m,n}$. We show that ${K_n}'$ is an $3(n-3)$-regular hamiltonian graph for all $n \geq 4$. Furthermore, ${K_n}'$ is eulerian if and only if $n \geq 3$ is odd. For the complete bipartite graph, we prove that ${K_{2,n}}'$ is hamiltonian for $n \geq 3$ and $2(n-2)$-regular eulerian for $n \geq 2$. In general, for $m, n \geq 2$, ${K_{m,n}}'$ is a $2(2mn - 3n - 3m + 4)$-regular eulerian graph.

Keywords: graph, cycle derivative of a graph, hamiltonian, eulerian

1 Preliminaries

We shall give only the definitions of important concepts, especially new concepts. The definitions of some common concepts in graph theory will not be given anymore. The readers may refer to some graph theory books for these definitions when necessary.

Definition 1.1. By a *graph* we mean a pair $G = (V, E)$, where V is a finite set of elements (hence, V is possibly empty) called *vertices* and E is a set of $2-$*subsets* of V called *edges*.

An edge joining two non-consecutive vertices of a cycle is called a *chord*. A cycle in a graph is called a *induced cycle* (or a *prime cycle*) if it is chordless.

Definition 1.2. Let G be graph. The graph G' whose vertices are the induced cycles of G and where two vertices are adjacent if the corresponding induced cycles have a common edge is called the *(first) cycle derivative* of G.

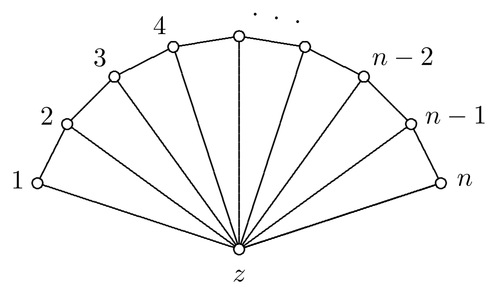

Figure 1: Labeling the vertices of the fan F_n.

Example 1.1. *The fan F_n is the sum $P_n + K_1$ of the path P_n of order n and the trivial graph K_1 with one vertex. Shown in Figure 1 is the fan F_n.*

By definition, $F_n = K_1 + P_n$. For $n = 2$, we have $F_2 = K_1 + P_2 = K_3$. Hence, ${F_2}' = {K_3}' = K_1 = P_1 = P_{2-1}$. In general, by Theorem 1.1, F_n has exactly $n - 1$ prime cycles. Furthermore, the proof of the said result states that these induced cycles are all $C_3's$ or triangles. Let F_n be the labeled graph shown in Figure 1. Evidently, each of the $n - 1$ induced cycles of F_n is of the form $c_i = [i, i+1, z, i]$, where $i = 1, 2, \ldots, n-1$. Observe that $\forall i$, $c_i c_{i+1} \in E({F_n}')$.

Consequently, ${F_n}' = P_{n-1}$.

1.1 Some Preliminary Results

First, we present initial and some known results. The known results can be found in [4] and [7]. In the first theorem, by $nic(G)$ we mean the number of induced cycles of a graph G.

Theorem 1.1. [7]
(a) $nic(F_n) = n - 1$, where $n \geq 2$;

(b) $nic(W_n) = n + 1$, where $n \geq 3$;

(c) $nic(K_n) = \binom{n}{3}$, where $n \geq 3$; and

(d) $nic(K_{m,n}) = \binom{m}{2}\binom{n}{2}$, where $m, n \geq 2$.

Theorem 1.2. [4] *The cycle derivative of the wheel W_n is itself, i.e., ${W_n}' = W_n$, where $n \geq 3$.*

In view of Theorem 1.2, the wheel behaves like the function e^x, since $D_x(e^x) = e^x$. It is a good exercise to find other graphs whose cycle derivatives are themselves. Tan [6] treated this problem in her doctoral dissertation in 1987. Tan used the term *cycle graph* for *cycle derivative* and the notation $C(G)$ instead of G'.

By *cycle derivative* of a graph we refer to the *first cycle derivative* of the graph.

The following remark involving the cycle derivative of the ladder follows similarly as in Example 1.1. Figure 2 shows the ladder $P_2 \times P_6$.

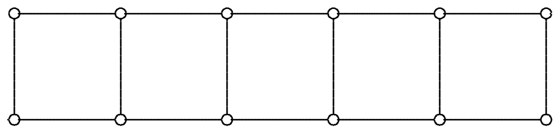

Figure 2: The ladder $P_2 \times P_6$.

