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On (k,l)-Chromatic Edge Colorings of Platonic Solids

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Abstract

A (k,l)-*chromatic coloring* is a coloring where the colors form two orbits, one with k colors and the other with l colors. In this paper, we will present two methods of obtaining (k,l)-chromatic colorings of the edges of Platonic solids; one by considering right cosets of a subgroup H of the symmetry group G of a Platonic solid and the other, by using the H-orbits of the edges of a Platonic solid. In particular, we will use the second method mentioned above to obtain some (k,l)-chromatic colorings of the edges of the tetrahedron, cube and dodecahedron.

1 Preliminaries

In their book *Tilings and Patterns* [1], Grunbaum and Shephard presented the idea of (k,l)-chromatic patterns. These are colored patterns where the colors form two orbits, one with k colors and the other with l colors. Two examples of (k,l)-chromatic patterns on the plane as given by Grunbaum and Shephard can be found in Figure 1. Under each pattern is the corresponding value of the pair (k,l). It was mentioned in the book that this topic has wide applicability and presents challenging problems. In this article we look at (k,l)-chromatic patterns formed when the edges of a Platonic solid are colored. We show how to arrive at such patterns.

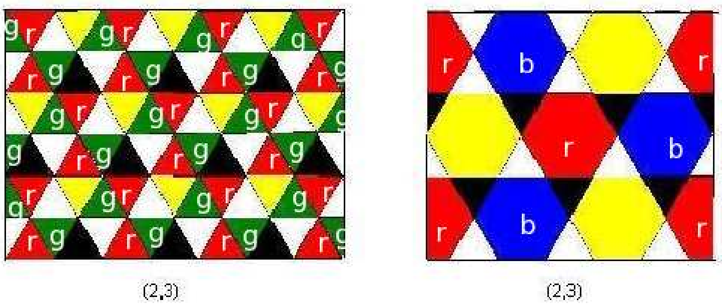


Figure 1: Examples of (k,l)-chromatic colorings in the plane.

There are three groups that play a significant role in the analysis of a colored pattern. These groups are:

G : symmetry group of the pattern when the colors are ignored
H : subgroup of elements of G which permute the colors
K : subgroup of elements of G which fix the colors.

Let C be the set of colors of the pattern. Then H acts on C and this induces a homomorphism f from C to the group of permutations of C. For $h \in H$, $f(h)$ is the color permutation induced by h. The kernel of f is K and the resulting group of color permutations $f(H) \cong H/K$.

To arrive at (k, l)-chromatic colorings of the edges of a Platonic solid, we let G be the symmetry group of the solid and take a subgroup H of G. We then color the edges so that the elements of H will permute the colors. We consider two approaches. In the first approach, we color triangular patches on the faces of the solid. In the second, we start by considering the orbits of the edges under the action of H.

2 Coloring the Edges by Coloring Faces

Let G be the symmetry group of a Platonic solid. Then each face of the solid may be divided into triangular patches where each patch is the intersection of the face with a fundamental domain for G. By associating one of the triangular patches with the identity element 1 of G, $g \in G$ may be associated with the triangular patch which is the image under g of the patch associated with 1. This association is a one-to-one correspondence. This way we arrive at a labeling of the triangular patches by the elements of G. We illustrate this in Figure 2 for the regular tetrahedron. Its symmetry group is of type $\overline{4}3m$, a group isomorphic to the symmetric group on 4 letters, S_4. The isomorphism arises from assigning to each symmetry the permutation it induces on the four vertices of the tetrahedron.

Coloring the edges of a Platonic solid corresponds to coloring the triangular patches such that the four patches adjacent to an edge have the same color. Figure 3 illustrates this for the tetrahedron. In [3] and [4] de las Peñas, Felix and Quilinguin developed a framework which can be used to color symmetrical patterns. Based on the framework, a coloring where there are two orbits of colors partition the symmetry group G into sets $\{hJ_1Y_1 : h \in H\} \bigcup \{hJ_2Y_2 : h \in H\}$ where $J_1,\ J_2 \leq H \leq G$ and $Y_1 \bigcup Y_2$ is a complete set of right coset representatives of H in G.

As an illustration, let $G = S_4$, $H = \{1,\ (123),\ (132),\ (13),\ (12),\ (23)\}$, $J_1 = H$ and $J_2 = \{1,\ (13)\}$. The right cosets of H in G are H, $H(34)$, $H(14)$ and $H(124)$. Let $Y_1 = \{1,\ (34)\}$ and $Y_2 = \{(14),\ (124)\}$. The partition of G given by $\{HY_1\} \bigcup \{h\{1,\ (13)\}\{(14),\ (124)\} : h \in H\}$ gives the $(1, 3)$-chromatic edge coloring in Figure 4.