Remark 1. *The cycle derivative of the ladder $P_2 \times P_n$ is P_{n-1}, i.e., $(P_2 \times P_n)' = P_{n-1}$.*

Lemma 1.1. *Let G be a connected graph. If a pendant edge is removed from G to obtain a graph H, then $G' = H'$.*

Proof. Removing a pendant edge from a connected graph G to obtain a connected graph H does not affect the derivative of G, since G and H will have the same induced cycles.

Therefore, $G' = H'$. □

By repeatedly applying Lemma 1.1 to an arbitrary graph G, we obtain the following corollary.

Corollary 1.2.1. *Let G be a graph and let H be the largest subgraph of G such that H has no pendant vertices. Then $G' = H'$.*

1.2 Main Results

Here, we present our main results involving the cycle derivative of K_n and the complete bipartite graph $K_{2,n}$.

For convenience, we shall denote or label by $1, 2, 3, \ldots, n$ the vertices of the complete graph K_n, where $n \geq 4$. The graph of K_6 is shown in Figure 3 with a labelling of its vertices.The induced cycles in K_n are the cycles of length 3. We shall denote by abc an induced cycle containing the vertices a, b, c. We shall always arrange the vertices in our notation such that $a < b < c$.

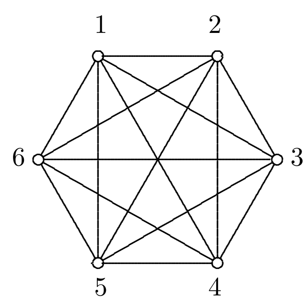

Figure 3: The complete graph K_6.

First, let us have the following lemma and its corollary:

Lemma 1.2. *Let $n \geq 4$. Then the induced cycles in K_n of the form $1bc$ i.e. $(1 < b < c \leq n)$ form a cycle in ${K_n}'$.*

Proof. The induced cycles of the form $1bc$ are:

$$\begin{array}{ccccc} 123 & 124 & 125 & \cdots & 12n \\ & 134 & 135 & \cdots & 13n \\ & & \ddots & & \vdots \\ & & & & 1(n-1)n \end{array}$$

Observe that two distinct cycles in the same row share a common edge; two distinct cycles in the same column share a common edge. Also, two consecutive cycles in the main diagonal share a common edge. Here is how to form a cycle in K_n' using only the induced cycles in our list: We start in row 1 from the extreme left to the extreme right. Then go vertically down to the last cycle. Then move up along the main diagonal. Then go up to the highest unused cycle. Then go one step to the left. Then move vertically down to the last cycle. And continue until all cycles are used. $\square$

Illustration: The construction above is illustrated in the case $n = 8$ below:

$$\begin{array}{ccccccccccc} 123 & \to & 124 & \to & 125 & \to & 126 & \to & 127 & \to & 128 \\ & \nwarrow & & & & & & & & & \downarrow \\ & & 134 & \leftarrow & 135 & & 136 & \leftarrow & 137 & & 138 \\ & & & & \uparrow & & \downarrow & & \uparrow & & \downarrow \\ & & & & 145 & & 146 & & 147 & & 148 \\ & & & & & \nwarrow & \downarrow & & \uparrow & & \downarrow \\ & & & & & & 156 & & 157 & & 158 \\ & & & & & & & & \uparrow & & \downarrow \\ & & & & & & & & 167 & & 168 \\ & & & & & & & & & \nwarrow & \downarrow \\ & & & & & & & & & & 178 \end{array}$$

Corollary 1.2.1. *Let v be a vertex of K_n, where $v \in \{1, 2, \ldots, n-2\}$. Then the induced cycles of the form vij, where $v < i < j \leq n$ in K_n form a cycle in ${K_n}'$.*

Proof. Let $n' = n - v + 1$ and rename the vertices $v, v+1, \ldots, v+n'-1$ as $1, 2, 3, \ldots, n'$, respectively. Thus, the renaming which is fairly straightforward, is as follows:

$$\begin{array}{ccc} v & \to & 1 \\ v+1 & \to & 2 \\ v+2 & \to & 3 \\ & \vdots & \\ v+n'-2 & \to & n'-1 \\ v+n'-1 & \to & n' \end{array}$$