In using the framework to arrive at (k, l)-chromatic colorings of the edges of a Platonic solid, we need a labelling of the triangular patches on the faces of the solid and we have to make sure that the patches adjacent to an edge are assigned the same color. We present in the next section another method of coloring the edges of a Platonic solid. This method does not rely on any labelling and we work directly with the edges.

3 Coloring the Edges Using the Orbits of Edges

Instead of coloring the faces to obtain the corresponding coloring of the edges, we use a second approach. The approach makes use of the H-orbits of edges. It is based on the theorem that follows.

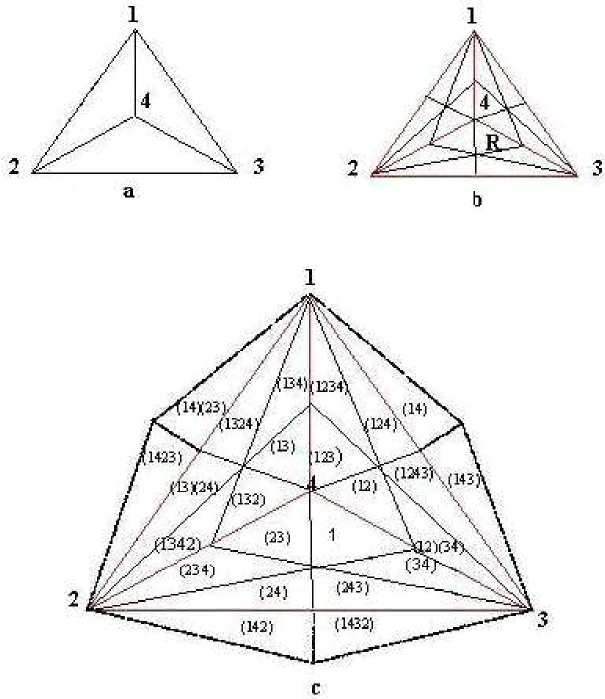


Figure 2: The labelling of the tetrahedron.

Theorem 1. *Let G be the symmetry group of a given Platonic solid. Assume a coloring of the edges, where the edges form a (k, l)-chromatic pattern. Denote by H the subgroup of elements of G which permute the colors. Then the coloring induces a partition of the set E of edges of the solid given by the union of*

$$\begin{aligned} O_1 &= \{hJ_1\{e_{11}, e_{12}, ..., e_{1m}\} : h \in H\} \textit{ and} \\ O_2 &= \{hJ_2\{e_{21}, e_{22}, ..., e_{2n}\} : h \in H\} \end{aligned}$$

where $J_1,\ J_2 \leq H$ and $e_{11}, e_{12}, ..., e_{1m}, e_{21}, e_{22}, ..., e_{2n}$ are edges of the Platonic solid.

Proof.

Consider the orbits of edges of the Platonic solid under the action of H. Then the set of edges E is partitioned into two sets E_1 and E_2 where E_1 consists of the edges in E which are colored using colors in the first orbit of colors and E_2 consists of the edges in E which are colored using colors found in the second orbit of colors. Moreover, each of E_1 and E_2 is a union of H-orbits of edges. Let $E_1 = E_{11} \cup E_{12} ... \cup E_{1m}$ and $E_2 = E_{21} \cup E_{22} ... \cup E_{2n}$ where $m + n$ is the number of H-orbits of edges and E_{1i} $(i = 1, ..., m)$ are the H-orbits of edges contained in E_1 ; E_{2j} $(j = 1, ..., n)$ are the H-orbits of edges contained in E_2.

Let c_1 be a color in the first orbit. Then for each E_{1i} $(i = 1, ..., m)$ there is an edge e_{1i} whose color is c_1. Let J_1 be the stabilizer in H of the color c_1. Then $J_1 e_{1i}$ = set of edges in E_{1i} colored c_1. For if $j \in J_1$ then je_{1i} must have color c_1. In the other direction, let e be an edge in E_{1i} which has color c_1. Since e and e_{1i} are in the same H-orbit of edges, there

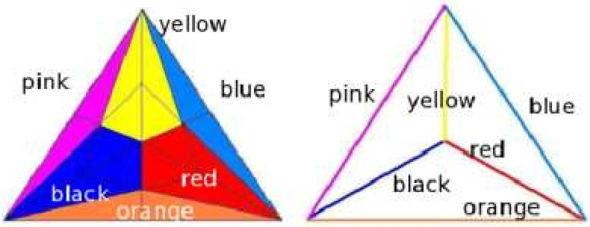


Figure 3: The correspondence between the colorings of the faces and the edges of the tetrahedron.