In the above renaming, note that $v + n' - 1 = n$. Then the induced cycles of the form vij, where $v < i < j \leq n$ become cycles of the form $1ij$, where $1 < i < j \leq n'$. Since $n' = n - v + 1 \geq 4$, then by Lemma 1.2, they form a cycle in ${K_n}'$. By giving back the original names of the vertices, the corollary follows. $\square$

Theorem 1.3. *${K_n}'$ is $3(n-3)$-regular for all $n \geq 4$. Furthermore, it is hamiltonian for all $n \geq 4$.*

Proof. The induced cycles in ${K_n}'$ are all the cycles of length 3. If C is a cycle of length 3, then for each vertex x not in C, we can form exactly 3 other cycles of length 3 sharing an edge in common with C. Thus, the number of induced cycles in K_n sharing an edge with C is $3(n-3)$.

Now, let us prove that ${K_n}'$ is hamiltonian. For each $v \in \{1, 2, \ldots, n-2\}$, let $C^{(v)}$ be the cycle in ${K_n}'$ formed by the induced cycles of the form vij, where $v < i < j \leq n$. Note that each induced cycle in K_n belongs to exactly one $C^{(v)}$. What we need to do now is form a hamiltonian cycle by interconnecting the cycles $C^{(v)}$. Consider the case when $n = 4$, then $C^{(1)}$ is the cycle $[123, 124, 134]$. On the other hand, $C^{(2)}$ is $[234]$. Now, remove the edge $[124, 134]$ from $C^{(1)}$ and add the edges $[124, 234]$ and $[234, 134]$. Thus we have formed the hamiltonian cycle $[123, 124, 234, 134]$ in ${K_4}'$ using all the induced cycles in $C^{(1)}$ and $C^{(2)}$. Thus ${K_4}'$ is hamiltonian.

Assume that $n > 4$, say $n = 5$. Then $C^{(1)} = [123, 124, 125, 135, 145, 134]$, $C^{(2)} = [234, 235, 245]$, and $C^{(3)} = [345]$. What we do next is to expand the hamiltonian cycle obtained when $n = 4$. Consider $C^{(1)}$ and $C^{(2)}$. Remove the edges $[123, 124]$, $[134, 234]$, $[124, 234]$ and $[234, 245]$. Then add the edges $[123, 234]$ and $[124, 245]$. Thus, we have formed one hamiltonian cycle using all the induced cycles in $C^{(1)}$ and $C^{(2)}$. We continue to expand this cycle using $C^{(3)}$. Remove the edge $[234, 235]$ from the previously obtained cycle. Then add the edges $[234, 345]$ and $[235, 345]$. Hence, we obtained a hamiltonian cycle in ${K_5}'$ using the induced cycles in $C^{(1)}, C^{(2)}$ and $C^{(3)}$. So if there is a $C^{(4)}$, we continue to expand our cycle obtained from $C^{(1)}, C^{(2)}$ and $C^{(3)}$. We can continue the process up to $C^{(n-2)}$, i.e., until we have created a hamiltonian cycle in ${K_n}'$. □

It should be noted that the hamiltonian cycle in ${K_n}'$ is not necessarily unique. That is, the hamiltonian cycle constructed in the proof of Theorem 1.3 is not the only one.

Remark 2. *From the proof of Theorem 1.3, we have*

$$o({K_n}') = \sum_{i=1}^{n-2} o(C^{(i)}.$$

Corollary 1.3.1. *${K_n}'$ is an eulerian graph if and only if n is odd, $n \geq 3$.*

Proof. Assume n is odd. Then $n = 2k+1$, for some $k \in \mathbf{Z}$. By Theorem 1.3, ${K_n}'$ is $3(n-3)$-regular. That is, ${K_n}'$ is $3[(2k+1)-3] = 2[3(k-1)]$-regular. Hence, $\forall u \in V({K_n}')$, $deg(u)$ is even. This means that ${K_n}'$ is eulerian.

Conversely, assume ${K_n}'$ is eulerian. Then, $deg(u)$ is even $\forall u \in V({K_n}')$. Suppose n is even. Then $n = 2k$ for some $k \in \mathbf{Z}$. Furthermore, by Theorem 1.3, ${K_n}'$ is $3(n-3)$- regular.

But

$$3(n-3) = 3(2k-3) = 6k-9 = 6k-10+1 = 2(3k-5)+1,$$

is an odd number. This contradicts the initial assumption that ${K_n}'$ is eulerian.

Therefore, n must be odd. □

Theorem 1.4. *${K_{2,n}}'$ is a $2(n-2)$-regular eulerian graph for $n \geq 2$.*

Proof. By Theorem 1.1, $K_{2,n}$ has exactly $\binom{n}{2}$ induced cycles and all are C_4's. Let $K_{2,n}$ be the labeled graph shown in Figure 4. Observe that every induced cycle in $K_{2,n}$ is of the form $c_{ij} = [a, z_i, b, z_{i+j}, a]$, where $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, n-1$, and $i + j = 2, 3, \ldots, n$. Let us now count the number of induced cycles which are adjacent to a cycle c_{ij}, for some i, j. Observe that two induced cycles in $K_{2,n}$ are adjacent whenever they have exactly two common edges. Now, observe also that there are $n-2$ vertices which can be paired with z_i, a and b to form an induced cycle adjacent to c_{ij}. Similarly, there are $n-2$ vertices which can be paired with z_{i+j}, a and b to form an induced cycle adjacent to c_{ij}. This means, by symmetry, that every c_{ij} is adjacent to $2(n-2)$ induced cycles.

Thus ${K_{2,n}}'$ is a $2(n-2)$-regular graph. Furthermore, since $2(n-2)$ is even, ${K_{2,n}}'$ is eulerian. □

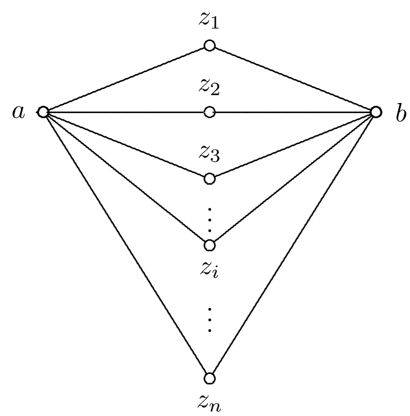

Figure 4: A labeling of $K_{2,n}$.

Theorem 1.5. *${K_{2,n}}'$ is a hamiltonian graph for each $n \geq 3$.*

Proof. Let the vertices of $K_{2,n}$ be the partite sets $A = \{a, b\}$ and $B = \{1, 2, 3, \ldots, n\}$. Any induced cycle in $K_{2,n}$ is of the form $[a, i, b, j, a]$, where $1 \leq i \neq j \leq n$. For convenience, we shall denote this induced cycle by ij. Now, when are two induced cycles ij and $i'j'$ adjacent in $K'_{2,n}$? It is easy to see that these two induced cycles are adjacent if and only if the sets $\{i, j\}$ and $\{i', j'\}$ have exactly one element in common. Our job now is to make a complete list of these induced cycles in such a way that every cycle in the list is adjacent to the cycle following it and that the first and last cycles in the list are adjacent. Let us first make a triangular tabulation of all the prime cycles in $K_{2,n}$ as follows:

$$\begin{array}{cccccc}
12 & 13 & 14 & 15 & \cdots & 1n \\
 & 23 & 24 & 25 & \cdots & 2n \\
 & & 34 & 35 & \cdots & 3n \\
 & & & \ddots & & \vdots \\
 & & & & & (n-1)n
\end{array}$$

To show that the above list is complete, notice that the number of elements in the rows, starting from the top row, has the following terms: $n-1$, $n-2$, $n-3$, ..., 3, 2, 1. Observe that we have an arithmetic sequence with common difference equal to 1. Hence, the sum, i.e., the total number of induced cycles, is $\frac{n(n-1)}{2}$, which is the value given in Theorem 1.1.

Note that in each row or column of the table, the cycles are mutually adjacent in ${K_{2,n}}'$. On the main diagonal, consecutive cycles are adjacent in ${K_{2,n}}'$. We can therefore create a hamiltonian cycle in ${K_{2,n}}'$ in exactly the same way as in Lemma 1.2. This is illustrated for the case $n = 6$ below.

12	$\rightarrow$	13	$\rightarrow$	14	$\rightarrow$	15	$\rightarrow$	16
	$\nwarrow$							$\downarrow$
		23		24	$\leftarrow$	25		26
			$\nwarrow$	$\downarrow$		$\uparrow$		$\downarrow$
				34		35		36
						$\uparrow$		$\downarrow$
						45		46
							$\nwarrow$	$\downarrow$
								56

□

As in the case of ${K_n}'$, it should be noted that the hamiltonian cycle in ${K_{2,n}}'$ is not necessarily unique.