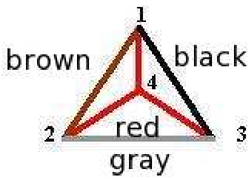


Figure 4: An example of a $(1,3)$-chromatic edge coloring.

exists $h \in H$ such that $e = he_{1i}$ and $h^{-1}e = e_{1i}$. Since e and e_{1i} are both colored c_1, then $h^{-1} \in J_1$ and so $h \in J_1$. Hence $e \in J_1 e_{1i}$. Thus the edges colored c_1 are those in the set $J_1 e_{11} \cup J_1 e_{12} \cup ... \cup J_1 e_{1m} = J_1\{e_{11}, e_{12}, ..., e_{1m}\}$.

If c_1' is any other color in the first orbit, then there exists $h' \in H$ such that $c_1' = h'c_1$. This means the edges colored c' are those in the set $h'J_1\{e_{11}, e_{12}, ...e_{1m}\}$. Thus each color in the first orbit of colors corresponds to a set of edges $hJ_1\{e_{11}, e_{12}, ..., e_{1m}\}$. Hence there is a one-to-one correspondence between the set of colors in the first orbit and the set $\{hJ_1\{e_{11}, e_{12}, ..., e_{1m}\} : h \in H\}$. The number of colors is equal to the cardinality of the preceding set which is $[H : J_1]$. Hence $[H : J_1] = k$.

Similarly, it can be shown that the second orbit of colors is in one-to-one correspondence with $\{hJ_2\{e_{21}, e_{22}, ..., e_{2m}\} : h \in H\}$ where J_2 is the stabilizer in H of a color in the second orbit. □

We now present the method for arriving at (k,l)-chromatic colorings of the edges for which the subgroup of elements of G which permutes the colors is a specified subgroup H.

Let H be a subgroup of the symmetry group G of a Platonic solid. Determine the orbits of edges of the solid under the action of H. There are 3 possible cases:

Case 1. There is only one H-orbit of edges.
Case 2. There are exactly two H-orbits of edges.
Case 3. There are more than two H-orbits of edges.

For Case 1, the action of H on the edges of the solid is transitive. This means that if we take any two edges of the solid, there is an element of H which will send one to the other.

For this reason, we cannot form a (k, l)-chromatic coloring of the edges of the solid where H permutes the colors.

For Case 2, we do the following to obtain (k, l)-chromatic colorings of the edges:

1. Let E_1 and E_2 be the H-orbits of edges
2. For each E_i, choose a particular edge e_i and determine its stabilizer in H. Let S_i be the stabilizer of e_i in H.
3. For $i = 1, 2$, consider all subgroups J_i of H containing S_i. Color the edges in the set E using the left cosets of J_i in H. Thus an orbit of colors is now of the form $O_i = \{hJ_ie_i : h \in H\}$. For each coloring the number of colors is $[H : J_i]$. The number of colors divides $\mid E_i \mid$ since $\mid E_i \mid = [H : S_i]$ and $[H : S_i]$=$[H : J_i] \cdot [J_i : S_i]$.

For Case 3, we need to take the union of some of the H-orbits to come up with only two orbits of colors.

1. Let F_1 and F_2 be the two sets of edges which are a union of some H-orbits. If F_1 consists only of a single H-orbit then color the edges in F_1 as in Case 2.
2. If F_1 consists of two or more H-orbits, then we color the edges in F_1 as follows:

   a. Choose an edge from each H-orbit in F_1. These will all be assigned the same color. They will be called e_{11}, e_{12}, ..., e_{1m} where m is the number of H-orbits contained in F_1.
   b. Determine the stabilizer S_{1i} in H of each e_{1i} .
   c. Take the join S_1 of all the stabilizers S_{1i}, $(i = 1, 2, ...m)$.
   d. Use the left cosets of a subgroup J_1 in H which contains S_1 to color the edges in F_1. The orbit of colors is of the form $O_1 = \{hJ_1\{e_{11}, e_{12}, ..., e_{1m}\} : h \in H\}$.

   We color the edges in F_2 in the same manner.

We present next two theorems which are useful in listing all (k, l)-chromatic edge colorings of the Platonic solids.

Theorem 2. *Let G be the symmetry group of a Platonic solid. Let $H \leq G$. If $[G : H]$=2, then there is no (k, l)-chromatic coloring of the edges of the solid where H permutes the colors.*

Proof.