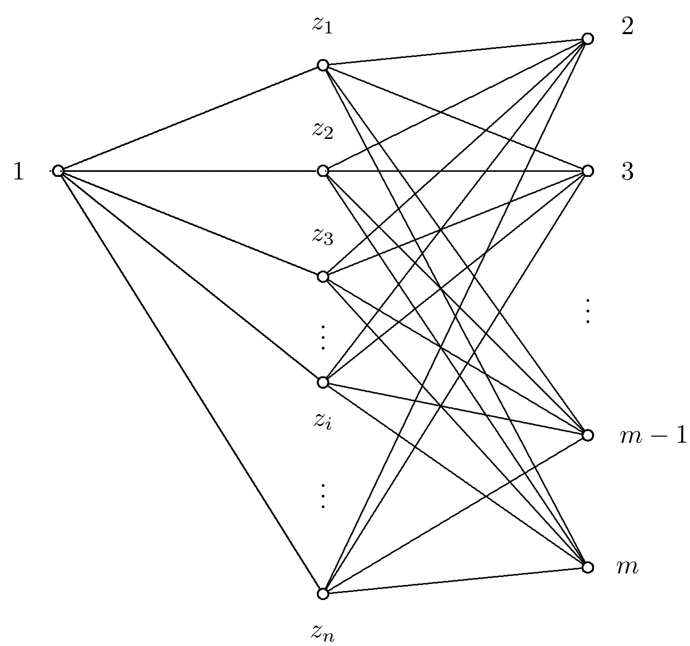

Figure 5: A labelling of $K_{m,n}$.

The next result is a generalization of Theorem 1.4.

Theorem 1.6. *For $m, n \geq 2$, ${K_{m,n}}'$ is a $2(n-2)+2(m-2)[2(n-2)+1] = 2(2mn-3n-3m+4)$-regular eulerian graph.*

Proof. By Theorem 1.1, the number of induced cycles of $K_{m,n}$ is equal to $\binom{m}{2}\binom{n}{2}$ and all are C_4's. Refer to Figure 5 and consider a particular cycle, say the cycle $c_1 = [1, z_1, 3, z_2, 1]$. Observe that with respect to the subgraph $K_{2,n}$, c_1 is adjacent to $2(n-2)$ induced cycles by Theorem 1.4. Thus, in addition, there are $m-2$ of the partite set of order m which are not accounted or considered yet. Note that for one vertex in $V(\overline{K}_2) = \{1, 2\}$ and two vertices in $V(\overline{K}_n) = \{z_1, z_2 \ldots, z_n\}$, there are $m-2$ ways to form induced cycles, all C_4's, which are adjacent to c_1. Furthermore, for each of these ways, there are $2(n-2)+1$ induced cycles of $K_{m,n}$ that are adjacent to c_1. By a similar counting procedure, if we consider the other vertex of $\overline{K}_2$ the same number of induced cycles adjacent to c_1 is obtained.

Hence, c_1 is of degree $2(n-2)+2(m-2)[2(n-2)+1]$. By symmetry and since c_1 was arbitrarily chosen, every induced cycle in $K_{m,n}$, i.e., every vertex in ${K_{m,n}}'$, is of degree $2(n-2)+2(m-2)[2(n-2)+1] = 2(2mn-3n-3m+4)$.

Therefore, ${K_{m,n}}'$ is a $2(2mn-3n-3m+4)$- regular eulerian graph. □

We enumerate some observations regarding ${K_{m,n}}'$ in the following remark.

Remark 3. *1.* ${K_{m,n}}' = {K_{n,m}}'$.

2. The size of ${K_{m,n}}'$ *is equal to*

$$\sum_{i=1}^{\binom{m}{2}\binom{n}{2}} 2(2mn-3n-3m+4) = mn(m-1)(n-1)(2mn-3n-3m+4).$$

3. From (1), one can start constructing the graph of ${K_{m,n}}'$ *by either drawing* $\binom{m}{2}$ *copies of* ${K_{2,n}}'$ *or* $\binom{n}{2}$ *copies of* ${K_{2,m}}'$.

We conclude this section with an example.

Example 1.2. ${K_{2,7}}'$ *is of order* $\binom{7}{2} = 21$*. It is* $2(7-2) = 10$*-regular, and of size* 105*. Following the proof of Theorem 1.5, one can construct a hamiltonian cycle in* ${K_{2,7}}'$ *easily.*

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