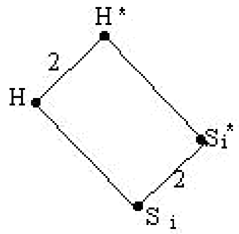
This follows from the fact that the symmetry group G of the solid is transitive on the set of edges of the solid. Thus, the edges of the solid form only one orbit under the action of H. But we need 2 orbits of colors. Therefore we cannot form a (k, l)-chromatic coloring of the edges where the group that permutes the colors is H. □

Theorem 3. *Let G be be the symmetry group of a Platonic solid. Let H^* and H be subgroups of G such that $[H^* : H]$=2. If there are only two H-orbits of edges and these are equal to the H^*-orbits of edges, then H^* permutes the colors in the (k, l)-chromatic colorings obtained under H.*

Proof.

Let E_i be an H^*-orbit of edges of the solid. This means E_i is also an H-orbit of edges. Let S_i^* be the stabilizer in H^* of an edge e_i in E_i and S_i be the stabilizer in H of the same edge. Thus we have

$$\begin{aligned} \mid E_i \mid &= [H^* : S_i^*] \\ &= [H : S_i] \end{aligned}$$



But $[H^* : H] = 2$ and $[H^* : S_i^*] = [H : S_i]$. By the Diamond Isomorphism Theorem,

$$\begin{aligned} \frac{H^*}{H} &\cong \frac{S_i^*}{H \cap S_i^*} \\ &\cong \frac{S_i^*}{S_i} \\ &\cong C_2 \end{aligned}$$

and thus $[S_i^* : S_i] = 2$.

Let $\alpha \in S_i^* \setminus S_i$. Let J_i be a subgroup of H containing S_i and J_i^* be the subgroup of H^* generated by J_i and α. Then

$$\begin{aligned} H^* &= H \cup H\alpha, \\ S_i^* &= S_i \cup S_i\alpha, \text{ and} \\ J_i^* &= J_i \cup J_i\alpha \end{aligned}$$

We need to show that α permutes the colorings of E_i under H.

Let $O_i = \{hJ_ie_i : h \in H\}$ be a coloring of E_i under H. Then a color in O_i is of the form hJ_ie_i. Thus

$$\begin{aligned} \alpha hJ_ie_i &= h'\alpha J_ie_i \quad h' \in H \\ &= h'J_i\alpha e_i \quad \text{since } \alpha \in S_i^* \subseteq J_i^* \\ &= h'J_ie_i \quad \text{since } \alpha \text{ stabilizes } e_i. \end{aligned}$$

Therefore, α permutes the colors in O_i. □

4 Examples

1. Let $H = 3m = \{1,\ 3,\ 3^{-1},\ m,\ m,\ m\}$, a group isomorphic to D_3. There are 3 H-orbits of edges of the cube. These are E_1, E_2, and E_3, in Figure 5b, c and d respectively. To come up with only two orbits of colors, we need to take the union of 2 of the 3 H-orbits as follows:

- $F_1 = E_1$ and $F_2 = E_2 \cup E_3$
- $F_1 = E_1 \cup E_3$ and $F_2 = E_2$
- $F_1 = E_1 \cup E_2$ and $F_2 = E_3$

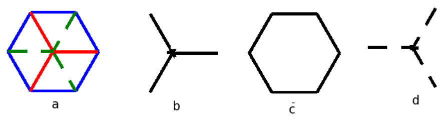


Figure 5: The H-orbits of edges of the cube where $H = 3m$.

The first and third partitions of edges above will yield equivalent colorings so we will only consider the second and third partitions.

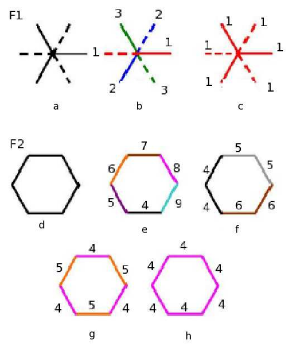


Figure 6: The resulting colorings where $F_1 = E_1 \cup E_3$ and $F_2 = E_2$.

a. Let $F_1 = E_1 \cup E_3$ and $F_2 = E_2$. In F_1, the stabilizer of an edge in H is of type m. If the edges chosen for each of the two H-orbit of edges has the same stabilizer, $S_1 = m$ then choices for J_1 are m and $H = 3m$. The corresponding colorings are in Figure 6b and c. In F_2, which contains only the H-orbit E_3, the stabilizer of an edge in $H = 3m$ is 1. Therefore, we may take J_2 to be any subgroup of H. The 4 possible colorings are in Figure 6e-h. Combining each of the 2 colorings in Figure 6b and c with each of the colorings in Figure 6e-h result in 8 (k,l)-chromatic colorings, where the elements of H permute the colors.

b. Let $F_1 = E_1 \cup E_2$ and $F_2 = E_3$. Then we can assign colors to the edges in F_1 and F_2 separately. Consider the edges in E_1. An edge in this set has stabilizer in H which is of type m. An edge in E_2 on the other hand, has a stabilizer in H which is of type 1. Thus we take $J_1 = m$ or $J_1 = 3m$. If $J_1 = m$, there are 3 different colorings of F_1 that arise. See Figure 7d, e and f. For F_2, there are only two ways of coloring the edges and these are found in Figure 7h and i. We get a (k,l)-chromatic edge coloring if we combine the colorings in Figure 7d-f with the colorings in Figure 7h and i, with the

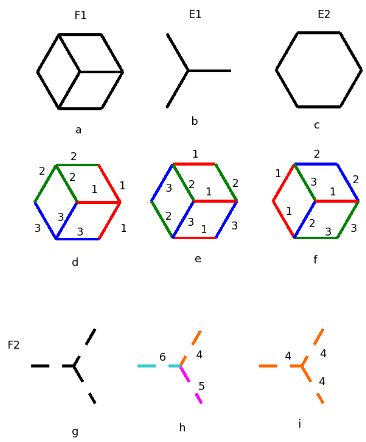


Figure 7: The resulting colorings where $F_1 = E_1 \cup E_2$ and $F_2 = E_3$.

exception of the coloring obtained when the colorings in Figure 7 d and i are combined, since in this coloring only one orbit of colors is formed.

2. Figure 8 illustrates a $(1,5)$-chromatic coloring of the dodecahedron where $H = \bar{5}m$, $J_1 = H$, $J_2 = 2/m$ which was obtained using the method described previously. The inner edges (not numbered) are all of the same color.

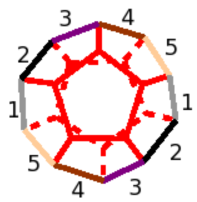


Figure 8: A $(1,5)$-chromatic coloring of the dodecahedron where $H = \bar{5}m$.

3. All the resulting (k,l)-chromatic colorings for the tetrahedron are in Figure 9 and the corresponding subgroups used are found in Table 1.

5 Conclusion

We used two methods of obtaining (k,l)-chromatic colorings of the edges of Platonic solids:

1. By using the right cosets of subgroups H of the symmetry group G of the solid and
2. By using H-orbits of the edges.

The first method is dependent on the labelling on the faces of the solid and in cases where the number of right cosets of H in G is already quite numerous, the method becomes tedious.

coloring	H	J_1	J_2	(k,l)
a	$\overline{4}2m$	$\overline{4}2m$	$\overline{4}2m$	(1,1)
b	$\overline{4}2m$	$\overline{4}2m$	2	(1,4)
c	$\overline{4}2m$	$mm2$	$\overline{4}2m$	(2,1)
d	$\overline{4}2m$	$mm2$	222	(2,1)
e	$3m$	$3m$	$3m$	(1,1)
f	$3m$	$3m$	m	(1,3)
g	$3m$	m	$3m$	(3,1)
h	$mm2$	$mm2$	$mm2$	(1,1)
i	$mm2$	$mm2$	m	(1,2)
j	m	m	m	(1,1)
k	m	1	1	(1,2)

Table 1: The subgroups used to color the edges of the tetrahedron.

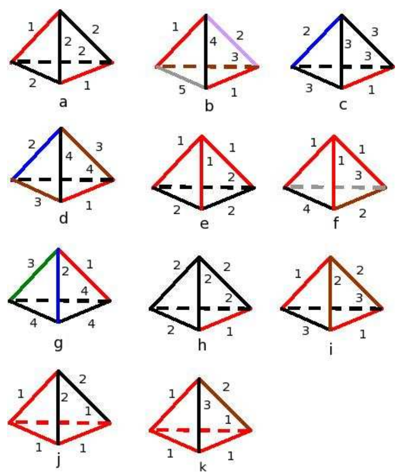


Figure 9: Summary of (k,l)-chromatic colorings of the edges of the tetrahedron.

The second method, on the other hand is already independent of the labels, so it has an advantage over the first method. This method may also be used to obtain (k,l)-chromatic colorings of other symmetrical objects. In particular this method may be used to obtain (k,l)-chromatic colorings of the vertices of Platonic and other solids.

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